A New Method of Generating Multivariate Weibull Distributed Data

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Abstract—**In order to fully test detector frameworks, it is important to have representative simulated clutter data readily available. While measured clutter data has often been fit to the Weibull distribution, generation of simulated complex multivariate Weibull data with prescribed covariance structure has been a challenging problem. As the multivariate Weibull distribution is admissible as a spherically invariant random vector for a specific range of shape parameter values, it can be decomposed as the product of a modulating random variable and a complex Gaussian random vector. Here we use this representation to compare the traditional method of generating multivariate Weibull data using the Rejection Method to a new approximation of the modulating random variable that lends itself to efficient computer generation.**

I. Introduction

Classical detector theory relies on the assumption of a null hypothesis distributed as independent, identically distributed Gaussian data [1]. However, sensing applications, radar in particular, often encounter heavy-tailed distributions with a higher frequency of outliers than is described by the Gaussian distribution (e.g. [2]–[5]). In the particular case of radar detection, the Gaussian assumption is commonly violated by measured radar clutter. Further complicating matters, measured clutter can often be fit to multiple possible statistical models equally well [5]. In addition, measured data is necessarily limited in scope and new experimental data is expensive to obtain. Hence, it is important to have a library of clutter models with which to test potential robust radar detectors before they are validated with experimental data [6]–[9]. A library of clutter models is also useful to validate the emerging field of cognitive radar detection, which continues the legacy of knowledge-aided detector research [8], [10]. Specifically, data under test may be compared to the library of clutter models to provide a null hypothesis suggestion [8], [9], [11]. This null hypothesis suggestion may be used to inform either a distribution specific detector or to estimate directly the detection threshold [9], [11]–[14].

The family of spherically invariant random processes (SIRPs) has been shown to encompass the majority of the models for non-Gaussian radar clutter, including the K, Pareto/Student-*t*, and Weibull distributions [6], [7], [15]–[17]. However, it should be noted that the lognormal distribution, while experimentally validated, is not admissable as a SIRP [6], [15], [18]. A spherically invariant random vector (SIRV) is a sample drawn from a SIRP. By definition, a SIRV may be formed as a Gaussian random vector (real or complex) multiplied by a positive random variable [6], [7], [17]. In other words, a length L, zero mean, white SIRV $\mathbf{X}$ is formed as

$$\mathbf{X} = V\mathbf{Z}, \tag{1}$$

where $\mathbf{Z} \sim CN(0, \mathbf{I})$ is a length L vector and $\mathbf{I}$ is the $L \times L$ identity matrix. Therefore, clutter modeled as a SIRV is locally Gaussian (i.e. a single draw from V), but globally non-Gaussian. As such, the modulating random variable provides a power modulation over the scene, resulting in an increased number of outlier samples as compared to the Gaussian assumption.

In addition to being experimentally validated, the SIRV model possesses several convenient properties. Perhaps the most useful property of SIRVs is the property of closure under linear transforms [6], [7]. Specifically, linearly transforming a SIRV results in a SIRV with the same modulating random variable V, but with a different covariance matrix and mean vector. Therefore, if the modulating random variable V is *both known and can be generated* (e.g. via Matlab), then it is straightforward to generate efficiently the desired data with an arbitrary mean and covariance matrix of any dimensionality. Such simulation capability allows for the quick comparison of detector performance with multiple models.

The Weibull distribution has been suggested as a good fit for heavy tailed, non-Gaussian clutter since at least 1969 [19]. While Waloddi Weibull noted that the Weibull distribution had no physical justification in general [20], the admissibility of the Weibull distribution as a SIRV provides a physical justification for the Weibull distribution to model radar clutter returns [6], [7], [15]–[17].

For SIRVs such as the K and Pareto distribution, the modulating random variables are known and can be formed from transformed Gamma distributed random variables [6], [7], [21]. Unfortunately, the general closed form solution to the modulating variable for the Weibull distribution is only known in the form of two equivalent infinite summations [17]. Previous work has used the norm of the Weibull SIRV to generate Weibull data via the Rejection Method [6], [7]. However, this method is dependent on the dimensionality of the desired vector, and can be computationally infeasible for high dimensionalities or low values of the shape parameter. In contrast, here we bound a novel approximation of the modulating random variable using one of the infinite summations from [17]. This bound may then be used in conjunction with the Rejection Method to generate previously computationally infeasible instantiations of Weibull distributed data.

II. Traditional Weibull Data Generation

A. The Weibull Distribution

The univariate envelope of the Weibull distribution (i.e. the amplitude of a complex Weibull random variable) is defined as

$$f_R(r) = abr^{b-1}\exp(-ar^b), \quad r > 0, \tag{2}$$

where $a > 0$ is the scale parameter of the distribution, and the shape parameter is $0 < b \leq 2$. Notably, for $b = 1$ the Weibull coincides with the exponential distribution, and for $b = 2$ the Weibull coincides with the Rayleigh distribution.

It can be shown that the envelope of the complex multivariate Weibull distribution is [6], [7], [17]

$$f_R(r) = \frac{2\,(-1)^L}{\Gamma(L)}\sum_{k=1}^{L} C_k \frac{a^k}{k!} r^{kb-1}\exp\left(-ar^b\right), \tag{3}$$

where

$$C_k = \sum_{m=1}^{k} (-1)^m \binom{k}{m} \frac{\Gamma\left(1+\frac{mb}{2}\right)}{\Gamma\left(1+\frac{mb}{2}-L\right)} \tag{4}$$

and $\Gamma(t)$ is the Gamma function. To simplify the examination of the two parameter Weibull distribution, consider a normalized form of the distribution where the scale parameter is restricted to enforce $E[r^2] = \sqrt{L}$. This normalization can be accomplished by relating the scale parameter to the shape parameter b as

$$a = \left[\Gamma\left(\frac{2}{b}+1\right)\right]^{b/2}. \tag{5}$$

Unless noted otherwise, hereafter the scale parameter is assumed to be set according to (5).

For the univariate case, Weibull distributed data is readily generated via a zero-memory non-linear (ZMNL) transform [22]. However, the ZMNL transform does not lend itself well to generating correlated vector valued data. The difficulty in generating vector valued data is due to the fact that the ZMNL does not afford independent control over the first order pdf and correlation structure. Therefore, the Method of Norms was used in [6], [7], [9] to generate Weibull distributed data.

B. Method of Norms

SIRVs are notable in that a length L SIRV may be generalized into a set of spherical coordinates. These spherical coordinates are constructed from a set of L independent random variables, of which only one random variable in the set has a distribution dependent on the SIRV distribution [6]. As might be expected from the form of (1), there is thus one "degree of freedom" that differentiates individual SIRV distributions. This random variable is by definition the norm of the distribution. Therefore, if the exact distribution of the modulating random variable is not known or able to be generated, the Method of Norms may be used to generate the random variable. The Method of Norms requires the ability to generate random variables distributed according to the norm of the desired random variable and the generation of random vectors from a different SIRV distribution (typically the Gaussian distribution). The method is summarized as [6], [9]

1) Generate a white, zero mean complex Gaussian random vector with identity covariance matrix, denoted as $\mathbf{z}$.
2) Compute the norm of $\mathbf{z}$ as $r_Z = ||\mathbf{z}|| = \sqrt{\mathbf{z}^H\mathbf{z}}$.
3) Generate a random variable distributed according to the desired norm of the SIRV $\mathbf{x}$, $r_X = ||\mathbf{x}|| = \sqrt{\mathbf{x}^H\mathbf{x}}$.
4) Generate $\mathbf{x}$ as $\mathbf{x} = \mathbf{z}\frac{r_X}{r_Z}$.

While the norm of the multivariate Weibull distribution is given in (3), the generation of random variables distributed according to (3) is not a straightforward task.

C. Generating Data from an Unknown Distribution via the Rejection Method

In practice, the most common methods of generating arbitrary random variables are via the transformation of a random variable that can be generated (e.g. transformation of Gaussian or Uniform distributed data) or by using the inverse of the cumulative distribution function (cdf) [23]. Let R be the random variable to be generated, and assume that the inverse cdf of R is unavailable. Further, let U_1 be a random variable that can be readily generated and whose scaled pdf bounds the pdf of R. In other words, $f_R(r) \leq kf_{U_1}(r), \quad \forall r, \quad k > 0$. The Rejection Method [6], [9], [23] is then:

1) Generate a sample u_1.
2) Generate $u_2 \sim U(0, kf_{U_1}(u_1))$.
3) If $u_2 \leq f_R(u_1)$, then accept the point (i.e. $r = u_1$)
4) Otherwise, reject u_1.

The overall process is illustrated in Figure 1. The Rejection Method thus performs a uniform sampling under $kf_{U_1}(u_1)$, and rejects the points that fall between $kf_{U_1}(u_1)$ and $f_R(r)$. For efficient use of the Rejection Method, it is important to minimize the quantity $kf_{U_1}(r) - f_R(r)$.

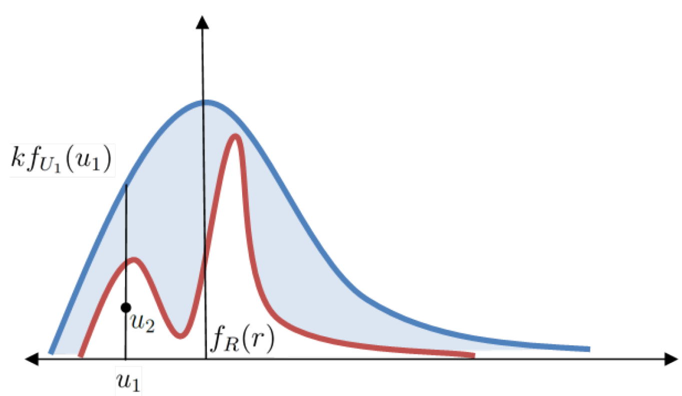

Fig. 1: Rejection Method

D. Generating Weibull Norm Values via the Rejection Method

In order to use the Rejection Method, a suitable bounding distribution must be found. In [6], [9] the Uniform distribution was suggested as a bounding distribution. However, note that truly accurate data generation via the Rejection Method requires accurate bounding of both the domain and range of the desired random variable. By definition, the Uniform distribution has limited support, and the pdf of (3) has support $r > 0$. Therefore, it is impossible to bound the distribution with a finite Uniform random variable.

In order to mitigate the impact of the support mismatch, in [9] the support of U_1 and the value of k are set by balancing the desired accuracy of the data generation and number of points rejected. Fortunately, for $b > 1$ the pdf of (3) has a finite maximum value that can be determined as the solution to a root finding problem [9]. In contrast, for $b \leq 1$ the maximum value of the pdf is $f_R(r \to 0) \to \infty$. Therefore, k must be selected carefully to effectively bound the pdf while maintaining a reasonable rate of rejecting samples. Finally, (3) also depends on the length of the desired random vector. As such, for realistic dimensionality (e.g. $L > 16$) the Rejection Method becomes very difficult to implement at low values of the shape parameter [6]. As an example, in [9] random variables distributed according the norm of a length $L = 4$ complex Weibull distribution with shape parameter $b = 0.7$ and scale parameter $a = 1$ were generated. It was found that after generating 10^5 points, the probability of rejecting a point was $\approx 99.92\%$.

To provide further context, consider the norm of length $L = 64$ Weibull distributed SIRVs whose scale parameter has been normalized as defined in (5). In the style of Figure 1, the graphical depiction of the chosen pdfs for $b = 0.7$ and $b = 1.5$ is given in Figures 2 and 3, respectively. For the bounding Uniform distribution for both $b = 0.7$ and $b = 1.5$, U_1 is chosen to be distributed as $U_1 \sim U(0, c)$, where c is found as

$$1 - F_R(c) < 10^{-4}. \tag{6}$$

Therefore, the support of the bounding distribution covers 99.99% of the desired distribution. For the choice of $U_1 \sim U(0, c)$, the value of k is found by solving

$$\begin{aligned} k f_{U_1}(u_1) &= f(k') \\ k &= c f(k'), \end{aligned} \tag{7}$$

where $f(k')$ is the maximum value of the pdf. For $b = 1.5$ the maximum value of the pdf is found numerically. However, for $b = 0.7$ the value of k' must be chosen to balance between the accuracy of the Rejection Method and computational feasibility (i.e. the number of rejected points). Here we follow the suggestion of [9] and find k' as the solution to

$$F_R(k') < 10^{-4}. \tag{8}$$

As a consequence of this approximation, close examation of Figure 2 shows that a small portion of the left tail of the pdf $f_R(r)$ extends above the bounding distribution $f_{U_1}(u_1)$. Therefore, due to the mismatch between the desired distribution and the bounding distribution, it is expected that the resultant random variables will not fully adhere to the envelope of the desired Weibull distribution.

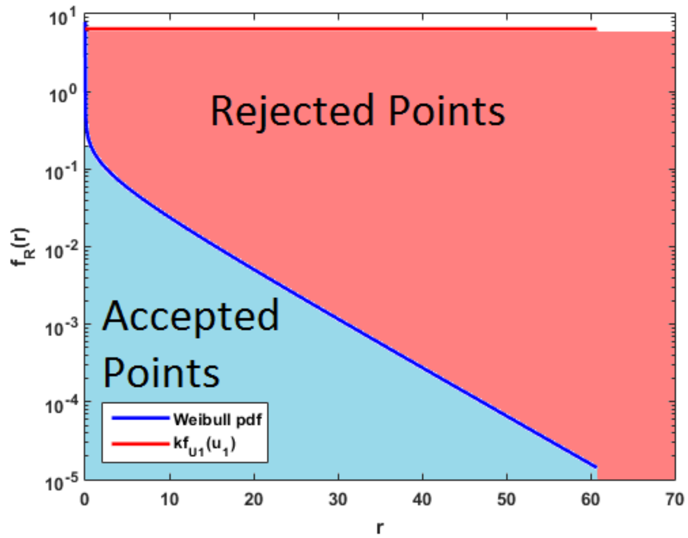

Fig. 2: Rejection Method for Estimating Norm, $b = 0.7$

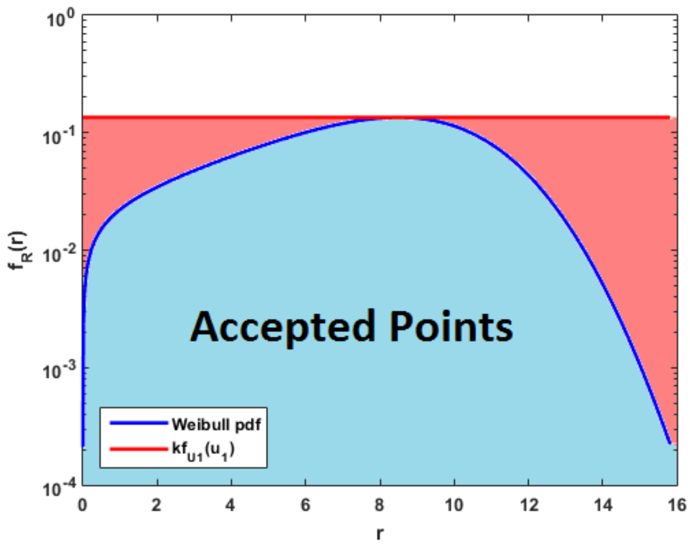

Fig. 3: Rejection Method for Estimating Norm, $b = 1.5$

It was noted in [9] that the rejection rate increased as the shape parameter decreased, to the point where it was computationally infeasible to generate data by this method for values of $b < 0.7$. In measured data, the Weibull distribution has been fit with estimated shape parameters as low as $b = 0.3$ [24]. Clearly, a more efficient method is needed to generate Weibull distributed data to better simulate measured clutter.

III. Approximating the Random Variable V

In contrast to the norm of an SIRV, the modulating random variable V by definition does not depend on the dimensionality of the SIRV. Therefore, if V can be generated, then SIRV data of the desired distribution and dimensionality can easily be generated from Gaussian distributed data. However, the modulating random variable of the Weibull distribution, denoted as V_W, is not obtainable in a form that is easily manipulated.

The characteristic function of V_W can be constructed via the moment generating function. However, determining the pdf of V_W via the inverse Laplace transform of the characteristic function is not tractable [17]. Alternately, the pdf of the square root of V_W may be found from the inverse Mellin Transform and the calculus of residues [17]. Setting $V_W \doteq \sqrt{\tau}$, the distribution of τ is found to be [17]:

$$f_\tau(\tau) = ab(2\tau)^{b/2-1} \sum_{n=0}^{\infty} \frac{\left(-a\left(\sqrt{2\tau}\right)^b\right)^n}{n!\Gamma\left(1-\frac{b}{2}(n+1)\right)}, \quad 0 < b \leq 2. \tag{9}$$

An equivalent pdf may be found by modifying the scale parameter as

$$a' = (a)^{2/b}. \tag{10}$$

Substituting (10) into (9), the pdf of τ becomes

$$f_\tau(\tau) = a'b(2a'\tau)^{b/2-1} \sum_{n=0}^{\infty} \frac{\left(-\left(2a'\tau\right)^{b/2}\right)^n}{n!\Gamma\left(1-\frac{b}{2}(n+1)\right)}, \quad 0 < b \leq 2. \tag{11}$$

As $\tau > 0$, perform the one-to-one transformation

$$\begin{aligned} x &= g(\tau) = (2a'\tau)^{b/2} \\ \implies \tau &= g^{-1}(x) = \frac{1}{2a'}x^{2/b}, \end{aligned} \tag{12}$$

where

$$\left|\frac{\mathrm{d}g^{-1}(x)}{\mathrm{d}x}\right| = \frac{1}{a'b}x^{2/b-1}, \tag{13}$$

which results in

$$\begin{aligned} f_x(x) &= f_\tau\left(g^{-1}(x)\right)\left|\frac{\mathrm{d}g^{-1}(x)}{\mathrm{d}x}\right| \\ &= x^{2/b-1}(x)^{1-2/b} \sum_{n=0}^{\infty} \frac{\left(-\left(x^{2/b}\right)^{b/2}\right)^n}{n!\Gamma\left(1-\frac{b}{2}(n+1)\right)} \\ &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma\left(1-\frac{b}{2}(n+1)\right)}. \end{aligned} \tag{14}$$

Hence, the pdf of τ can be transformed into an infinite summation with a relatively simple form. In addition, the transformation of (12) is unambiguously invertible.

However, it is important to note that the reciprocal of the Gamma function displays oscillatory behavior tending towards $\pm\infty$ for increasingly negative arguments. This behavior is illustrated in Figure 4. The argument to the Gamma function in (14) is negative for

$$n > \frac{2}{b} - 1. \tag{15}$$

Therefore, when evaluating truncated instantiations of (14) care must be taken to remove any numerical instabilities that cause the expression to deviate from being a valid probability density function (i.e. negative probabilities, a total cumulative distribution function > 1). Inspection of (15) suggests that higher values of b will cause unstable behavior at lower values of n.

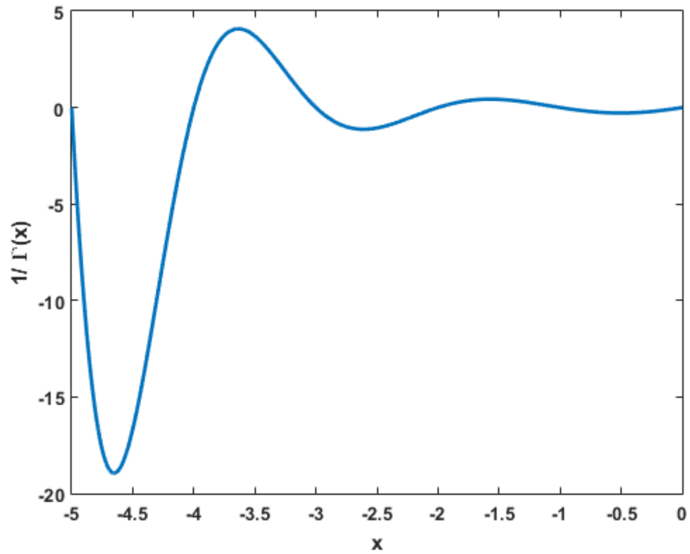

Fig. 4: Reciprocal of the Gamma function for negative arguments

Upon examination of (14), the exponential function appears to be a possible bound. In other words, we wish to find an ϵ such that

$$\begin{aligned} f_X(x) &\leq f_Y(y;\epsilon) \\ \implies \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma\left(1-\frac{b}{2}(n+1)\right)} &\leq \exp\left(-\frac{y}{\epsilon}\right). \end{aligned} \tag{16}$$

Using the Taylor series expansion for e^{-y}, (16) becomes

$$\begin{aligned} f_X(x) \leq f_Y(y;\epsilon), &\quad y = x, \epsilon > 0 \\ \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma\left(1-\frac{b}{2}(n+1)\right)} &\leq \sum_{n=0}^{\infty} \frac{\left(-\frac{x}{\epsilon}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\epsilon^n}. \end{aligned} \tag{17}$$

As the exponential pdf is defined as

$$f_Y(y;\lambda) = \lambda \exp(-\lambda x), \tag{18}$$

if the bound of (17) holds, the Rejection Method may be used with $k = \epsilon = \frac{1}{\lambda}$, yielding

$$\begin{aligned} kf_{U_1}(u_1) &= \epsilon f_Y(y;\epsilon) \\ &= \exp\left(-\frac{y}{\epsilon}\right). \end{aligned} \tag{19}$$

A potential bound has been found whose shape parameter is determined with respect to the shape parameter of the desired Weibull distribution as

$$\epsilon = \Gamma\left(1-\frac{b}{2}\right), \quad b < 1.6. \tag{20}$$

In other words, the bound is obtained using the value of the Gamma function in (14) evaluated at $n = 0$. A formal investigation of this bound is ongoing.

The summation of (14) and the bound of (19)-(20) were evaluated for $b = 0.7$ and $b = 1.5$, and the results shown in Figures 5 and 6, respectively. Values of $n = 49$ for $b = 0.7$, and $n = 145$ for $b = 1.5$ were used to evaluate the summation. From examination of Figures 2 and 5, both the norm and the transformed modulating random variable of the Weibull distribution agree by producing their maximum value at 0 for b = 0.7. From a similar examination of Figures 3 and 6, both the norm and the transformed modulating random variable reach their maximum probability value at a point greater than zero.

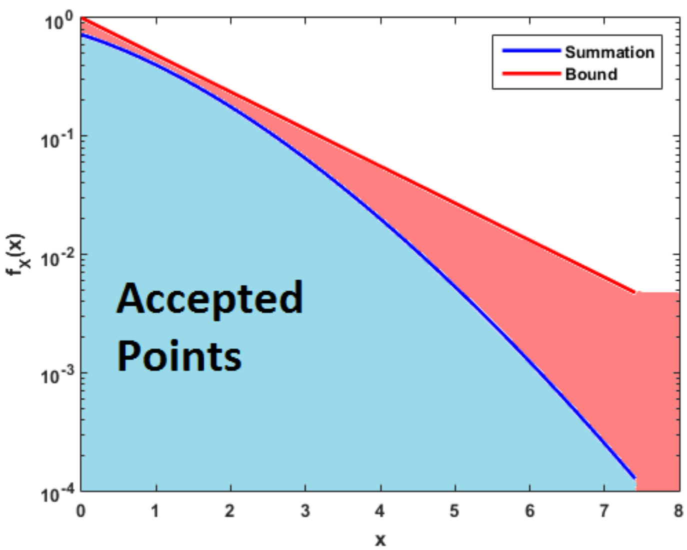

Fig. 5: Rejection Method for Estimating X, $b = 0.7$

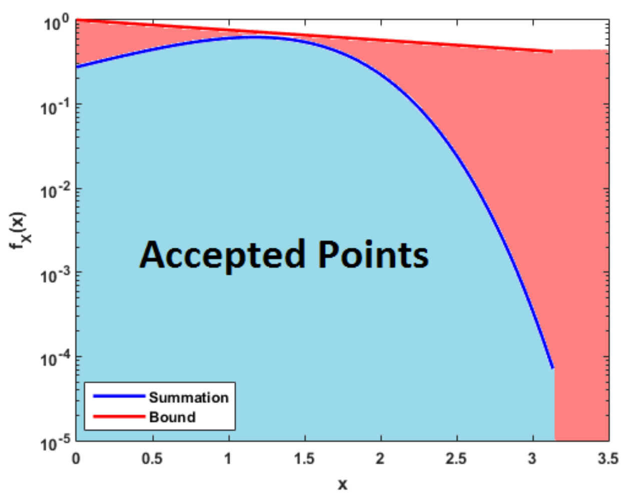

Fig. 6: Rejection Method for Estimating X, $b = 1.5$

Armed with an approximation of the distribution of the transformed modulating random variable and a viable bound, the transformed modulating random variable can be generated via the Rejection Method. Therefore, we denote the following method of generating multivariate Weibull distributed data as the Transform Method:

1) Generate the transformed random variable X distributed according to (14) via the Rejection Method using bounding random variables defined in (19)-(20).
2) Invert the transformations of (10) and (12) to generate τ.
3) Generate a white, zero mean complex Gaussian random vector with identity covariance matrix, denoted as $\mathbf{z}$.
4) Generate the complex Weibull vector $\mathbf{x}$ as $\mathbf{x} = \sqrt{\tau}\mathbf{z}$.

IV. Comparsion of Techniques

As noted in Sections II-D and III, both the previously developed Method of Norms (MoN) and the new Transform Method (TxM) use approximations in order to implement the Rejection Method. Specifically, the MoN uses a bounding distribution with a finite support to bound a distribution with an infinite support. Hence approximations are made using equations (6) and (8) to balance the computational cost of the MoN with the accuracy of the resulting distribution. In contrast to the bounds used in the MoN, the empirically determined bound used in the TxM fully bounds the distribution of the transformed modulating random variable. However, the numerical instability and the infinite nature of the sum form of the pdf in (14) requires the use of a truncated sum and a limited support to ensure the approximating expression is a valid pdf.

To evaluate the accuracy of the competing methods, 10^7 values distributed according to the Weibull norm were generated by each algorithm for the parameters used in Sections II-D and III (i.e. $L = 64$, a normalized according to (5), $b = 0.7, 1.5$). For the MoN, only the envelope values were generated. For the TxM the envelope was determined from the generated complex vector. The pdf was calculated via histogram with 5×10^4 bins. The cdf was then estimated via the cumulative sum of the histogram. The true CDF was calculated by numerically integrating (3), and the square of the difference between the true CDF and the estimated CDF is plotted in Figures 7 and 8. Figures 7 and 8 show that the TxM produces more accurate results than the MoN.

As a final point of comparison, Table I provides two points of comparison for the techniques. First, the percentage of generated points that are accepted by the Rejection Method is shown for the two methods for both values of shape parameter. Second, the total integrated error (i.e. cumulative sum of Figures 7 and 8) is given for each case. Note that while significant improvement in acceptance percentage is shown for the low shape parameter case of $b = 0.7$, the MoN has a better bound than the TxM for the high shape parameter case. However, in both cases the TxM results in a more accurate final distribution.

V. Conclusions

Methods of generating multivariate Weibull distributed random data were presented. A new method based on a trans-

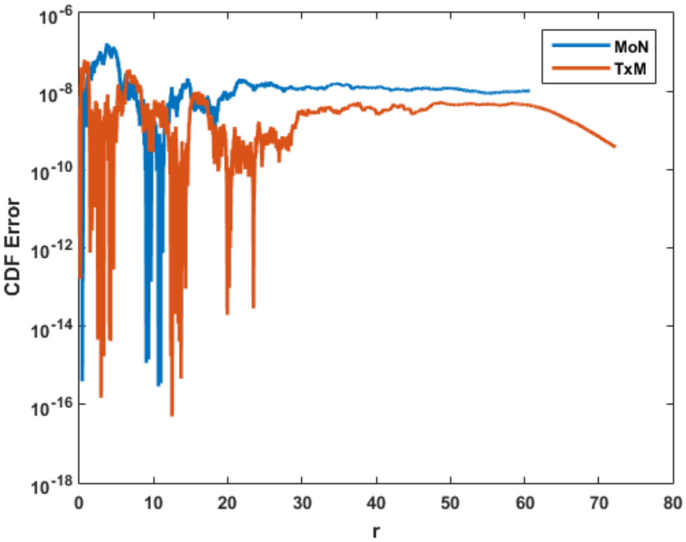

Fig. 7: CDF error for $b = 0.7$

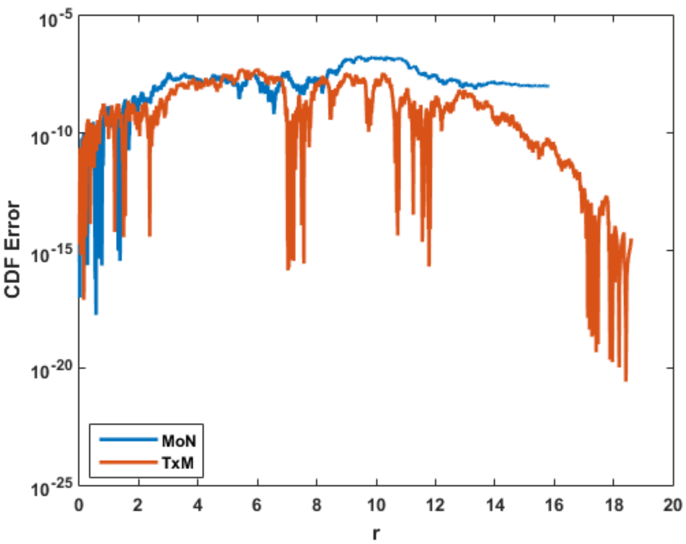

Fig. 8: CDF error for $b = 1.5$

formation of the infinite sum formula for the modulating random variable of the Weibull distribution was developed and tested. The new method produces extreme reduction in the number of samples required to generate data for low shape parameter (i.e. heavy tailed) Weibull distributions, but suffers a performance penalty for high shape parameter data relative to traditional methods. However, for both high and low shape parameter values, the transform method produces data that is more accurately distributed.

b	Method	Percentage Accepted	Integrated Error
0.7	Meth. of Norms	0.26%	7.3e-4
0.7	Transform Meth.	72.2%	1.9e-4
1.5	Meth. of Norms	46.8%	1.7e-3
1.5	Transform Meth.	27.6%	3.4e-4

TABLE I: Comparison of Methods

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