Practical Considerations for Optimal Mismatched Filtering of Nonrepeating Waveforms

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Motivation

- **Random Frequency Modulation** (RFM) represents a class of waveforms yielding considerable **design freedom**
  - Spectral shaping of RFM waveforms addresses both spectral containment and matched filter range sidelobes
  - Nonrepeating RFMs have been experimentally demonstrated [1-3] for a variety of design approaches
- Here we examine how a particular form of **mismatched filtering** (MMF) is impacted by **spectral shaping** of RFM waveforms
- Specifically, the spectrally-shaped inverse filter (SIF) is examined in the context of iterative RFM optimization based on spectrum template matching

[1] C.A. Mohr, P.M. McCormick, C.A. Topliff, S.D. Blunt, J.M. Baden, "Gradient-based optimization of PCFM radar waveforms," *IEEE Trans. Aerospace & Electronic Systems*, vol. 57, no. 2, pp. 935-956, Apr. 2021.
[2] J. Jakabosky, S.D. Blunt, B. Himed, "Spectral-shape optimized FM noise radar for pulse agility," *IEEE Radar Conf.*, Philadelphia, PA, May 2016.
[3] M.B. Heintzelman, T.J. Kramer, S.D. Blunt, "Experimental evaluation of super-Gaussian-shaped random FM waveforms," *IEEE Radar Conf.*, New York City, NY, Mar. 2022.

Mismatched Filtering & the Convolution Model

The Convolution Model

- In some situations, traditional linear convolution can be **approximated** as circular convolution, with low error
  - Circular convolution is efficiently implemented in the frequency domain
  - Permits convenient design of waveform / filter cross-power spectral density (CPSD)

- The linear convolution model, with Toeplitz matrix $\mathbf{S}$, is represented as

$$\mathbf{y} = \mathbf{S}\mathbf{x} + \mathbf{v}$$

- Receive filtering is represented via

$$\hat{\mathbf{x}} = \mathbf{W}\mathbf{y} = \mathbf{W}\mathbf{S}\mathbf{x} + \mathbf{W}\mathbf{v}$$

for Toeplitz matrix $\mathbf{W}$ containing the matched/mismatched filter.

Circulant Approximation

- The circular convolution model, with circulant matrix $\breve{\mathbf{S}}$, is represented as

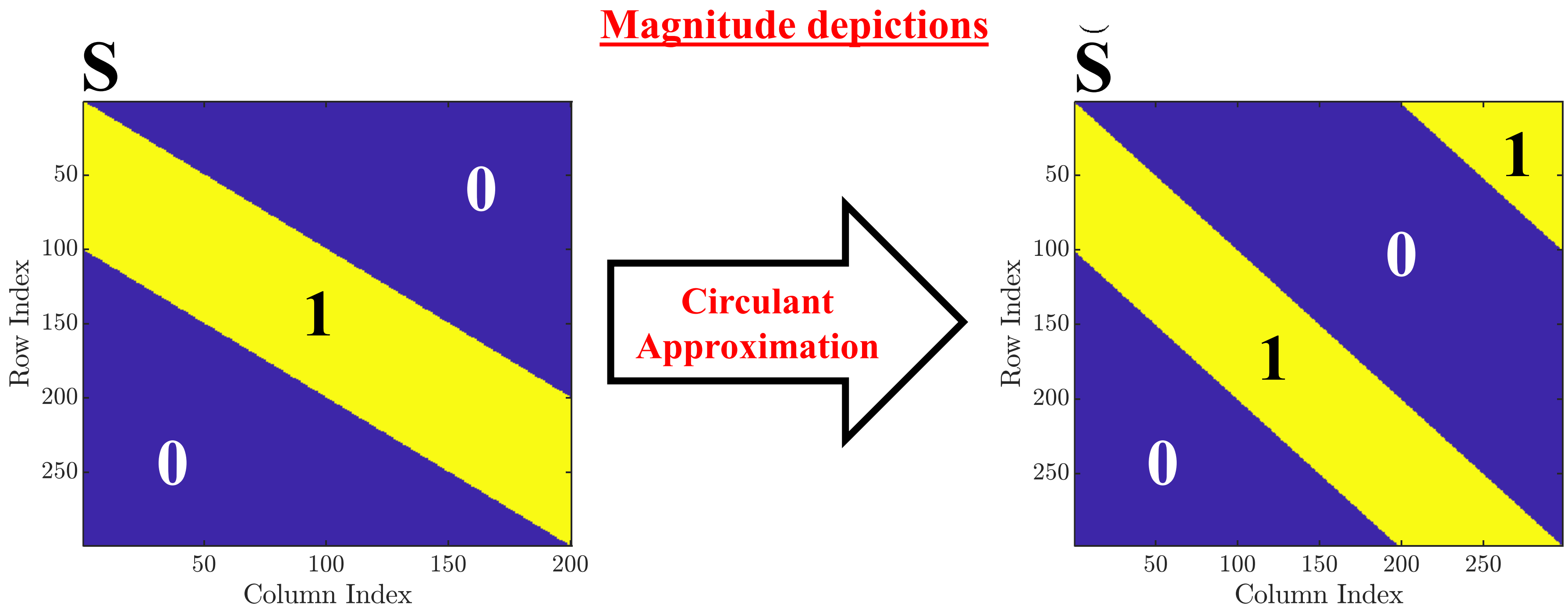
$$\mathbf{y} \approx \breve{\mathbf{S}}\bar{\mathbf{x}} + \mathbf{v}$$
$$= \mathbf{A}^H((\mathbf{A}\bar{\mathbf{s}}) \odot (\mathbf{A}\bar{\mathbf{x}})) + \mathbf{v},$$

- Receive filtering is represented via

$$\hat{\mathbf{x}} \approx \mathbf{A}^H((\mathbf{A}\bar{\mathbf{y}}) \odot \mathbf{A}\bar{\mathbf{w}})$$

The Convolution Model

- Compare:
  - (Left) Linear convolution matrix, used in matched filter & least-squares (LS) mismatched filter (MMF) construction
  - (Right) Circulant matrix approximation, used in spectrum inverse filter (SIF) construction
- Circulant approximation assumes that low-lag convolution outputs "wrap-around"
  - Could produce significant error, depending on application



The Circulant Approximation

- The circulant matrix decomposition

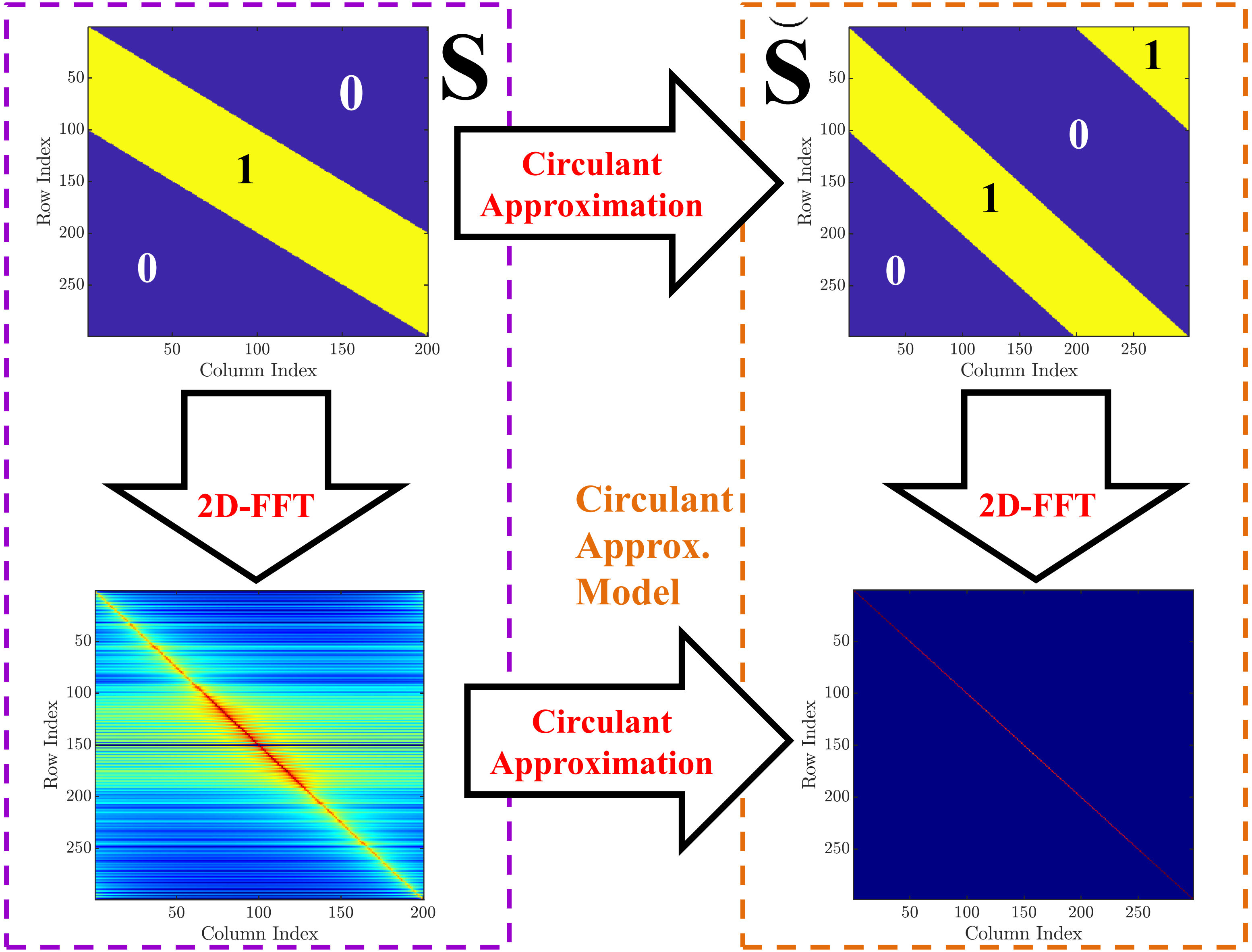
$$\breve{\mathbf{S}} = \mathbf{A}^H \mathbf{D}_S \mathbf{A}$$

  ensures that $\breve{\mathbf{S}}$ can be diagonalized via the DFT

- For the linear model

$$\mathbf{S} \neq \mathbf{A}^H \mathbf{D}_S \mathbf{A}$$

- Thus, approximation results in residual error, but can be controlled by spectral shape



Least-Squares Mismatched Filter (LS-MMF)

- **LS-MMF** relies on the linear convolution model, taking the form [4]

$$\min_{\mathbf{w}} \|\mathbf{g} - \mathbf{S}\mathbf{w}\|_2^2 \quad \xrightarrow[\text{Time Domain}]{\text{Freq Domain}} \quad \min_{\mathbf{w}} \|\mathbf{A}\mathbf{g} - \mathbf{A}(\mathbf{S}\mathbf{w})\|_2^2$$

- The regularized **solution** is

$$\mathbf{w}_{\text{LS}} = (\mathbf{S}^H\mathbf{S} + \gamma\mathbf{I})^{-1}\mathbf{S}^H\mathbf{g}$$

- Sidelobes, mismatch loss (MML), and CPSD shape are controlled via choice of $\mathbf{g}$
  - For $\mathbf{g} = \mathbf{e}_{\text{m}}$, the LS-MMF minimizes sidelobes, at the cost of mismatch loss

- Baseline complexity is $O(M^3)$ due to **matrix inverse**, but can be reduced to $O(M^2)$

[4] M.H. Ackroyd, F. Ghani, "Optimum mismatched filters for sidelobe suppression," *IEEE Trans. Aerospace & Electronic Systems*, vol. AES-9, no. 2, pp. 214-218, March 1973.

Spectrally Shaped Inverse Filter (SIF)

- The **SIF** employs the circular convolution model [5], thereby posing a **modified** version of the **LS-MMF** problem as

$$\min_{\mathbf{w}} \left\| \mathbf{A}\mathbf{g} - \mathbf{A}(\mathbf{S}\mathbf{w}) \right\|_2^2 \quad \xrightarrow{\text{Circulant Approximation}} \quad \min_{\mathbf{w}} \left\| \mathbf{A}\mathbf{g} - \mathbf{A}(\breve{\mathbf{S}}\mathbf{w}) \right\|_2^2$$

- Noting that a circulant matrix is **diagonalizable** via the DFT, the regularized closed form **solution** is

$$\begin{aligned}\mathbf{w}_{\text{SIF}} &= (\mathbf{D}_{\text{S}}^{H}\mathbf{D}_{\text{S}} + \gamma\mathbf{I})^{-1}\mathbf{D}_{\text{S}}^{H}\mathbf{g}_{\text{f}} \\ &= \left((\mathbf{A}\breve{\mathbf{s}})^{*} \odot (\mathbf{A}\mathbf{g})\right) \oslash (|\mathbf{A}\breve{\mathbf{s}}|^{2} + \gamma)\end{aligned}$$

$\odot$: Hadamard product

$\oslash$: Hadamard division

L : MMF length

- Sidelobe levels, MML, and CPSD shape now controlled by choice of $\mathbf{g}_{\text{f}}$
- Use of FFTs allows computational **reduction** to $O(L \times \log(L))$

[5] S. Senmoto, D.G. Childers, "Signal resolution via digital inverse filtering," *IEEE Trans. Aerospace and Electronic Systems*, vol. AES-8, no. 5, pp. 633-640, Sept. 1972.

SIF applied to RFM waveforms

SIF applied to RFM waveforms

- **Pulse-agile** waveforms exhibit range sidelobe modulation (**RSM**), introducing slow-time **nonstationarity** to clutter response **[6]**
  - Is a result of slow-time/fast-time coupling
  - Manifests as increased clutter floor that spreads in Doppler
  - CPI of M pulses yields matched filter sidelobes of $\sim 10 \log_{10}(MTB)$ dB

- **Mismatched filtering** greatly reduces RSM **[2]**
  - LS-MMF is effective, but has high computational cost (*new filter for each waveform*)
  - SIF is a pragmatic alternative solution

[2] J. Jakabosky, S.D. Blunt, B. Himed, "Spectral-shape optimized FM noise radar for pulse agility," *IEEE Radar Conf.*, Philadelphia, PA, May 2016.

[6] S.D. Blunt, J.K. Jakabosky, C.A. Mohr, P.M. McCormick, J.W. Owen, B. Ravenscroft, C. Sahin, G.D. Zook, C.C. Jones, J.G. Metcalf, and T. Higgins, "Principles & applications of random FM radar waveform design," *IEEE Aerospace & Electronic Systems Magazine*, vol. 35, no. 10, pp. 20-28, Oct. 2020.

SIF applied to RFM waveforms

- Let $S_q(f)$ and $W_q(f)$ be the respective **waveform** and **filter** spectra for the qth unique **RFM** waveform (of Q), yielding desired **cross-spectrum** shape

$$S_q(f) \times W_q(f) = \left|G(f)\right|^2$$

- The qth **SIF** mismatched filter can then be posed as

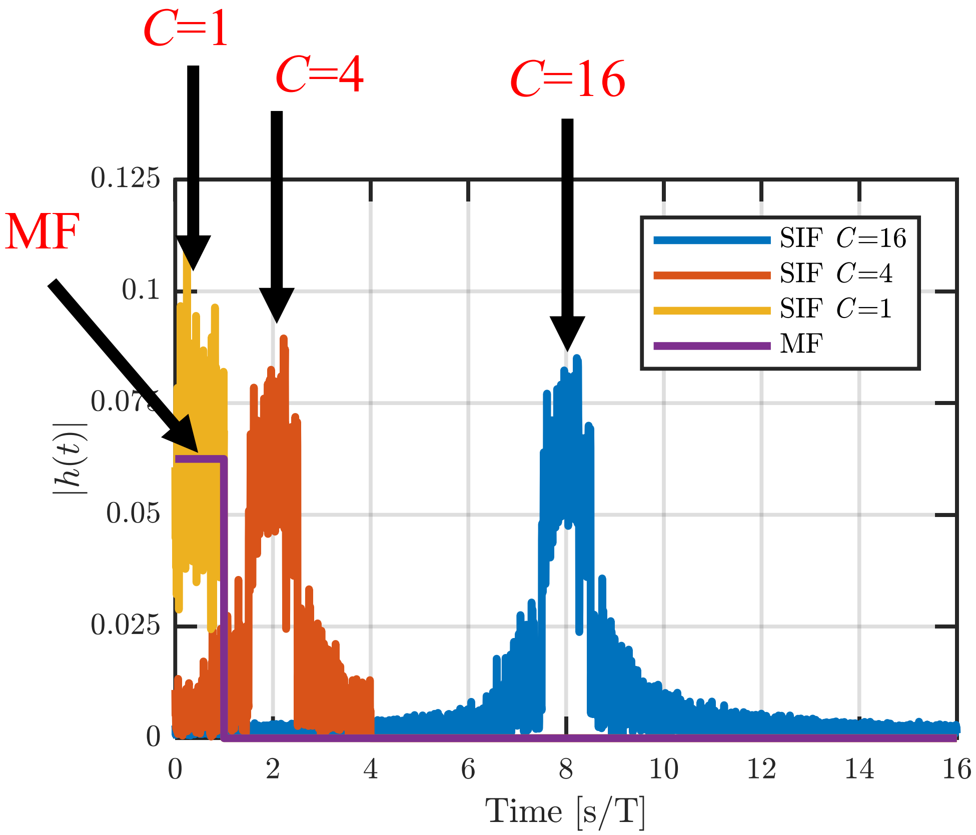
$$W_q(f) = \frac{\left|G(f)\right|^2}{S_q(f)}$$

- The existence of $S_q(f)$ in the denominator translates to poles away from the origin, meaning that $W_q(f)$ would theoretically have an **infinite impulse response** (IIR)

SIF Truncation

- While the **SIF** is theoretically **IIR**, its practical application requires truncation
- Consider the magnitude envelopes of different truncated SIFs, which incur more/less error
  - Here C indicates the SIF length increase factor (as CN) compared to the matched filter (MF) length N, achieved via zero-padding
- **Higher** value of C yields a closer approximation to **ideal SIF**

N = 100 waveform samples
M = CN filter samples



Magnitude Envelopes Comparison

SIF Simulation Results

Shaping Selection & Assessment Metrics

- Consider waveform **spectral shaping** via pseudo-random optimized FM (PRO-FM) [2]
  - Realizes FM (constant amplitude) signal for desired spectral template $|G(f)|^2 \rightarrow \mathbf{g}_\mathrm{f}$

Template SIF Desired Response

- Metric used for evaluation of shaped waveforms is total **mean-squared deviation** (MSD)

$$\Delta_k = \mathbf{1}^T \left(\left| \left[(\mathbf{A}\mathbf{s}^{(k)}) \odot (\mathbf{A}\mathbf{s}^{(k)})^* \right] - \left[\mathbf{g}_\mathrm{f} \odot \mathbf{g}_\mathrm{f}^* \right] \right|^2 \right)$$

where the spectrum template that minimizes MSD is

$$\bar{\mathbf{g}}_\mathrm{f}^{(k)} = \left(\frac{1}{Q} \sum_{q=1}^{Q} \left[(\mathbf{A}\mathbf{s}^{(k)}) \odot (\mathbf{A}\mathbf{s}^{(k)})^* \right] \right)^{0.5}$$

Average SIF Desired Response

- Also define **mismatch loss** (MML)

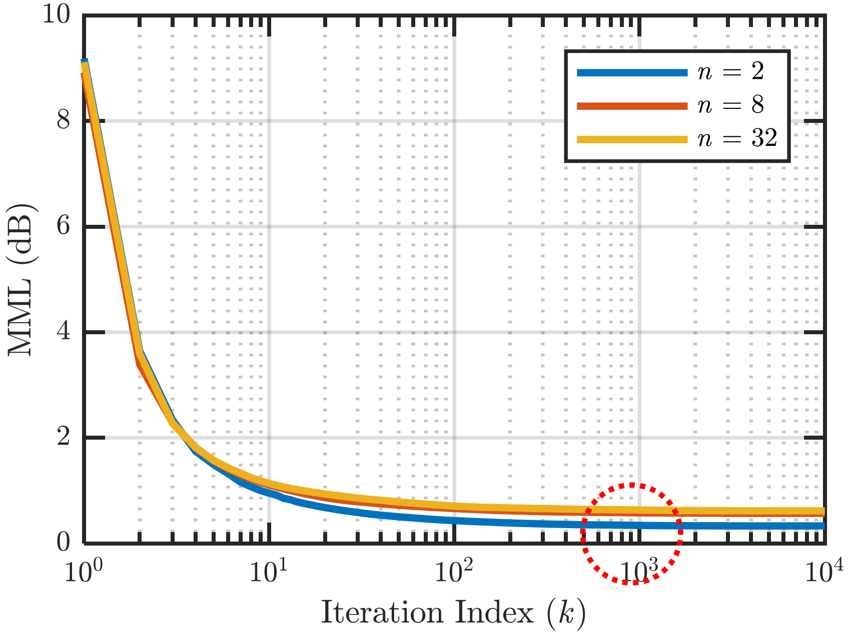
$$\Gamma_k = \frac{\left\| \mathbf{s}^{(k)} \right\|_2^2 \left\| \mathbf{w}^{(k)} \right\|_2^2}{\max \left\{ \left| s^{(k)}[n] * w^{(k)}[n] \right|^2 \right\}}$$

[2] J. Jakabosky, S.D. Blunt, B. Himed, "Spectral-shape optimized FM noise radar for pulse agility," *IEEE Radar Conf.*, Philadelphia, PA, May 2016.

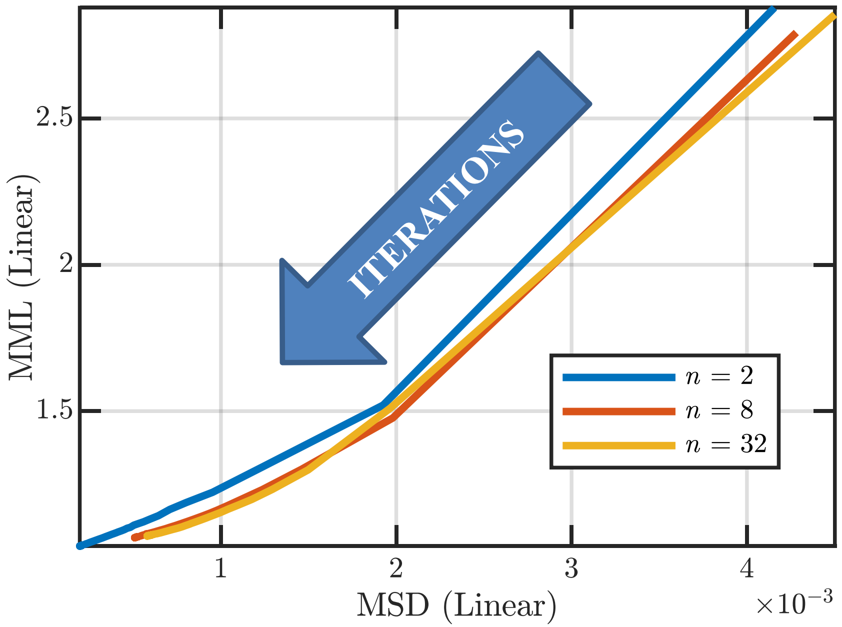
Reducing MML and MSD

- Consider an ensemble of 100 **PRO-FM** waveforms, optimized for 10^4 iterations based on **super-Gaussian** parameter n = 2, 8 and 32 [3]
- For **spectrally-shaped** waveforms pulse compressed via SIF, MML and MSD are closely related
  - In general, reducing MSD reduces MML (and vice versa)

MML vs PRO-FM iteration



MML vs MSD for each PRO-FM iteration



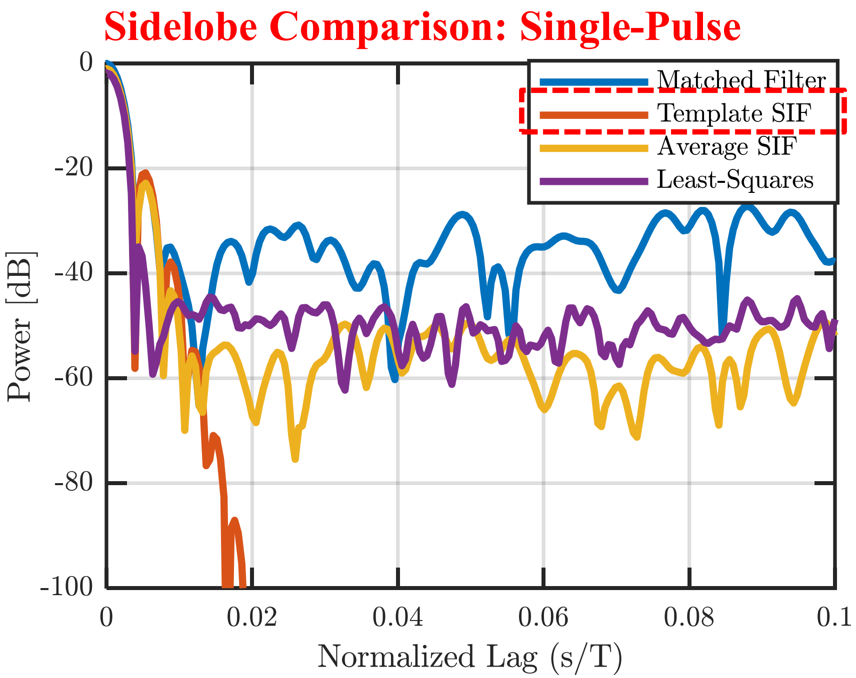
[3] M.B. Heintzelman, T.J. Kramer, S.D. Blunt, "Experimental evaluation of super-Gaussian-shaped random FM waveforms." *IEEE Radar Conf.*, New York City, NY, Mar. 2022.

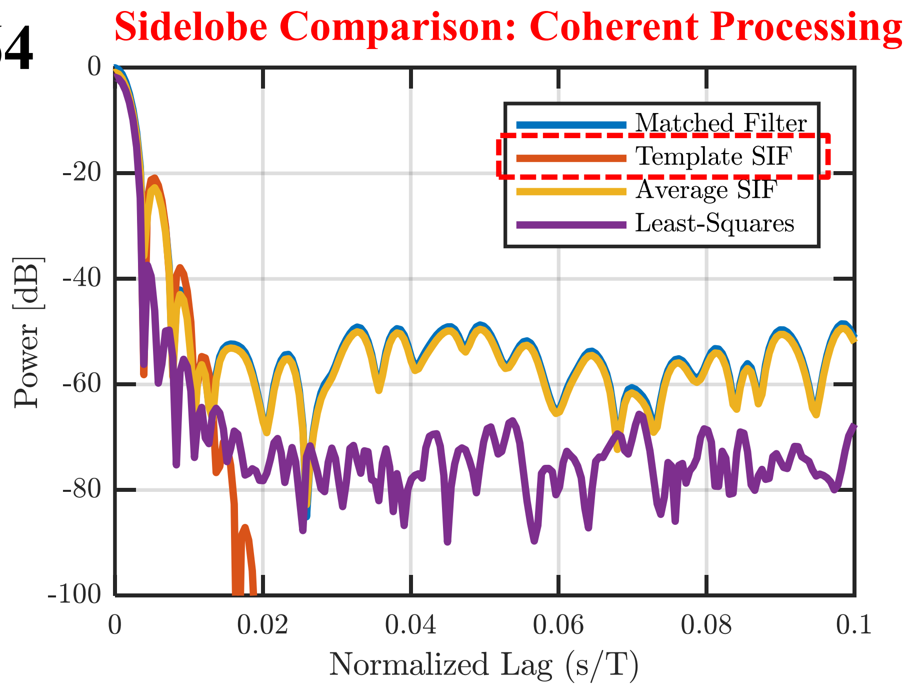
Sidelobe Level Comparison

- To assess SIF **performance**, relative to the MF and LS-MMF, consider PRO-FM optimized for 1000 iterations
- The "**Template SIF**" implementation yields no meaningful sidelobe floor, while "**Average SIF**" has lowest MML
- Shoulder lobe roll-off due to super-Gaussian template

MML Comparison (dB)		
	1 Pulse	**100 Pulses**
MF	0	0
Template SIF	1.03	1.09
Average SIF	0.99	0.91
LS-MMF	1.80	1.78

TB = 64

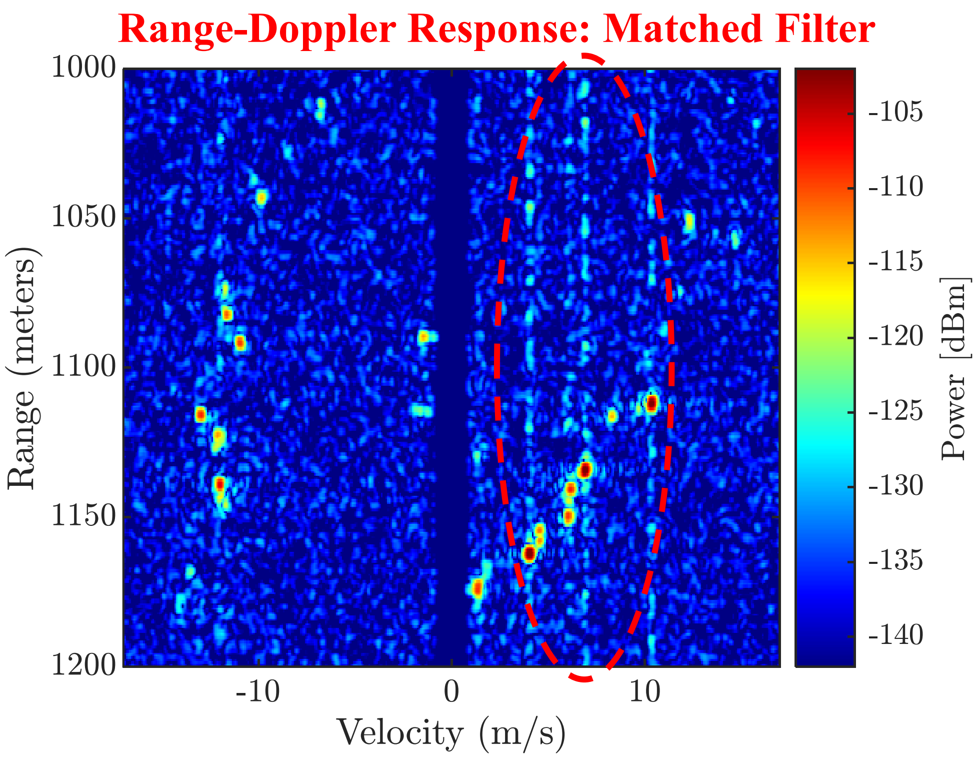


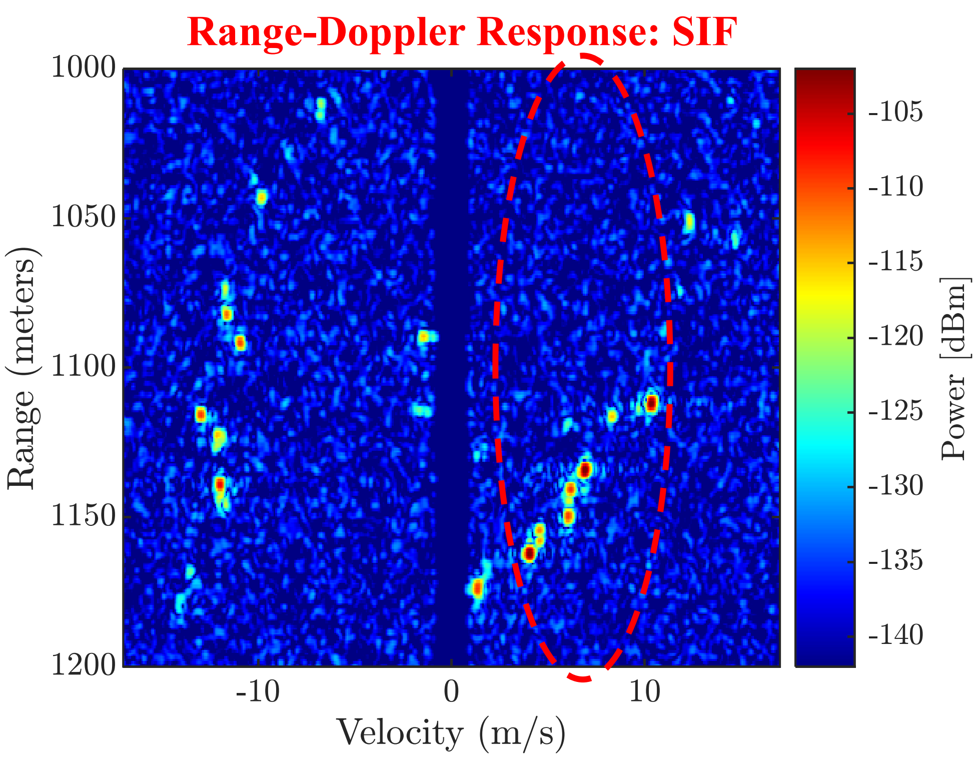


Open-Air Experimental Results

Pulse-Repeated RFM

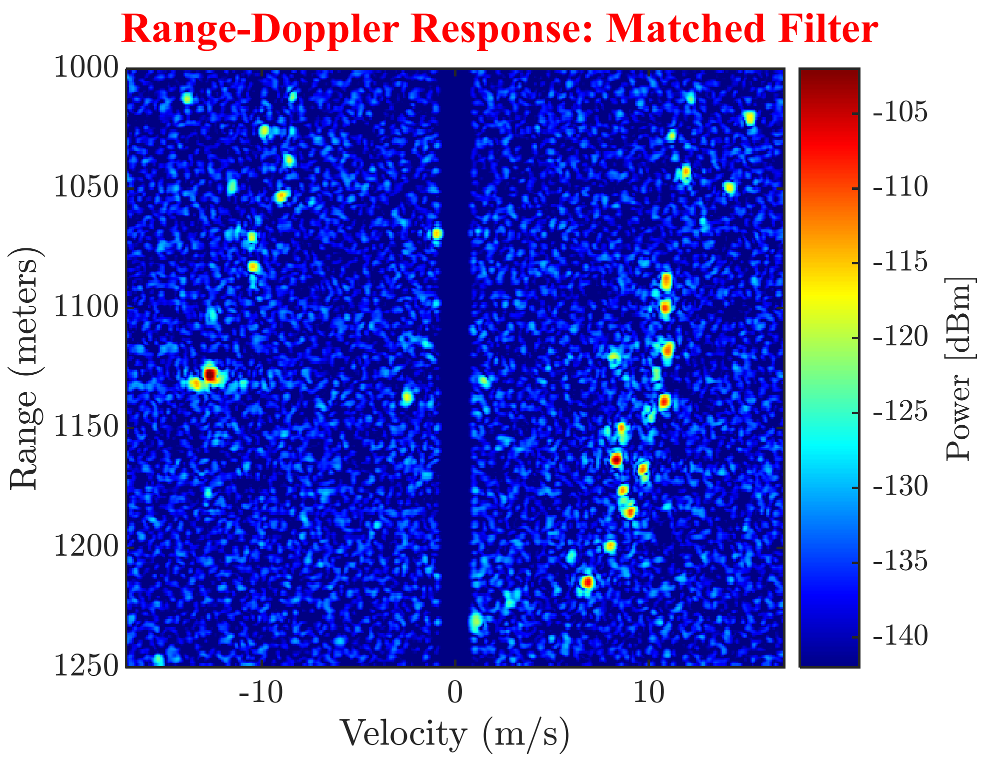
- Consider the **range-Doppler responses** for a CPI of 5000 **identical** PRO-FM pulses with $TB = 64$, transmitted toward a nearby traffic intersection in Lawrence, KS
- The **SIF** (right) shows considerable **improvement** over the matched filter
  - Caveat: not a high dynamic range scenario given relatively low transmit power

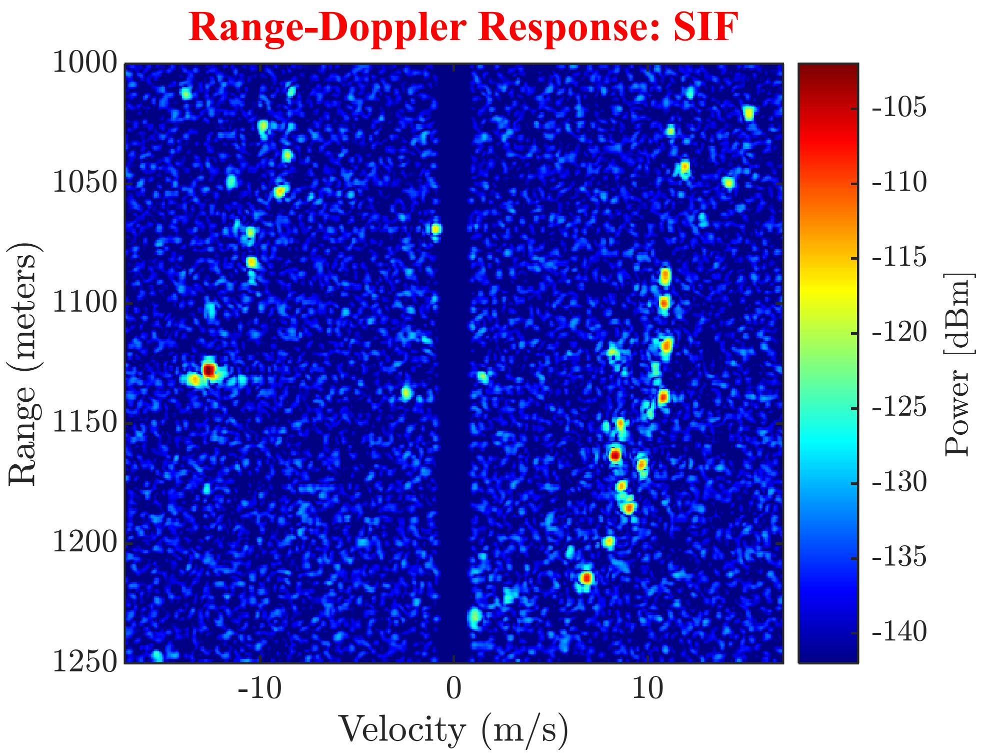




Pulse-Agile RFM

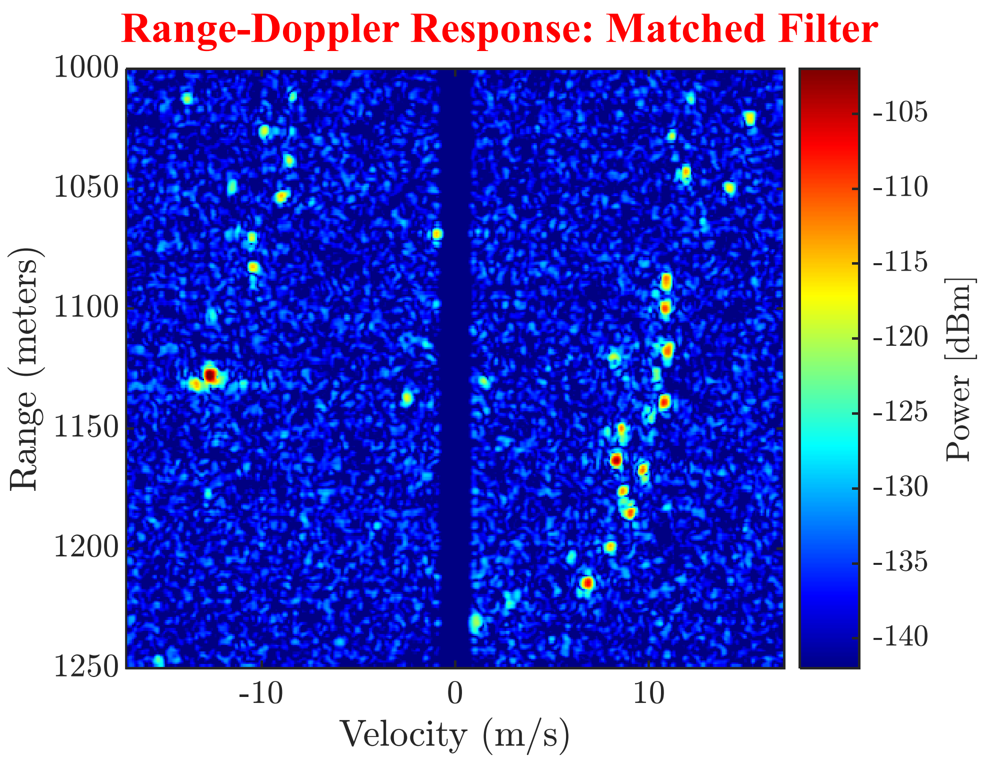
- Now consider a CPI of 5000 **unique** PRO-FM pulses with *TB* = 64
- **Higher dimensionality** (5000×64) pushes matched filter sidelobes below noise floor
  - While not evident, SIF still yields 1.1 dB RSM improvement (expect more at higher dynamic range)
  - Note: illuminated scene is different since collected back-to-back (but not simultaneously)

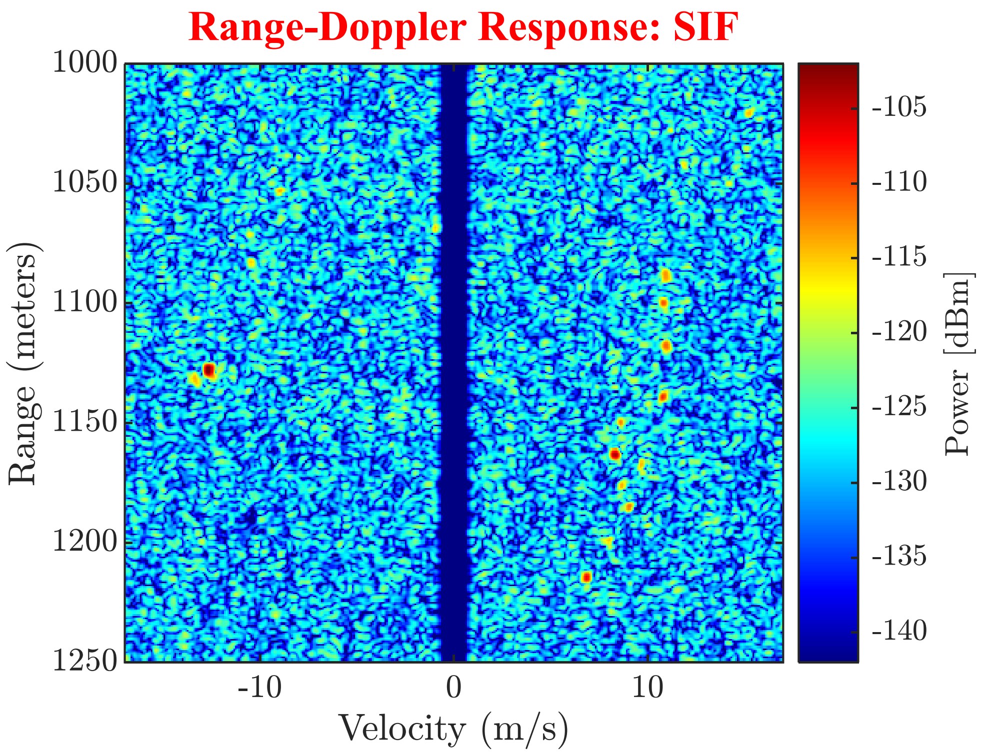




Emulation of Pulse Eclipsing

- Finally, using CPI of 5000 **unique** PRO-FM pulses, we artificially introduced a **pulse eclipsing** effect in the measured data (by zeroing half of the direct-path component)
- **Violates** "wrap-around" assumption of the **circulant** approximation
  - Results in ~10 dB degradation for SIF => suggests care must be taken when SIF is employed





Conclusions

- The **spectrally-shaped inverse filter** (SIF) is an attractive approach to obtain **optimal** MMFs in a computationally **efficient** manner
  - Approximates LS-MMF performance while being practical for large CPIs of unique pulses
- **Mismatch loss** is proportional to waveform spectrum **template deviation** when waveform and filter spectral shape have same desired structure
- **SIF** experimentally shown to **reduce RSM**, with greater potential utility for higher dynamic range
- However, if the **circulant** approximation is **violated** due to pulse **eclipsing**, significant **degradation** can be incurred

Questions?