On the Optimality of Spectrally Notched Radar Waveform & Filter Designs

Jonathan W. Owen, Patrick M. McCormick, Christian C. Jones, Shannon D. Blunt

Radar Systems & Remote Sensing Lab (RSL), University of Kansas

This work was sponsored by the Office of Naval Research under Contract #N00014-20-C-1006.

Motivation

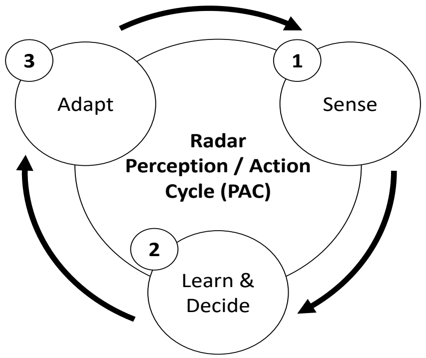
- The RF spectrum is becoming increasingly congested due to repeated spectrum auctions and subsequent 4G/5G roll-out
  - Designing radar waveforms with **notched spectral regions can mitigate mutual interference** with other proximate RF users, at the cost of degraded range-Doppler sidelobe performance [1]
  - To evaluate the limitations of correlation-based processing, the null-constrained power spectral density **(PSD) that globally minimizes correlation (range) sidelobe levels** is determined
  - The optimal null-constrained PSD and implied correlation (for a given spectral notch location) is **compared with waveform and pulse compression filter design methods**

[1] B. Ravenscroft, J.W. Owen, J. Jakabosky, S.D. Blunt, A.F. Martone, K.D. Sherbondy, "Experimental demonstration and analysis of cognitive spectrum sensing and notching for radar," *IET Radar, Sonar & Navigation*, vol. 12, no. 12, pp. 1466-1475, Dec. 2018.

Motivation

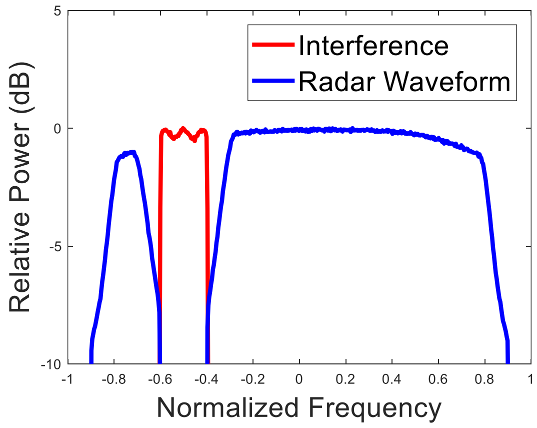
- Recent work demonstrated real-time cognitive radar for spectrum sharing [2]
  - The real-time cognitive radar utilized a sense-and-notch (SAN) framework for per-pulse RF interference mitigation.
  - Correlation-based matched filter processing was implemented for moving target indication (MTI)
  - **To determine the fundamental limitations of correlation-based processing**, the null-constrained power spectral density (PSD) that globally minimizes correlation (range) sidelobe levels is determined

[2] J. W. Owen, C. Mohr, B. Ravenscroft, S. Blunt, B. Kirk, A. Martone, "Real-Time Experimental Demonstration and Evaluation of Open-Air Sense-and-Notch Radar," IEEE Radar Conference, New York City, NY, March 2022.



1) **Sense the spectrum environment**
2) **Ascertain where interference is located**
3) **Generate physically realizable waveforms to mitigate mutual interference**

"sense & notch": places notches in the radar spectrum based on sensed RFI



Fundamentals

- Power spectral density (PSD) and autocorrelation are a Fourier transform pair for deterministic signals.

$$\mathbf{r} = \mathbf{A}^H \mathbf{g}$$

$\mathbf{r}$: Autocorrelation
$\mathbf{A}^H$: Inverse DFT Matrix
$\mathbf{g}$: Power Spectrum (non − negative)

Fundamentals:

1. The waveform autocorrelation determines the matched filter pulse compression response.
2. Waveforms designed to spectrally adhere to a PSD template have implicit autocorrelation properties.
3. Determining the PSD template that minimizes autocorrelation sidelobes (while constraining spectral nulls) implies global minimum boundaries for waveform/filter performance.

Problem Statement

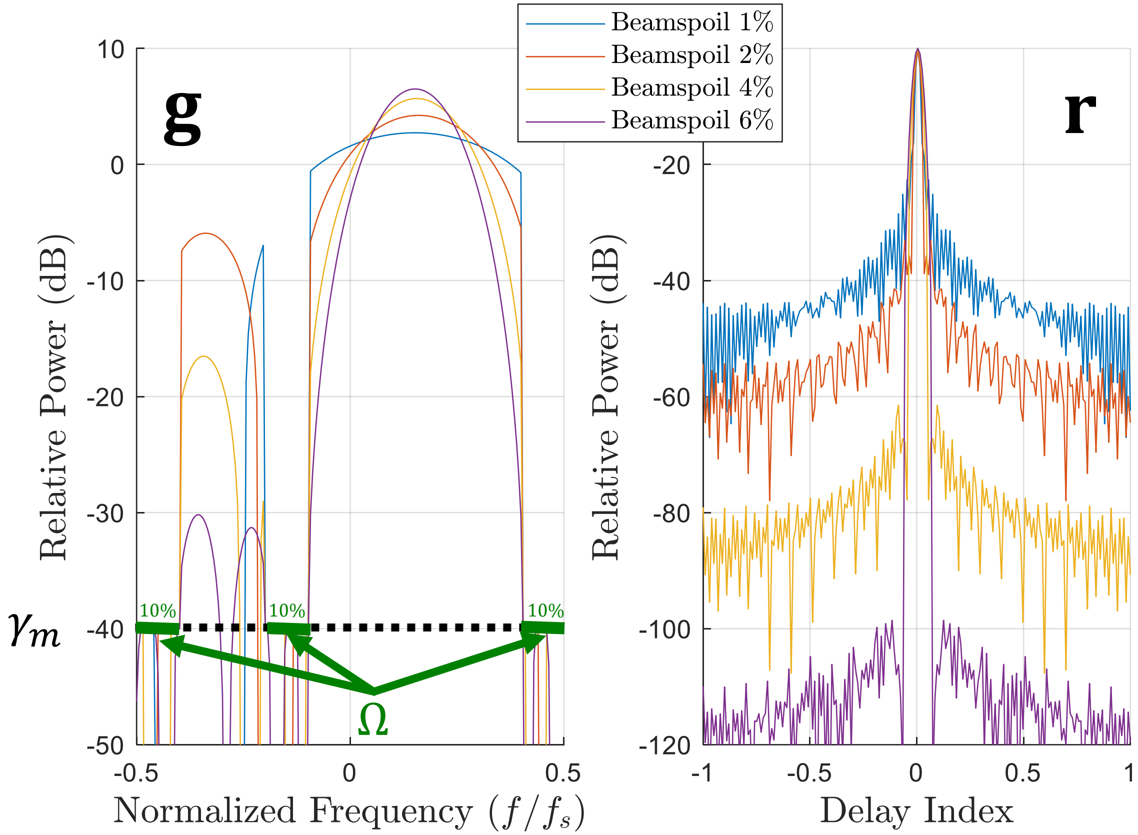
- The power spectral density (PSD) may be directly optimized to minimize autocorrelation **integrated sidelobes (ISL).**

$$\min_{\mathbf{g}} \ \|\mathbf{e} - \mathbf{A}^H \mathbf{g}\|_2^2$$
$$\text{s.t.} \quad g_m \leq \gamma_m \quad \text{for } m \in \Omega$$
$$0 \leq g_m \quad \text{for } m = 0,1,\ldots M-1$$

$\mathbf{r}$: Autocorrelation
$\mathbf{A}^H$: Inverse DFT Matrix
$\mathbf{g}$: Power Spectrum (non − negative)
g_m : m^{th} element of the power spectrum
γ_m : Constrained maximum value for $g_m \in \Omega$
Ω : Frequency indices to null constrain
$\mathbf{e}$: Desired Autocorrelation Response (impulse)

> The boxed least squares formulation provides a globally convex objective function to determine the power spectrum **g** (subject to null constraints) that **minimizes the integrated sidelobe level (ISL)** of the autocorrelation.

Global Minimum Integrated Sidelobes (ISL)



r : Autocorrelation

$\mathbf{A}^H$: Inverse DFT Matrix

g : Power Spectrum (non − negative)

g_m : m^{th} element of the power spectrum

γ_m : Constrained maximum value for $g_m \in \Omega$

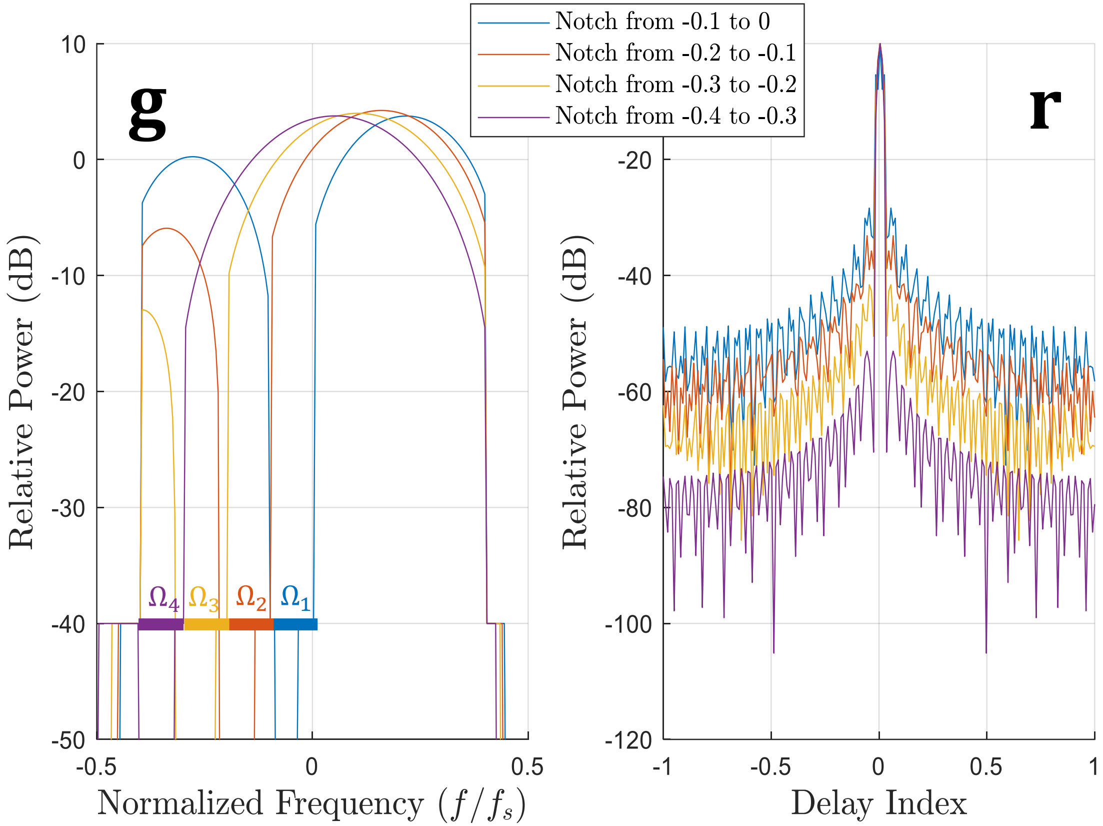
Ω : Frequency indices to null constrain

e : Desired Autocorrelation Response (impulse)

$M = 200$ window length

Different degrees of beamspoiling are achieved by replacing $\bar{M}$ rows of $\mathbf{A}^H$ (corresponding to autocorrelation mainlobe roll-off) with zeros, thus **permitting different mainlobe widths and achievable sidelobe levels**

Global Minimum Integrated Sidelobes (ISL)



$\mathbf{r}$: Autocorrelation
$\mathbf{A}^H$: Inverse DFT Matrix
$\mathbf{g}$: Power Spectrum (non − negative)
g_m : m^{th} element of the power spectrum
γ_m : Constrained maximum value for $g_m \in \Omega$
Ω : Frequency indices to null constrain
$\mathbf{e}$: Desired Autocorrelation Response (impulse)

$M = 200$ window length
Beamspoiling 2%

Spectral notching located closer to the power spectrum center
degrades the autocorrelation global minimum ISL floor

Problem Statement

KU

- The power spectral density (PSD) may be directly optimized to minimize autocorrelation **peak sidelobes (PSL).**

$$\min_{\mathbf{g}} \ \|\mathbf{e} - \mathbf{A}^H\mathbf{g}\|_p^p$$
$$\text{s.t.} \quad g_m \leq \gamma_m \quad \text{for } m \in \Omega$$
$$0 \leq g_m \quad \text{for } m = 0,1, \ldots M-1$$

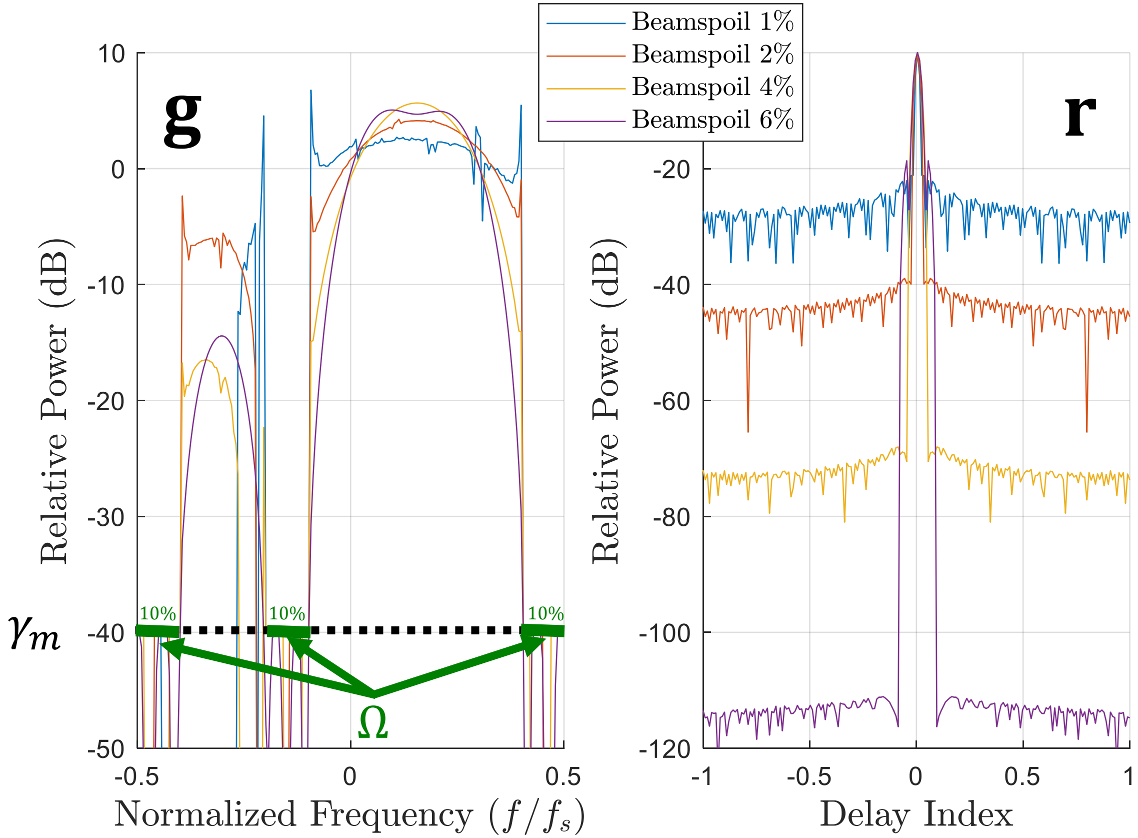
$\mathbf{r}$: Autocorrelation
$\mathbf{A}^H$: Inverse DFT Matrix
$\mathbf{g}$: Power Spectrum (non − negative)
g_m : m^{th} element of the power spectrum
γ_m : Constrained maximum value for $g_m \in \Omega$
Ω : Frequency indices to null constrain
$\mathbf{e}$: Desired Autocorrelation Response (impulse)

- The L_p-norm maintains convexity, thus preserving global optimality. **Sufficiently large p values well-approximate the peak sidelobe level (PSL) metric.**

- The gradient is

$$\nabla_{\mathbf{g}} \|\mathbf{e} - \mathbf{A}^H\mathbf{g}\|_p^p = -p\, \Re\left\{\mathbf{A}\left(|\mathbf{e} - \mathbf{A}^H\mathbf{g}|^{p-2} \odot (\mathbf{e} - \mathbf{A}^H\mathbf{g})\right)\right\}$$

Global Minimum Peak Sidelobes (PSL)



r : Autocorrelation

$\mathbf{A}^H$: Inverse DFT Matrix

g : Power Spectrum (non − negative)

g_m : m^{th} element of the power spectrum

γ_m : Constrained maximum value for $g_m \in \Omega$

Ω : Frequency indices to null constrain

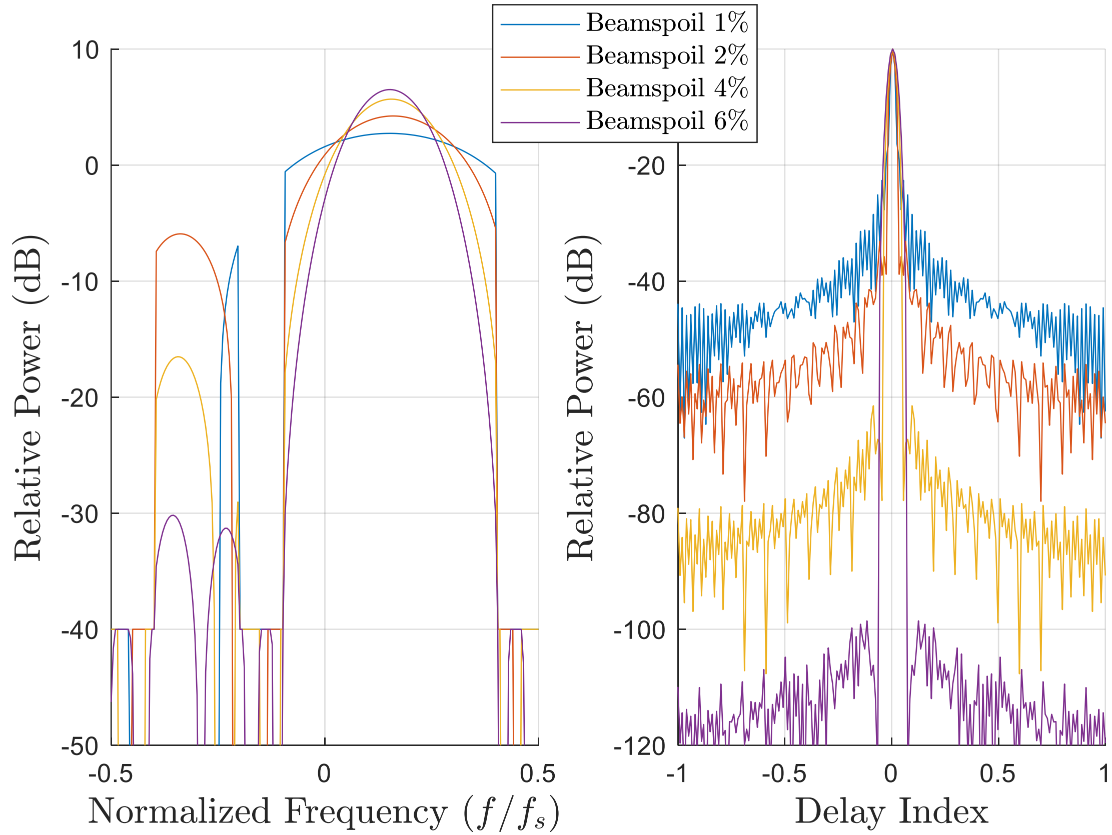
e : Desired Autocorrelation Response (impulse)

$M = 200$ window length

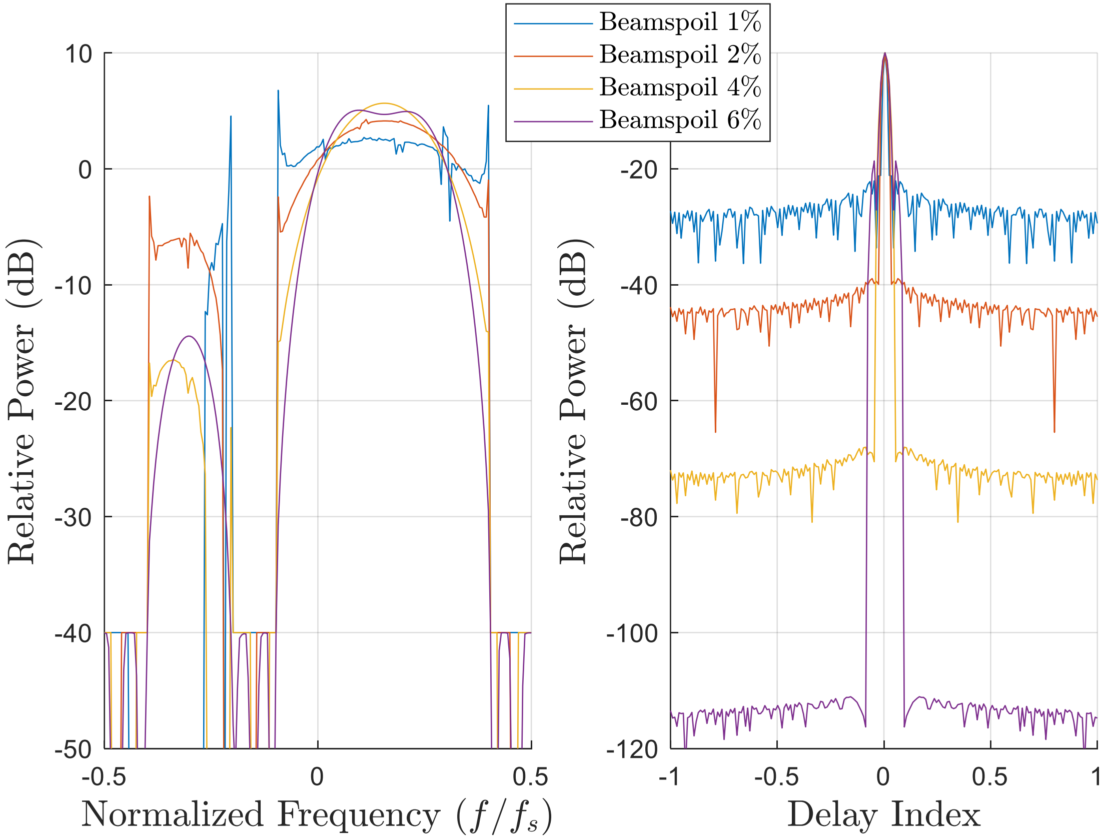
Different degrees of beamspoiling are achieved by replacing $\bar{M}$ rows of $\mathbf{A}^H$ (corresponding to autocorrelation mainlobe roll-off) with zeros, thus **permitting different mainlobe widths and achievable sidelobe levels**

Comparison of Optimum

Minimize ISL



Minimize PSL

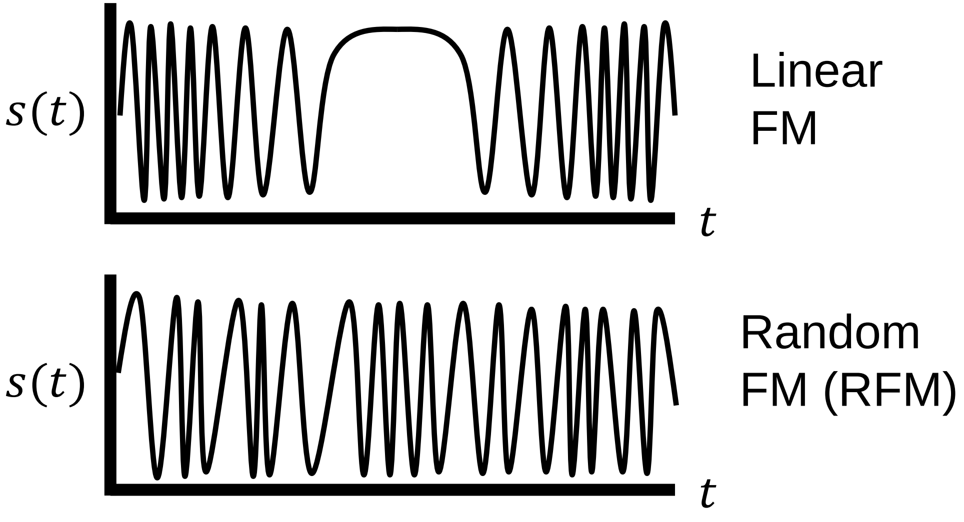


$M = 200$ window length

Waveform Design

Nonrepeating Spectrally Notched FM Waveforms

- Various methods have been **experimentally demonstrated** to realize **spectrally shaped** forms of nonrepeating random FM (RFM) waveforms [3]
- Notched versions of RFM can achieve **> 50 dB notch depth**, while preserving **transmitter-amenable FM structure** [4]



Here, waveforms are spectrally shaped to the notched PSD template that globally minimizes ISL

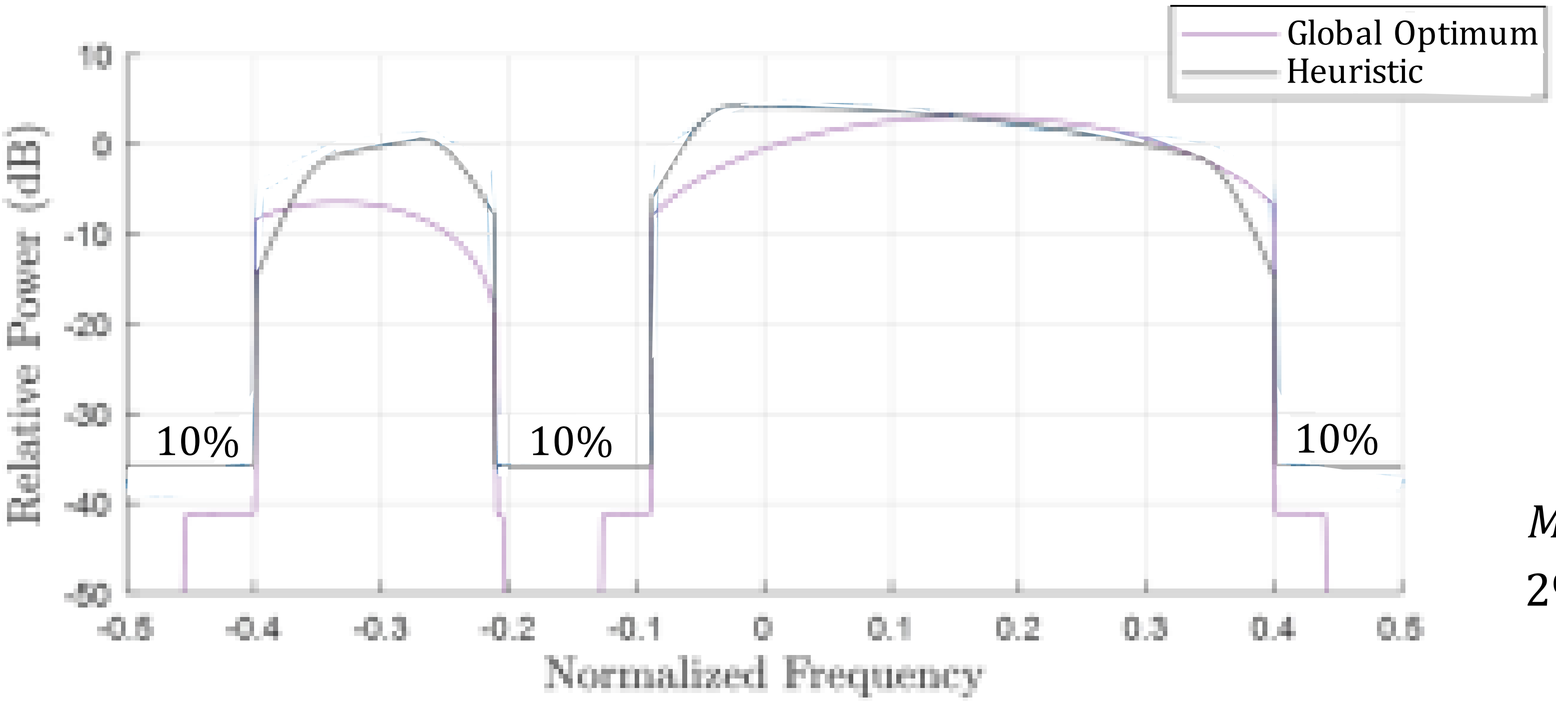
[3] S.D. Blunt, J.K. Jakabosky, C.A. Mohr, P.M. McCormick, J.W. Owen, B. Ravenscroft, C. Sahin, G.D. Zook, C.C. Jones, J.G. Metcalf, T. Higgins, "Principles & applications of random FM radar waveform design," *IEEE Aerospace & Electronic Systems Magazine*, vol. 35, no. 10, pp. 20-28, Oct. 2020.

[4] C.A. Mohr, S.D. Blunt, "Analytical spectrum representation for physical waveform optimization requiring extreme fidelity," IEEE Radar Conf., Boston, MA, Apr. 2019.

Waveform Spectrum Templates

1. In [5] the waveform design method mitigated correlation sidelobes by **matching the waveform PSD to a heuristic template** having tapered spectral null borders

2. Here, the waveform design method mitigates correlation sidelobes by **matching the waveform PSD to the least-squares optimal PSD template** that minimizes ISL



$M = 200$ window length

2% Beamspoil

[5] J. Jakabosky, B. Ravenscroft, S. Blunt, A. Martone, "Gapped spectrum shaping for tandem-hopped radar/communications & cognitive sensing," IEEE Radar Conf., Philadelphia, PA, May 2016.

Notched FM Waveform Generation

KU

The waveform optimization is executed in two stages:

1. Perform K iterations of alternating time-frequency projections to produce a pseudo-random optimized FM (PRO-FM) waveform [5, 6]
   - The desired spectrum **g** is **the heuristic or optimal PSD template**, producing waveforms with shallow spectral notches over Ω
2. Then apply L iterations of the zero-order reconstruction optimization of waveforms (ZOROW) [7] to significantly deepen spectral notches over Ω

- K=200 and L=1000 iterations to guarantee **full convergence, ensuring a modest waveform spectrum match to the template**

PRO – FM

$$\tilde{\mathbf{s}}^{(k)} = \tilde{\mathbf{A}}^H\{\mathbf{g}^{1/2} \odot \exp(j\angle\tilde{\mathbf{A}}\{\mathbf{s}^{(k-1)}\})\}$$

$$\mathbf{s}^{(k)} = \mathbf{u} \odot \exp(j\angle\,\tilde{\mathbf{s}}^{(k)})$$

ZOROW

$$[\mathbf{s}]_n = e^{j\phi_n}$$

$$\boldsymbol{\phi} = [\phi_1 \quad \phi_2 \quad \cdots \quad \phi_N]^T$$

$$S(f_m,\boldsymbol{\phi}) = \frac{\sin(\pi f_m T_s)}{\pi f_m}\sum_{n=1}^{N}\exp(-j(2\pi f_m(n-.5)T_s+\phi_n))$$

$$\min_{\boldsymbol{\phi}} \sum_{m\in\Omega} |S(f_m,\boldsymbol{\phi})|^2$$

[5] J. Jakabosky, B. Ravenscroft, S. Blunt, A. Martone, "Gapped spectrum shaping for tandem-hopped radar/communications & cognitive sensing," IEEE Radar Conf., Philadelphia, PA, May 2016.

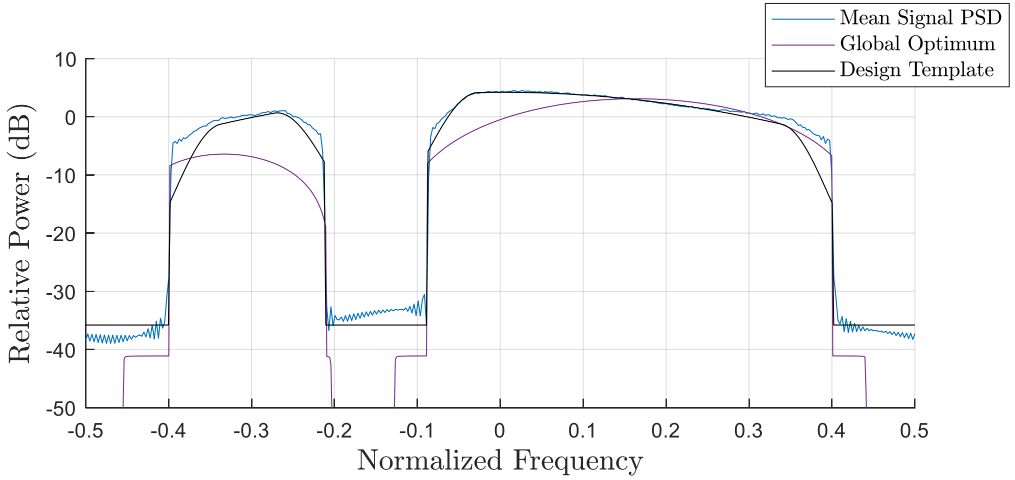
[6] J. Jakabosky, S.D. Blunt, B. Himed, "Spectral-shape optimized FM noise radar for pulse agility," IEEE Radar Conf., Philadelphia, PA, May 2016.

[7] C. Mohr, J.W. Owen, S.D. Blunt, "Zero-order reconstruction optimization of waveforms (ZOROW) for modest DAC-rate systems," *IEEE Intl. Radar Conf.*, Washington, DC, Apr. 2020.

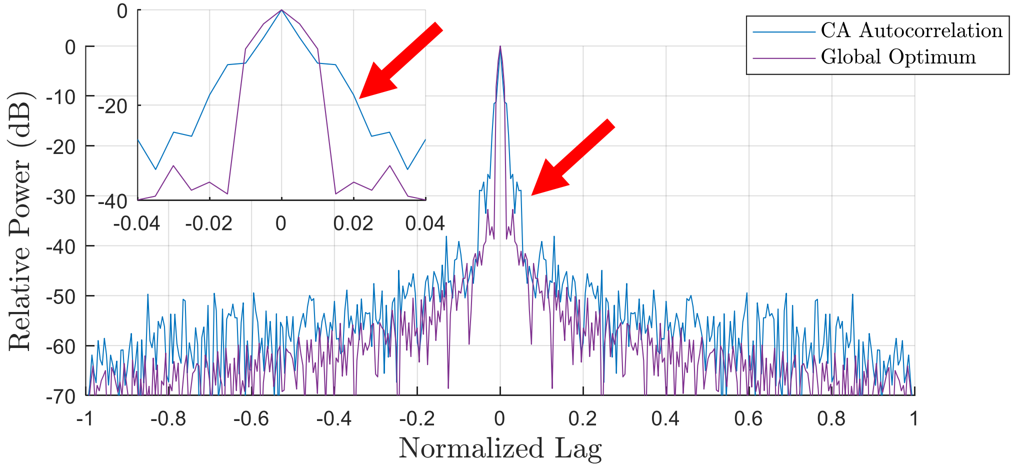
Heuristic Spectral Template

$Q = 1000$ pulses
$N = 200$ pulse parameters
$M = 800$ PSD window samples
2% Beamspoiling

Mean PSD



Coherent Average Autocorrelation



$$\mathbf{r} = \mathbf{A}^H \mathbf{g}$$

$$\sum_q \mathbf{r}_q = \sum_q \mathbf{A}^H \mathbf{g}_q$$

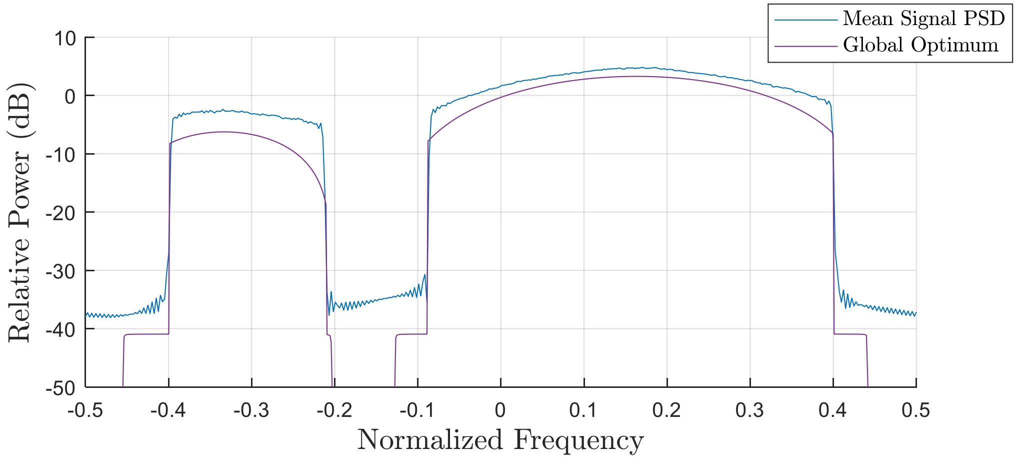
$$\sum_q \mathbf{r}_q = \mathbf{A}^H \sum_q \mathbf{g}_q$$

Waveforms designed to the heuristic template exhibit mainlobe broadening and "shoulder" lobes compared to the global optimum

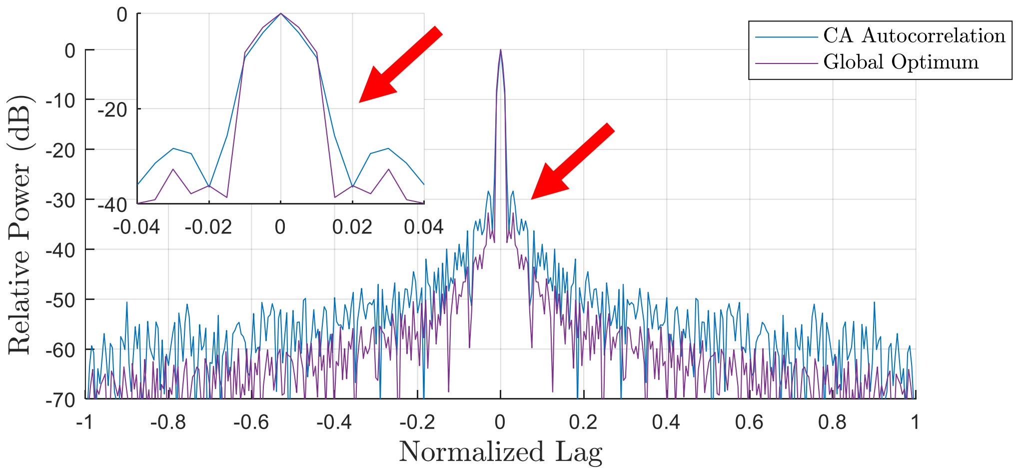
Optimal Spectral Template

$Q = 1000$ pulses
$N = 200$ pulse parameters
$M = 800$ PSD window samples
2% Beamspoiling

Mean PSD



Coherent Average Autocorrelation



$$\mathbf{r} = \mathbf{A}^H \mathbf{g}$$

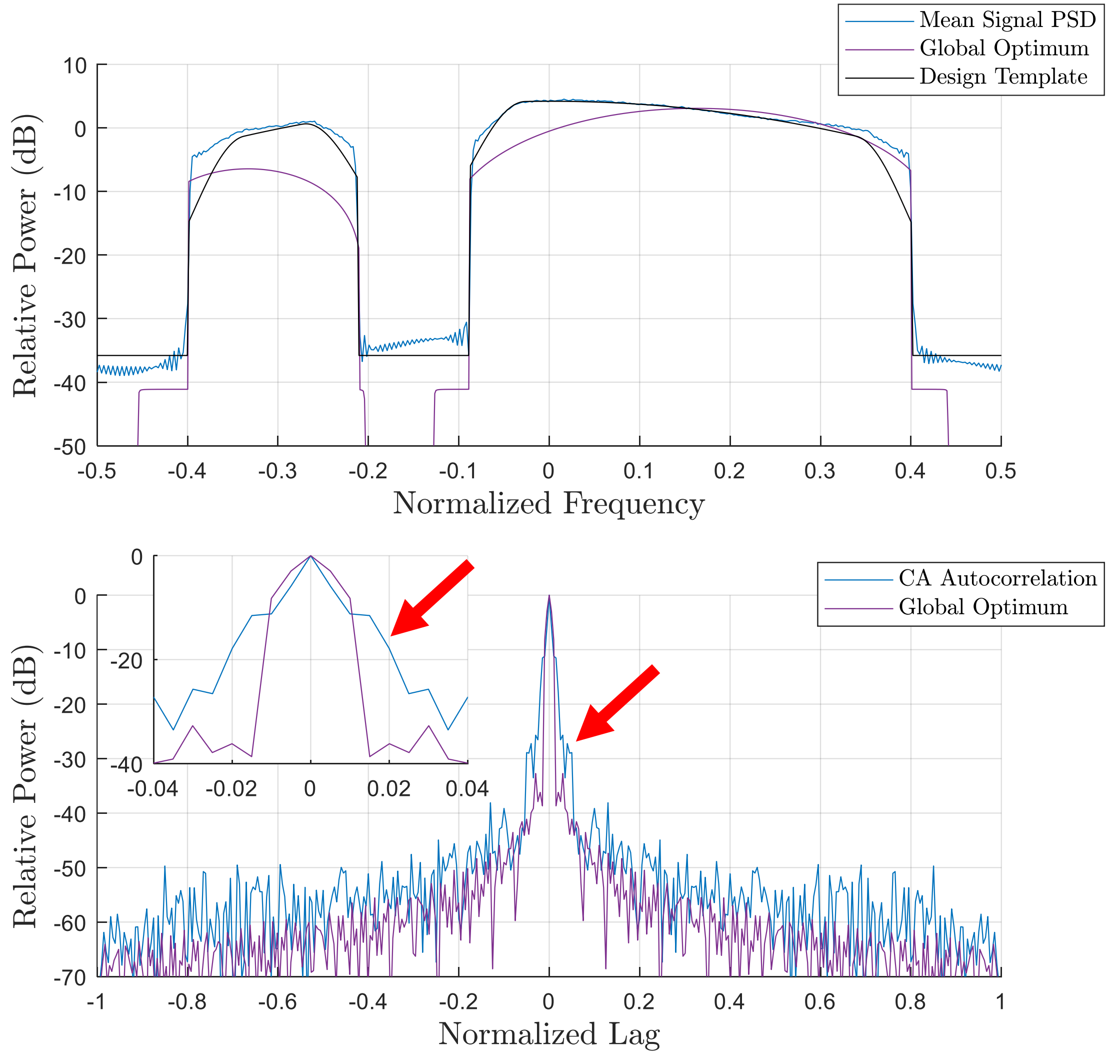
$$\sum_q \mathbf{r}_q = \sum_q \mathbf{A}^H \mathbf{g}_q$$

$$\sum_q \mathbf{r}_q = \mathbf{A}^H \sum_q \mathbf{g}_q$$

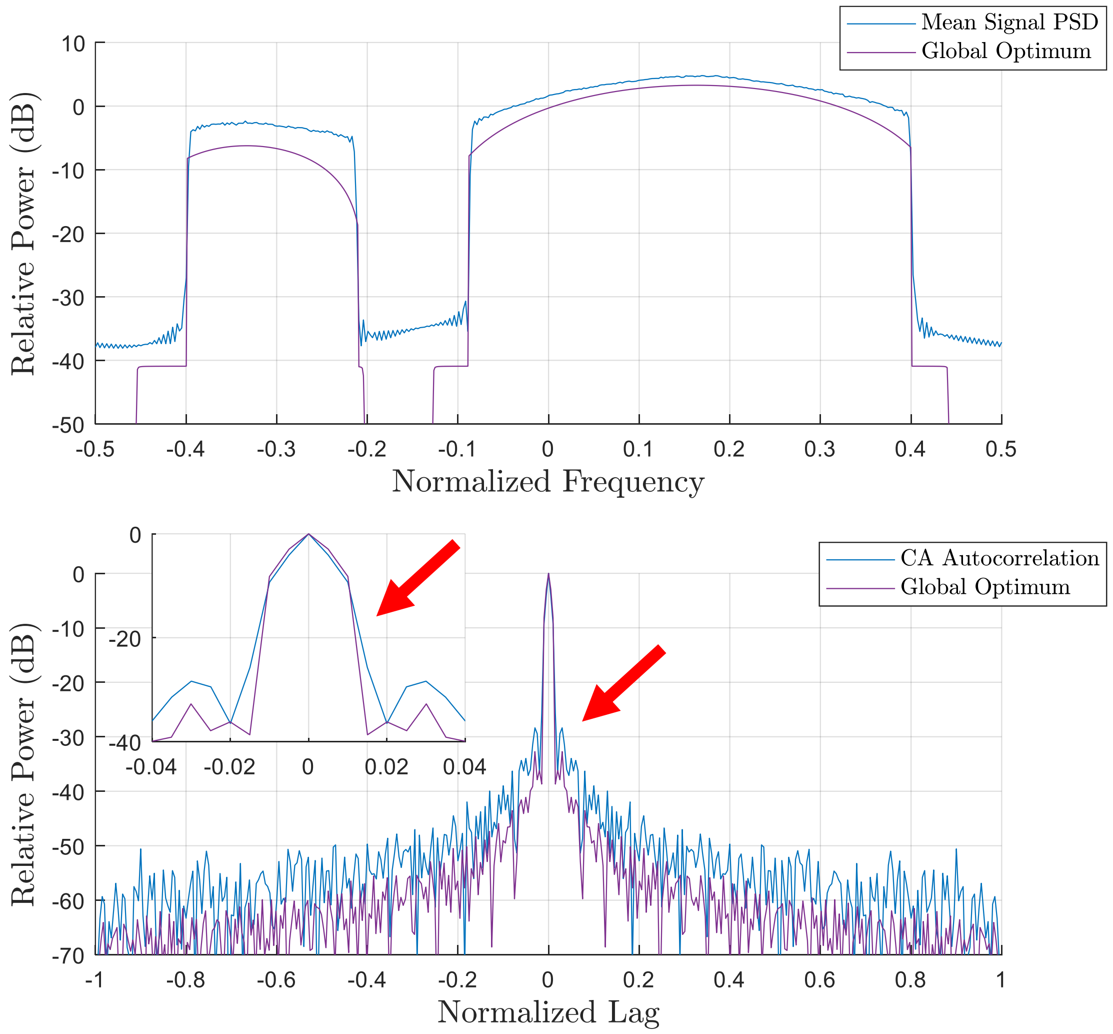
Waveforms designed to the optimal template match closely, with residual sidelobes due to range sidelobe modulation

Heuristic vs. Optimal Spectral Template

Heuristic



Optimal



Mismatched filter design

Least Squares Filter Formulation

KU

Because the ISL optimum spectral template is based on least squares in a 2-norm sense, it is logical to apply the least squares mismatched filter (MMF) [8-10] to the same waveform sets

$$\mathbf{w}_{\text{LS}} = (\mathbf{S}^H\mathbf{S} + \sigma\mathbf{I})^{-1}(\mathbf{S}^H\mathbf{e})$$

$$\mathbf{S} = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ \vdots & s_1 & \ddots & \vdots \\ s_N & \vdots & \ddots & 0 \\ 0 & s_N & & s_1 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & s_N \end{bmatrix}$$

- $\mathbf{S}$: Convolution matrix of signal $\mathbf{s}$
- $\mathbf{e}$: Desired Correlation Response
- $\mathbf{g}$: Desired Power Spectrum
- σ : Regularization term

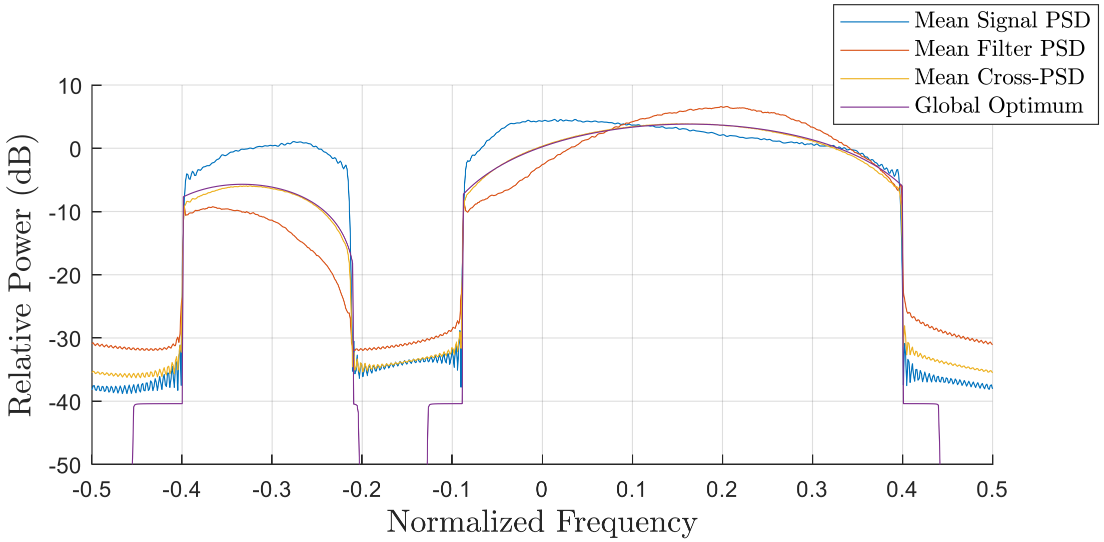
The desired correlation response is the IDFT of the optimal spectrum $\mathbf{e} = \mathbf{A}^H\mathbf{g}$

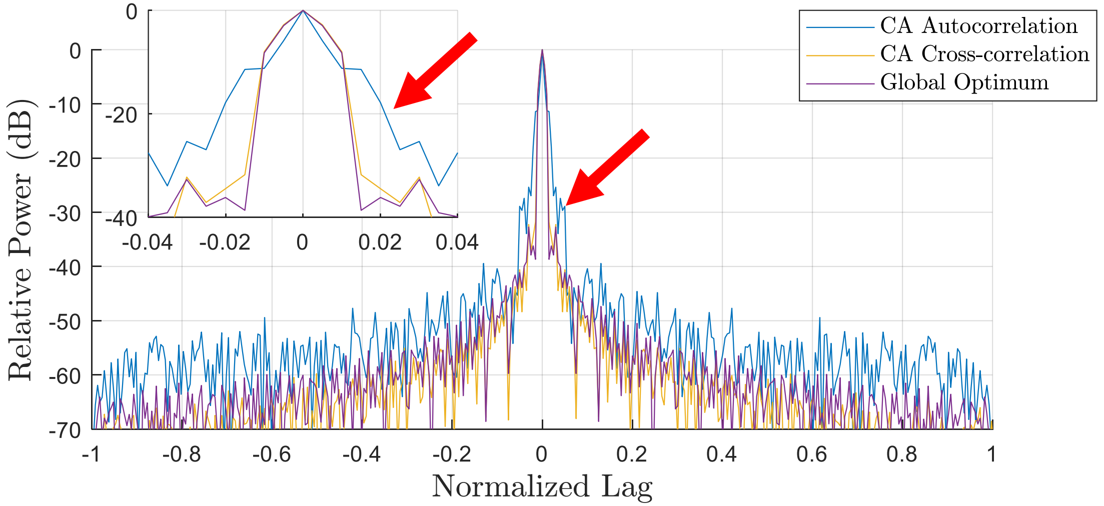
[8] M. H. Ackroyd and F. Ghani, "Optimum Mismatched Filters for Sidelobe Suppression," in IEEE Transactions on Aerospace and Electronic Systems, vol. AES-9, no. 2, pp. 214-218, March 1973.

[9] D. Henke, P. McCormick, S. D. Blunt, T. Higgins, "Practical aspects of optimal mismatch filtering and adaptive pulse compression for FM waveforms," IEEE Radar Conf., Washington, DC, May 2015.

[10] B. Ravenscroft, J. Owen, S. Blunt, A. Martone, K. Sherbondy, "Optimal mismatched filtering to address clutter spread from intra-CPI variation of spectral notches", IEEE Radar Conf., Boston, MA, Apr. 2019.

Heuristic Spectral Template



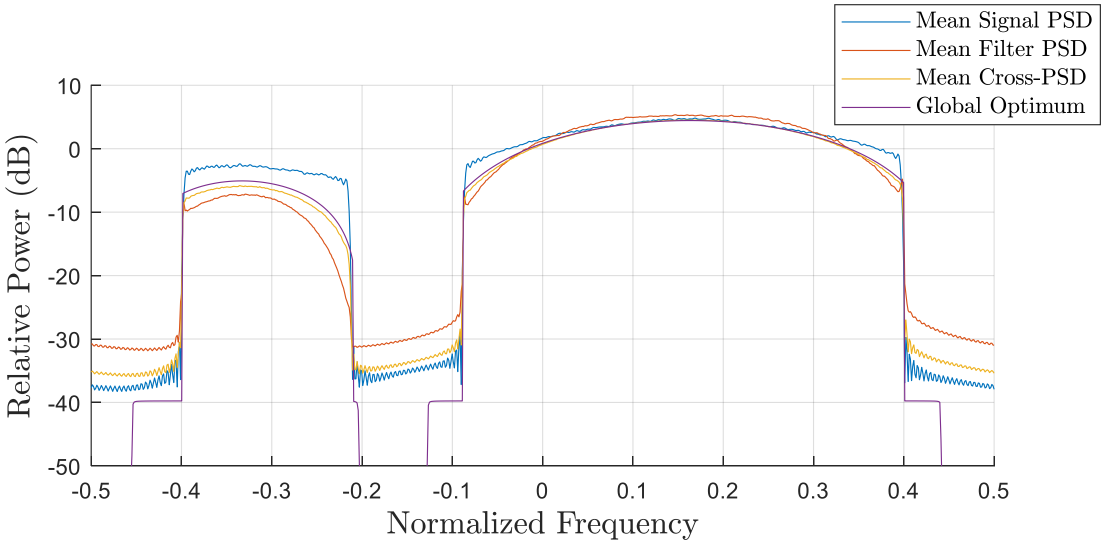


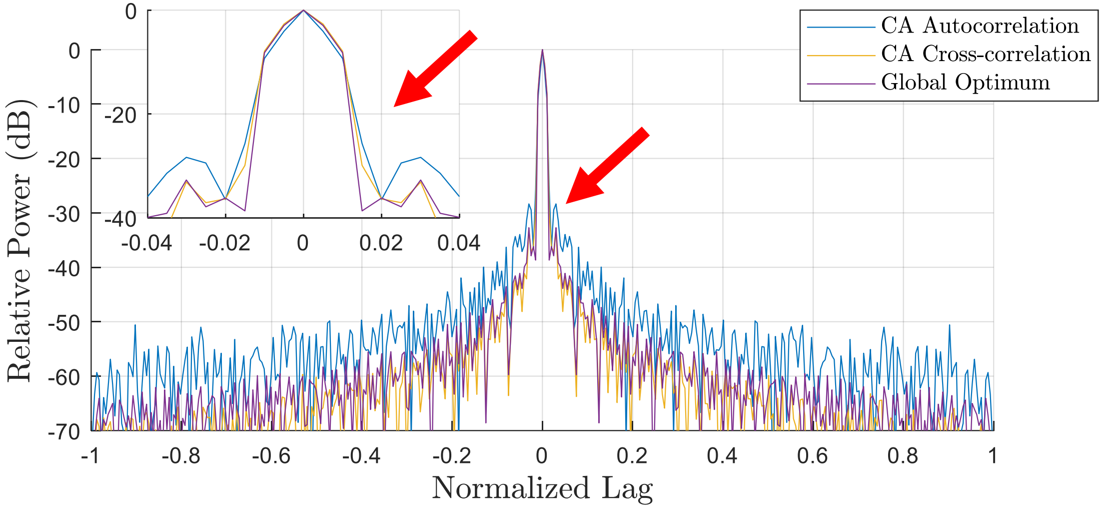
$Q\ = 1000$ pulses
$N\ = 200$ pulse parameters
$P\ = 600$ filter parameters
$M\ = 800$ CPSD window samples
2% Beamspoiling

2.59 dB mismatch loss (SNR loss)

Least squares filtering compensates for the heuristic spectrum mainlobe broadening and "shoulder" lobes, closely matching the global optimum

Optimal Spectral Template





$Q = 1000$ pulses

$N = 200$ pulse parameters

$P = 600$ filter parameters

$M = 800$ CPSD window samples

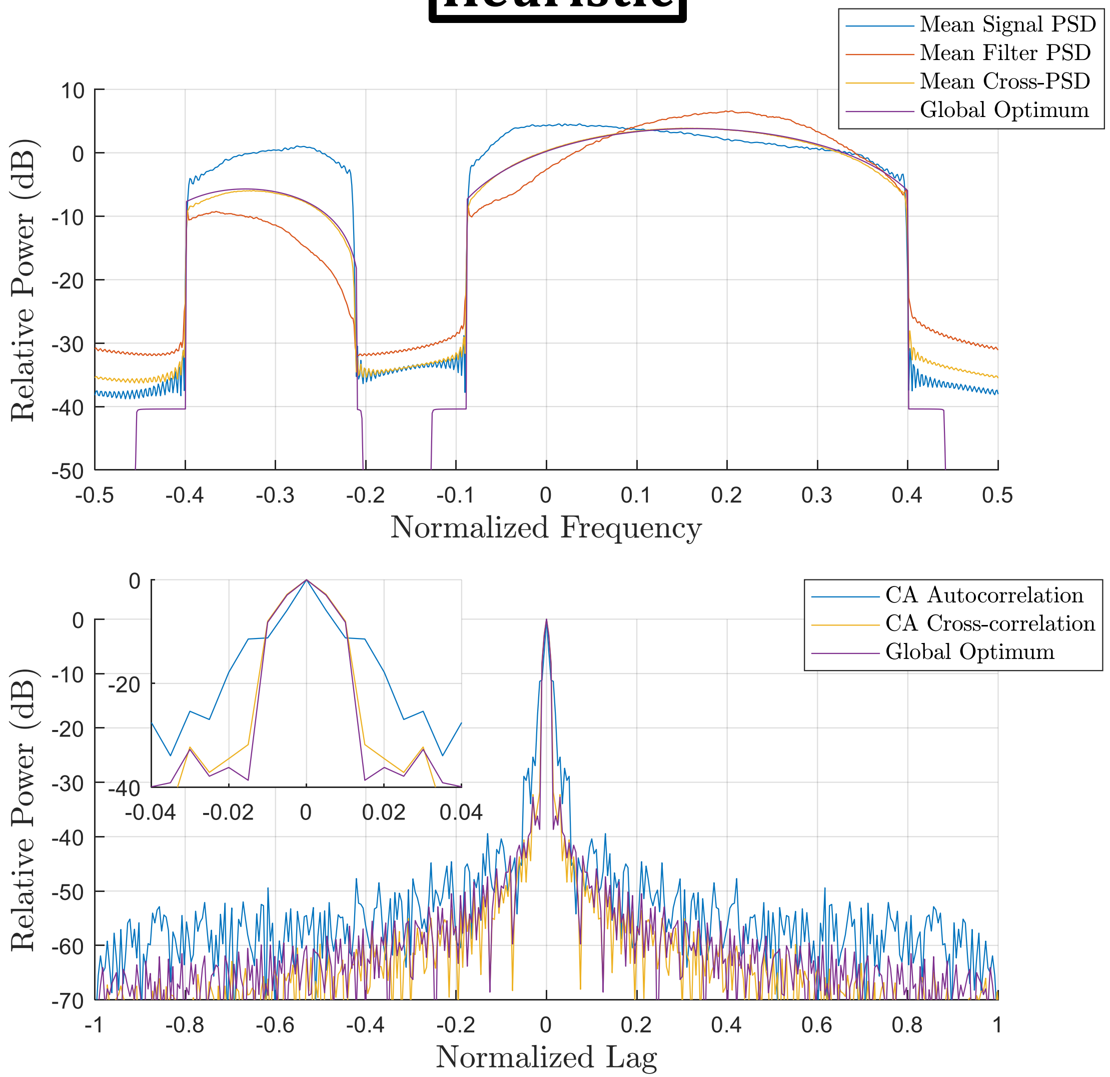
2% Beamspoiling

1.37 dB mismatch loss (1.22 dB improvement)

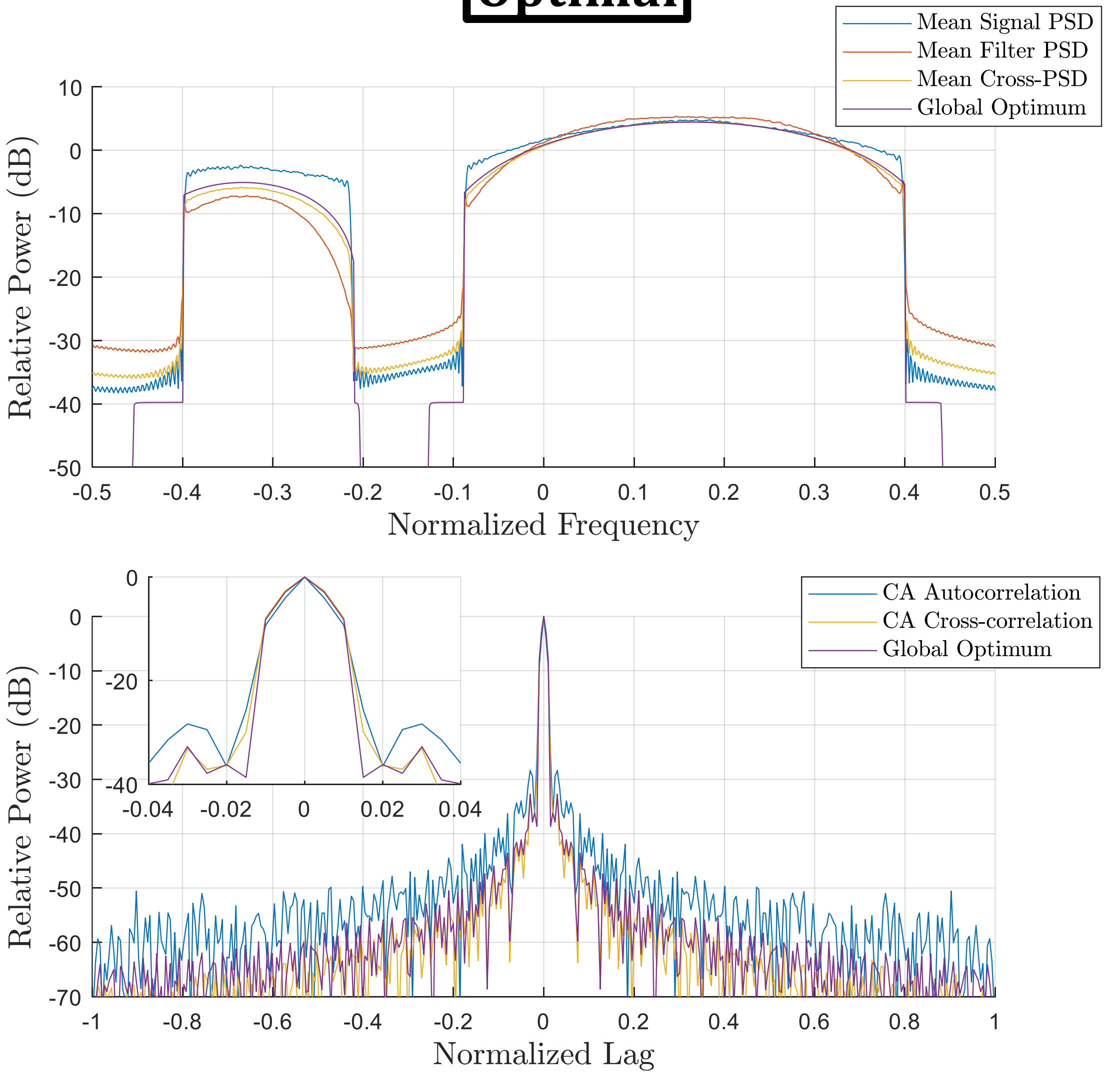
Waveforms matched to the optimal spectral template already exhibit near-optimality, such that least squares filtering incurs minimal mismatch loss

Heuristic vs. Optimal Spectral Template

Heuristic



Optimal



Conclusions

- **The least squares global optimum power spectrum has been determined to minimize ISL and PSL when portions of the spectrum are null constrained.**
  - ✓ **By designing waveform spectra to closely match the optimal template, their attendant sidelobes also approach the optimal level.**
  - ✓ **Application of the least-squares mismatched filter then closes the remaining sidelobe difference, with mismatch loss in trade.**
  - ✓ **The heuristic PSD template design involving simple tapering of notch edges is determined to achieve near-optimal performance with a computational cost that is low enough for real-time implementation.**

Thank You!