On the Optimality of Spectrally Notched Radar Waveform & Filter Designs

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Motivation

• The RF spectrum is becoming increasingly congested due to repeated spectrum auctions and subsequent 4G/5G roll-out

  • Designing radar waveforms with notched spectral regions can mitigate mutual interference with other proximate RF users, at the cost of degraded range-Doppler sidelobe performance [1]

  • To evaluate the limitations of correlation-based processing, the null-constrained power spectral density (PSD) that globally minimizes correlation (range) sidelobe levels is determined

  • The optimal null-constrained PSD and implied correlation (for a given spectral notch location) is compared with waveform and pulse compression filter design methods

Motivation

• Recent work demonstrated real-time cognitive radar for spectrum sharing [2]
  • The real-time cognitive radar utilized a sense-and-notch (SAN) framework for per-pulse RF interference mitigation.
  • Correlation-based matched filter processing was implemented for moving target indication (MTI)
  • To determine the fundamental limitations of correlation-based processing, the null-constrained power spectral density (PSD) that globally minimizes correlation (range) sidelobe levels is determined


1) Sense the spectrum environment
2) Ascertain where interference is located
3) Generate physically realizable waveforms to mitigate mutual interference

"sense & notch": places notches in the radar spectrum based on sensed RFI
• Power spectral density (PSD) and autocorrelation are a Fourier transform pair for deterministic signals.

\[ r = A^H g \]

**Fundamentals:**

1. The waveform autocorrelation determines the matched filter pulse compression response.
2. Waveforms designed to spectrally adhere to a PSD template have implicit autocorrelation properties.
3. Determining the PSD template that minimizes autocorrelation sidelobes (while constraining spectral nulls) implies global minimum boundaries for waveform/filter performance.
Problem Statement

• The power spectral density (PSD) may be directly optimized to minimize autocorrelation integrated sidelobes (ISL).

\[
\min_g \|e - A^H g\|^2_2
\]
\[\text{s.t. } g_m \leq \gamma_m \text{ for } m \in \Omega
\]
\[0 \leq g_m \text{ for } m = 0, 1, \ldots, M - 1
\]

The boxed least squares formulation provides a globally convex objective function to determine the power spectrum \( g \) (subject to null constraints) that minimizes the integrated sidelobe level (ISL) of the autocorrelation.

\( r \) : Autocorrelation
\( A^H \) : Inverse DFT Matrix
\( g \) : Power Spectrum (non-negative)
\( g_m \) : \( m^{th} \) element of the power spectrum
\( \gamma_m \) : Constrained maximum value for \( g_m \in \Omega \)
\( \Omega \) : Frequency indices to null constrain
\( e \) : Desired Autocorrelation Response (impulse)
Global Minimum Integrated Sidelobes (ISL)

\[ \mathbf{r} \cdot \mathbf{g} \cdot \gamma_m \]

\( \Omega \) : Frequency indices to null constrain
\( e \) : Desired Autocorrelation Response (impulse)

Different degrees of beamspoiling are achieved by replacing \( M \) rows of \( \mathbf{A}^H \) (corresponding to autocorrelation mainlobe roll-off) with zeros, thus permitting different mainlobe widths and achievable sidelobe levels.
Global Minimum Integrated Sidelobes (ISL)

Spectral notching located closer to the power spectrum center degrades the autocorrelation global minimum ISL floor

\[ r \]: Autocorrelation

\[ A^H \]: Inverse DFT Matrix

\[ g \]: Power Spectrum (non-negative)

\[ g_m \]: \( m^{th} \) element of the power spectrum

\[ \gamma_m \]: Constrained maximum value for \( g_m \in \Omega \)

\[ \Omega \]: Frequency indices to null constrain

\[ e \]: Desired Autocorrelation Response (impulse)

\[ M = 200 \] window length

Beamspoiling 2%
Problem Statement

- The power spectral density (PSD) may be directly optimized to minimize autocorrelation peak sidelobes (PSL).

\[
\min_g \| e - A^H g \|_p^p \\
\text{s.t. } g_m \leq \gamma_m \quad \text{for } m \in \Omega \\
0 \leq g_m \quad \text{for } m = 0, 1, \ldots, M - 1
\]

\[
r : \text{Autocorrelation} \\
A^H : \text{Inverse DFT Matrix} \\
g : \text{Power Spectrum (non-negative)} \\
g_m : m^{th} \text{ element of the power spectrum} \\
\gamma_m : \text{Constrained maximum value for } g_m \in \Omega \\
\Omega : \text{Frequency indices to null constrain} \\
e : \text{Desired Autocorrelation Response (impulse)}
\]

- The \( L_p \)-norm maintains convexity, thus preserving global optimality. Sufficiently large \( p \) values well-approximate the peak sidelobe level (PSL) metric.

- The gradient is

\[
\nabla_g \| e - A^H g \|_p^p = -p \, \Re \{ A (| e - A^H g |^{p-2} \odot (e - A^H g)) \}
\]
Global Minimum Peak Sidelobes (PSL)

Different degrees of beamspoiling are achieved by replacing $\bar{M}$ rows of $A^H$ (corresponding to autocorrelation mainlobe roll-off) with zeros, thus permitting different mainlobe widths and achievable sidelobe levels.

$r$ : Autocorrelation
$A^H$ : Inverse DFT Matrix
$g$ : Power Spectrum (non-negative)
$g_m$ : $m^{th}$ element of the power spectrum
$\gamma_m$ : Constrained maximum value for $g_m \in \Omega$
$\Omega$ : Frequency indices to null constrain
$e$ : Desired Autocorrelation Response (impulse)

$M = 200$ window length
Comparison of Optimum

Minimize ISL

Minimize PSL

$M = 200$ window length
Waveform Design
Nonrepeating Spectrally Notched FM Waveforms

- Various methods have been experimentally demonstrated to realize spectrally shaped forms of nonrepeating random FM (RFM) waveforms [3]

- Notched versions of RFM can achieve > 50 dB notch depth, while preserving transmitter-amenable FM structure [4]


1. In [5] the waveform design method mitigated correlation sidelobes by matching the waveform PSD to a heuristic template having tapered spectral null borders.

2. Here, the waveform design method mitigates correlation sidelobes by matching the waveform PSD to the least-squares optimal PSD template that minimizes ISL.

Notched FM Waveform Generation

The waveform optimization is executed in two stages:

1. Perform $K$ iterations of alternating time-frequency projections to produce a pseudo-random optimized FM (PRO-FM) waveform [5, 6]
   - The desired spectrum $g$ is the heuristic or optimal PSD template, producing waveforms with shallow spectral notches over $\Omega$

2. Then apply $L$ iterations of the zero-order reconstruction optimization of waveforms (ZOROW) [7] to significantly deepen spectral notches over $\Omega$
   - $K=200$ and $L=1000$ iterations to guarantee full convergence, ensuring a modest waveform spectrum match to the template

\[
\begin{align*}
\mathbf{s}^{(k)} &= \mathbf{A}^H \{ \mathbf{g}^{1/2} \odot \exp(j\angle \mathbf{A} \{ \mathbf{s}^{(k-1)} \}) \} \\
\mathbf{s}^{(k)} &= \mathbf{u} \odot \exp(j\angle \mathbf{s}^{(k)}) \\
\end{align*}
\]

\[
\begin{align*}
\left[ \mathbf{s} \right]_n &= e^{j\phi_n} \\
\Phi &= [\phi_1 \phi_2 \cdots \phi_N]^T \\
S(f_m, \phi) &= \frac{\sin(\pi f_m T_s)}{\pi f_m} \sum_{n=1}^{N} \exp(-j(2\pi f_m(n - 0.5)T_s + \phi_n)) \\
\min_{\Phi} \sum_{m \in \Omega} |S(f_m, \Phi)|^2
\end{align*}
\]

Waveforms designed to the heuristic template exhibit mainlobe broadening and “shoulder” lobes compared to the global optimum.
Optimal Spectral Template

Waveforms designed to the optimal template match closely, with residual sidelobes due to range sidelobe modulation.

\[ r = A^H g \]

\[ \sum_q r_q = \sum_q A^H g_q \]

\[ \sum_q r_q = A^H \sum_q g_q \]

\[ \text{Mean PSD} \]

\[ \text{Coherent Average Autocorrelation} \]

\[ Q = 1000 \text{ pulses} \]

\[ N = 200 \text{ pulse parameters} \]

\[ M = 800 \text{ PSD window samples} \]

2% Beamspoiling
Heuristic vs. Optimal Spectral Template

**Heuristic**

- Mean Signal PSD
- Global Optimum
- Design Template

**Optimal**

- Mean Signal PSD
- Global Optimum

**Normalized Frequency**

- Relative Power (dB)

**Normalized Lag**

- Relative Power (dB)

- CA Autocorrelation
- Global Optimum
Mismatched filter design
Least Squares Filter Formulation

Because the ISL optimum spectral template is based on least squares in a 2-norm sense, it is logical to apply the least squares mismatched filter (MMF) \[8-10\] to the same waveform sets

\[
\mathbf{w}_{LS} = (\mathbf{S}^H \mathbf{S} + \sigma \mathbf{I})^{-1} (\mathbf{S}^H \mathbf{e})
\]

\[
\mathbf{s} = \begin{bmatrix}
 s_1 & 0 & \cdots & 0 \\
 \vdots & s_1 & \ddots & \vdots \\
 s_N & \vdots & \ddots & 0 \\
 0 & s_N & \cdots & s_1 \\
 \vdots & \ddots & \ddots & \vdots \\
 0 & \cdots & 0 & s_N
\end{bmatrix}
\]

- \( \mathbf{S} \) : Convolution matrix of signal \( \mathbf{s} \)
- \( \mathbf{e} \) : Desired Correlation Response
- \( \mathbf{g} \) : Desired Power Spectrum
- \( \sigma \) : Regularization term

The desired correlation response is the IDFT of the optimal spectrum \( \mathbf{e} = \mathbf{A}^H \mathbf{g} \)


$Q = 1000$ pulses
$N = 200$ pulse parameters
$P = 600$ filter parameters
$M = 800$ CPSD window samples
2% Beamspoiling

2.59 dB mismatch loss (SNR loss)

Least squares filtering compensates for the heuristic spectrum mainlobe broadening and “shoulder” lobes, closely matching the global optimum
Optimal Spectral Template

Waveforms matched to the optimal spectral template already exhibit near-optimality, such that least squares filtering incurs minimal mismatch loss.

1.37 dB mismatch loss (1.22 dB improvement)

\[ Q = 1000 \text{ pulses} \]
\[ N = 200 \text{ pulse parameters} \]
\[ P = 600 \text{ filter parameters} \]
\[ M = 800 \text{ CPSD window samples} \]

2% Beamspoiling
Heuristic vs. Optimal Spectral Template

**Heuristic**

- Mean Signal PSD
- Mean Filter PSD
- Mean Cross-PSD
- Global Optimum

**Optimal**

- Mean Signal PSD
- Mean Filter PSD
- Mean Cross-PSD
- Global Optimum

Normalized Frequency

Relative Power (dB)

Normalized Lag

Relative Power (dB)
Conclusions

• The least squares global optimum power spectrum has been determined to minimize ISL and PSL when portions of the spectrum are null constrained.

✓ By designing waveform spectra to closely match the optimal template, their attendant sidelobes also approach the optimal level.

✓ Application of the least-squares mismatched filter then closes the remaining sidelobe difference, with mismatch loss in trade.

✓ The heuristic PSD template design involving simple tapering of notch edges is determined to achieve near-optimal performance with a computational cost that is low enough for real-time implementation.
Thank You!