Design and Generation of Stochastically Defined, Pulsed FM Noise Waveforms

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This work was supported by the Office of Naval Research under Contract #N00014-16-C-2029 and by a subcontract with Matrix Research, Inc. for the Air Force Research Laboratory under Prime Contract #FA8650-14-D-1722. DISTRIBUTION STATEMENT A. Approved for Public Release.

Motivation

- The high dimensionality of noise waveforms provides tremendous degrees-of-freedom with which to achieve separability and sidelobe suppression

- The special class of FM (constant amplitude) noise waveforms extend this capability to high powered systems

- However, the need for spectral shaping of FM noise waveforms imposes an optimization requirement that can incur a high computational cost

- To greatly reduce this cost, here families of FM noise waveforms are obtained through the development of an off-line optimized generating function that can be driven by a simple stochastic process

  => hence, ***Stochastic Waveform Generation (StoWGe)***

Defining Stochastic FM Waveforms

Define a time-limited, stochastic process as :

$$s[m] = \begin{cases} \exp(j\phi[m]) & m = 1,2,\cdots,M \\ 0 & \text{otherwise} \end{cases}$$

where $\phi[m]$ is a real, random process

Each member function of $s[m]$ is a unique, FM waveform such that $\mathbf{s} = \exp(j\boldsymbol{\phi})$

The DFT of **s** is defined as:

$$\mathbf{s}_f = \mathbf{A}\bar{\mathbf{s}}$$

$$\bar{\mathbf{s}} = \left[\mathbf{s}^{\mathrm{T}}\ \mathbf{0}_{W-M}^{\mathrm{T}}\right]^{\mathrm{T}}$$

where $\mathbf{A}$ is a $W \times W$ DFT matrix, for $W \geq 2M - 1$

Spectral containment will be provided through the optimization process

Analyzing Stochastic Waveforms

The analysis of stochastic waveforms necessitates statistical tools

Aggregate Measures – Average response of an infinite number of waveforms

$E\left[|\mathbf{s}_f|^2\right]$ – Expected power spectral density

$\mathbf{A}^{\mathrm{H}} E\left[|\mathbf{s}_f|^2\right]$ – Expected coherent autocorrelation

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Individual Measures – Expected response of an individual waveform

$E\left[|\mathbf{s}_f|^4\right] - E\left[|\mathbf{s}_f|^2\right]$ – Mean squared error from the expected spectrum

$\left(E\left[\left|\mathbf{A}^{\mathrm{H}}|\mathbf{s}_f|^2\right|^2\right]\right)^{1/2}$ – Expected RMS autocorrelation response of a single waveform

Designing Stochastic Waveforms

We have defined Stochastic FM waveforms and how to analyze them.

But how to design them?

First, parameterize the phase

$$\mathbf{s} = \exp(j\boldsymbol{\phi})$$

$$\boldsymbol{\phi} = \mathbf{B}\mathbf{x} + \boldsymbol{\mu}$$

- $\boldsymbol{\mu}$: $M \times 1$ vector of constants
- $\mathbf{B}$: $M \times N$ matrix of constants
- $\mathbf{x}$: $M \times 1$ vector of $\mathcal{N}(0,1)$ random variables

such that

$$\phi_m \sim \mathcal{N}\left(\mu_m, \mathbf{b}_m \mathbf{b}_m^{\mathrm{T}}\right)$$

$\mathbf{b}_m$: m^{th} row of **B**

B and **μ** provide sufficient design freedom to optimize the <u>expected</u> behavior of **s**

Expected Frequency Template Error (FTE)

$$J = \left\| E\left[\left| \mathbf{s}_f \right|^2 \right] - \mathbf{u} \right\|_2^2$$

Measures the squared error between the expected spectrum and some desired spectrum

By minimizing J, the $E\left[\left| \mathbf{s}_f \right|^2 \right]$ term is made more similar to the $W \times 1$ length desired spectrum $\mathbf{u}$

Gradient-Based Optimization

- Because the cost function is a continuous function of **B** and **μ**, a gradient can be calculated.

- Here a Heavy Ball gradient-descent method is used to minimize the cost function
  - A good compromise between convergence rate, algorithmic complexity, and stability
  - Search direction can be reset to steepest descent direction if found to be an ascent direction

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Gradient-descent structure

$$\mathbf{q}_{i+1} = \mathbf{q}_i + \mu_i \mathbf{p}_i$$

$$\mathbf{p}_i = \begin{cases} -\nabla_{\mathbf{q}_i} J & \text{when } i = 0 \\ -\nabla_{\mathbf{q}_i} J + \beta \mathbf{p}_{i-1} & \text{otherwise} \end{cases}$$

$\mathbf{q}_i$ Parameters to be optimized at the i^{th} iteration

μ_i step-size at the i^{th} iteration

$\nabla_{\mathbf{q}_i} J$ gradient at the i^{th} iteration

β Heavy Ball parameter, where $0 \leq \beta < 1$

Evaluating the Gradient

$$\nabla_{\mathbf{B}} J = \nabla_{\mathbf{B}} \left(\left\| E\left[\left| \mathbf{s}_f \right|^2 \right] - \mathbf{u} \right\|_2^2 \right)$$

First, expand $E\left[\left| \mathbf{s}_f \right|^2 \right]$ into $E[\mathbf{A}\bar{\mathbf{s}} \odot (\mathbf{A}\bar{\mathbf{s}})^*]$ such that for a single sample $s_{f,w}$:

$$E\left[\left| s_{f,w} \right|^2 \right] = \sum_{m_1}^{W} \sum_{m_2}^{W} a_{w,m_1} a_{w,m_2}^* E\left[\bar{s}_{m_1} \bar{s}_{m_2}^* \right]$$

where $c_{m_1,m_2} = E\left[\bar{s}_{m_1} \bar{s}_{m_2}^* \right]$ is the element of the m_1^{th} row and m_2^{th} column of the waveform correlation matrix $\mathbf{C}$ and

$$c_{m_1,m_2} = \exp\left(j\left(\mu_{m_1} - \mu_{m_2} \right) - 0.5\left(\mathbf{b}_{m_1} - \mathbf{b}_{m_2} \right) \left(\mathbf{b}_{m_1} - \mathbf{b}_{m_2} \right)^{\mathrm{T}} \right)$$

The gradients with respect to **B** and μ can be calculated in terms of correlation matrix **C**

$$\frac{\partial J}{\partial b_{\ell,n}} = 2\left[\frac{\partial E\left[\left|\mathbf{s}_f\right|^2\right]}{\partial b_{\ell,n}}\right]^{\mathrm{T}}\left(E\left[\left|\mathbf{s}_f\right|^2\right] - \mathbf{u}\right)$$

$$\frac{\partial J}{\partial \mu_r} = 2\left[\frac{\partial E\left[\left|\mathbf{s}_f\right|^2\right]}{\partial \mu_r}\right]^{\mathrm{T}}\left(E\left[\left|\mathbf{s}_f\right|^2\right] - \mathbf{u}\right)$$

$$\frac{\partial E\left[\left|s_{f,w}\right|^2\right]}{\partial b_{\ell,n}} = \sum_{m=1}^{M} 2\Re\left\{a_{w.\ell}a_{w,m}^{*}c_{\ell,m}\left(b_{\ell,n} - b_{m,n}\right)\right\}$$

$$\frac{\partial E\left[\left|s_{f,w}\right|^2\right]}{\partial \mu_r} = \sum_{m=1}^{M} 2\Im\left\{a_{w,r}^{*}a_{w,m}c_{\ell,m}\right\}$$

Using these derivatives and the gradient descent approach, the expected FTE cost function can be minimized

Stochastic FM Design/Generation Process

Choose an initial $\mathbf{B}$, $\boldsymbol{\mu}$, and a desired spectrum $\mathbf{u}$

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Perform gradient descent to minimize J as a function of $\mathbf{B}$ and $\boldsymbol{\mu}$ ← Optimization is only performed **ONCE** per desired spectrum $\mathbf{u}$

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Instantiate Stochastic FM waveforms by drawing $\mathbf{x} \sim \mathcal{N}(0,1)$ and evaluating $\mathbf{s} = \exp(\mathbf{Bx} + \boldsymbol{\mu})$ ← Actual generation of these random FM waveforms has negligible computational cost

Demo: Initialization

B – initialized as a first order PCFM basis matrix (time shifted ramps) **[1]**

- $BT = 150$
- 6 times oversampling w.r.t. 3dB BW
- Size = 900×150

μ – initialized as a 900×1 vector of zeros

u – initialized as Gaussian for spectral containment and good autocorrelation

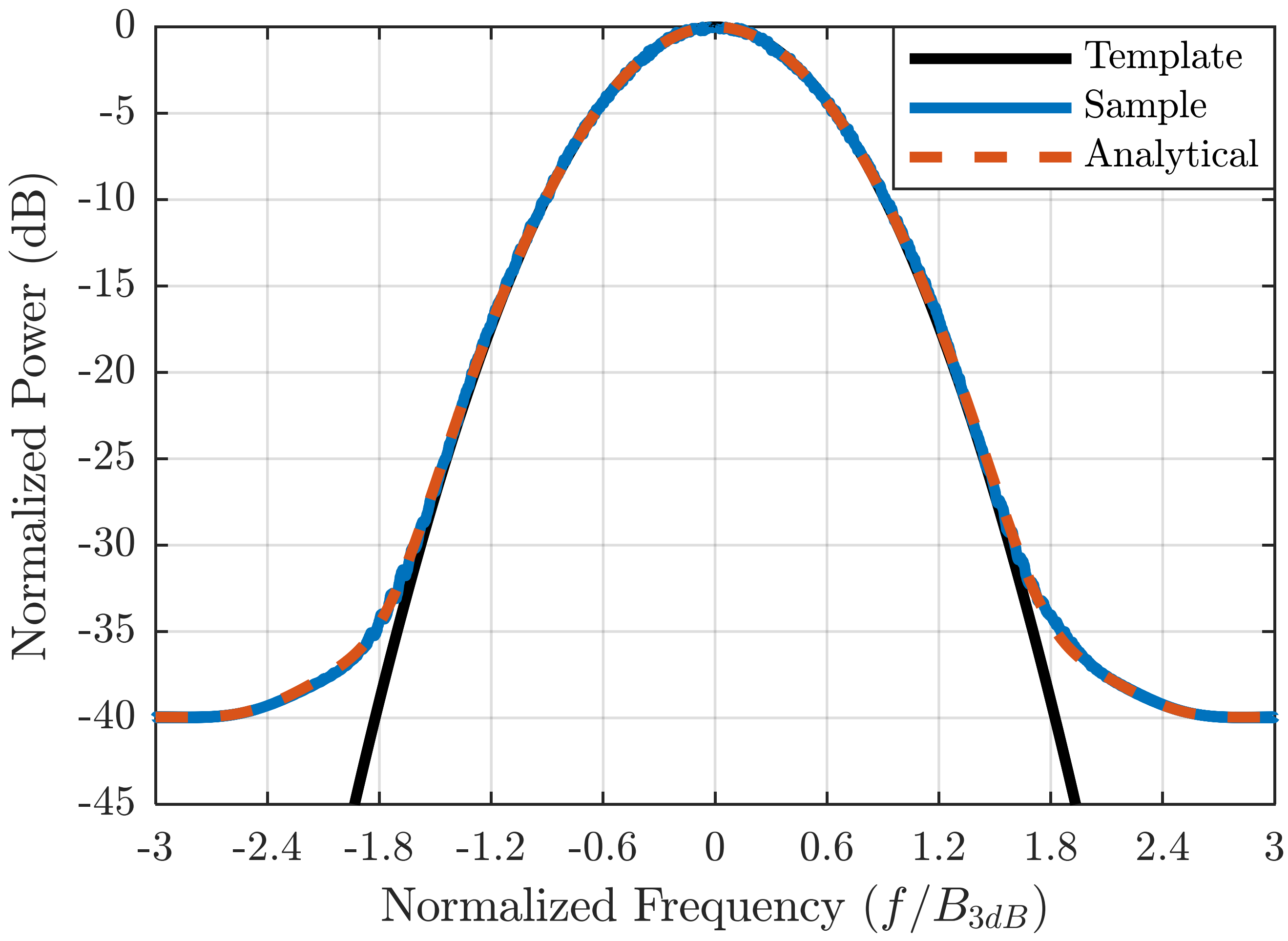
Relatively high oversampling factor ($\times$ 6) emphasizes spectral roll-off attributes

Choose an initial **B**, **μ**, and a desired spectrum **u**

[1] S.D. Blunt, M. Cook, J. Jakabosky, J.D. Graaf, E. Perrins, "Polyphase-coded FM (PCFM) radar waveforms, part I: implementation," *IEEE Trans. AES*, vol. 50, no. 3, pp. 2218–2229, July 2014.

Demo: Optimization Results

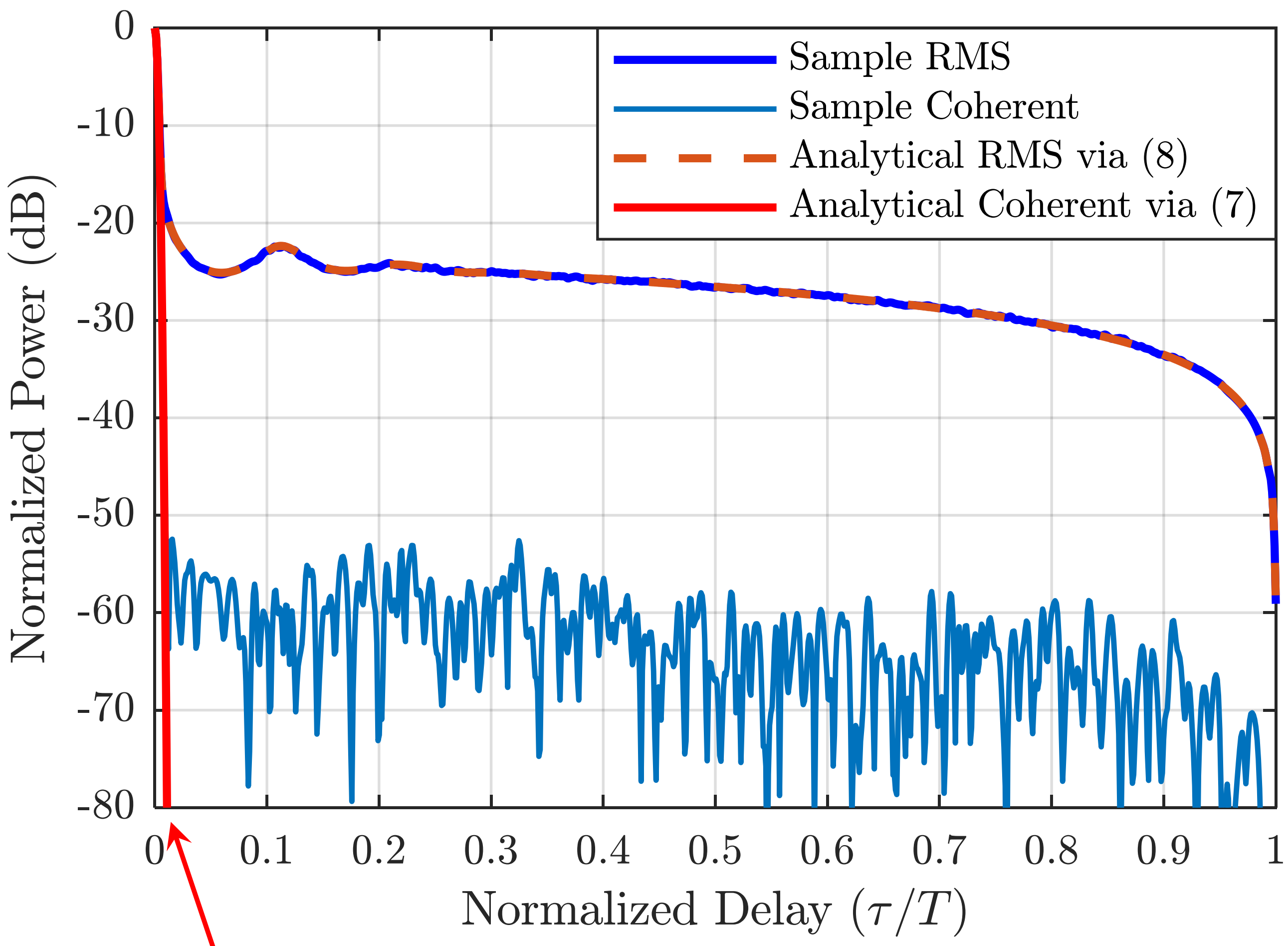
- 3000 waveforms were generated via $\mathbf{s} = \exp(\mathbf{Bx} + \boldsymbol{\mu})$ and their mean spectrum was calculated. (denoted "sample")
- The analytical trace was calculated directly from $E\left[\left|\mathbf{s}_f\right|^2\right]$ (denoted "analytical")
- Analytical and sample traces match the template very well down to about -30 dB (due to pulse shape)



Sample and analytical spectra for the OPTIMIZED **B** and **μ**

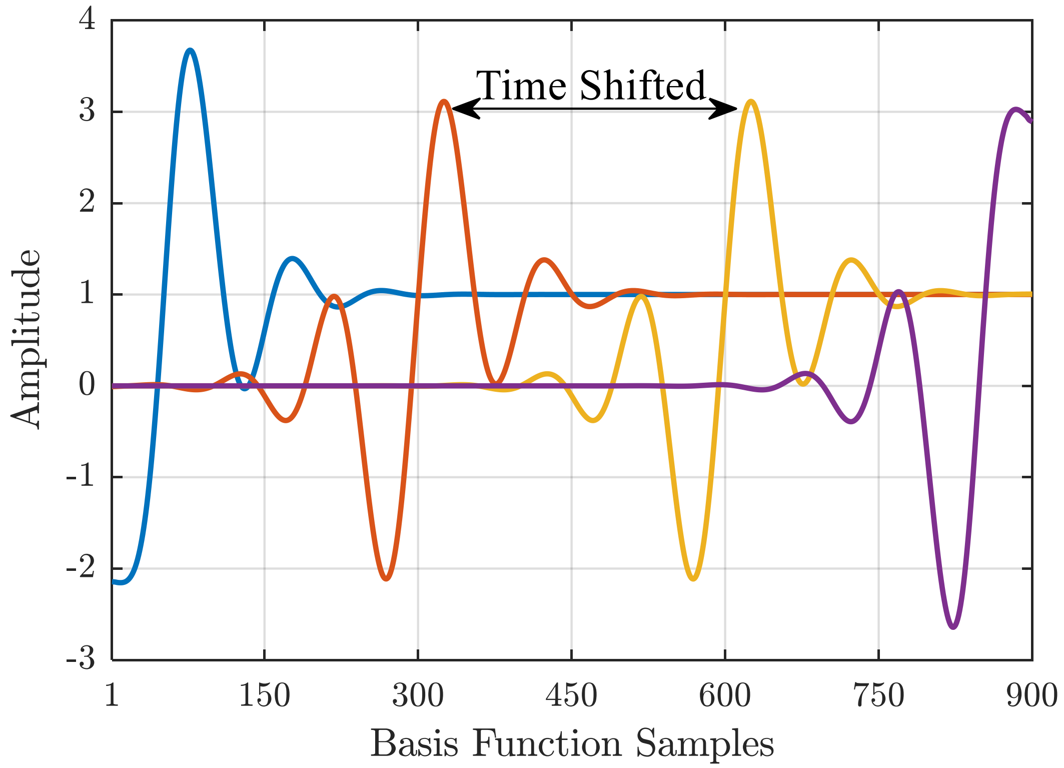
Demo: Optimization Results

- Same 3000 Waveforms
- Sample RMS matches Analytical RMS autocorrelation $\left(E\left[\left|\mathbf{A}^{\mathrm{H}}|\mathbf{s}_f|^2\right|^2\right]\right)^{1/2}$, with peak at about $-10\log_{10}(150) = -23$ dB
- Coherent combination of the 3000 unique autocorrelations reduces sidelobes by roughly 30 dB
- Analytical coherent response determined via $\left(\mathbf{A}^{\mathrm{H}}E\left[|\mathbf{s}_f|^2\right]\right)$ approaches no sidelobes at all

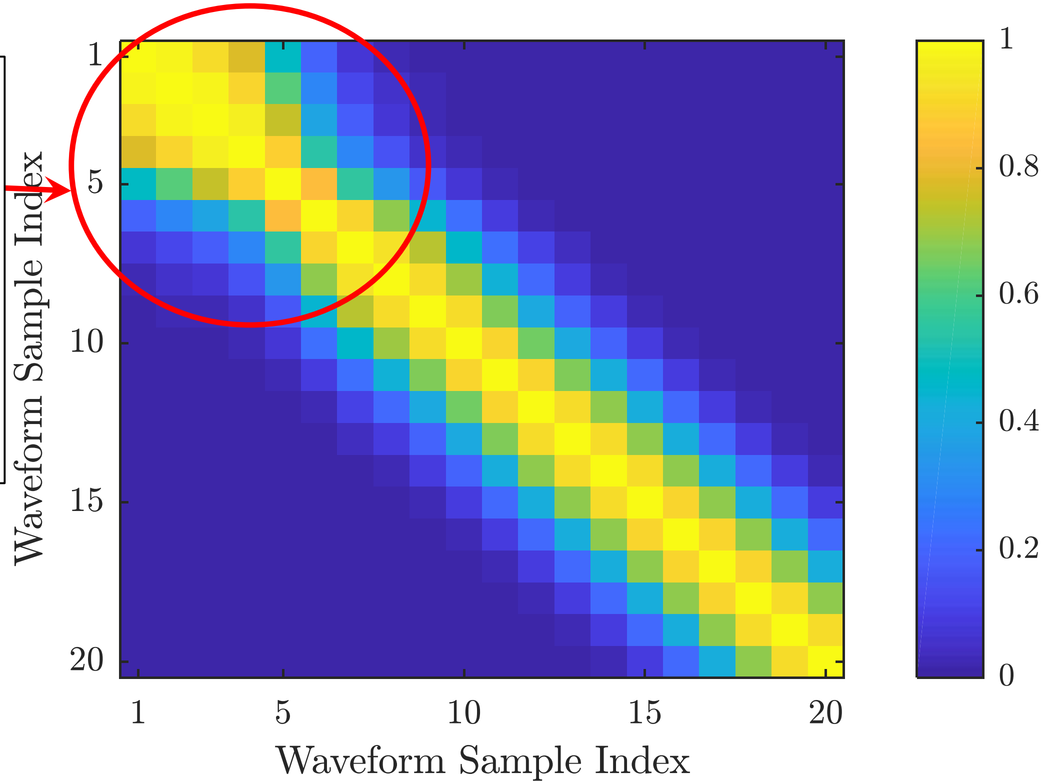


Expected Autocorrelation approaches complete absence of sidelobes

Demo: A closer look at **B** and **C**



First and last several samples exhibit unique correlation relative to all others.



Example basis functions (columns) in the optimized **B**

Upper left corner of the optimized **correlation matrix C**

- Central basis functions are time-shifted versions of each other
- First and last few basis functions are unique

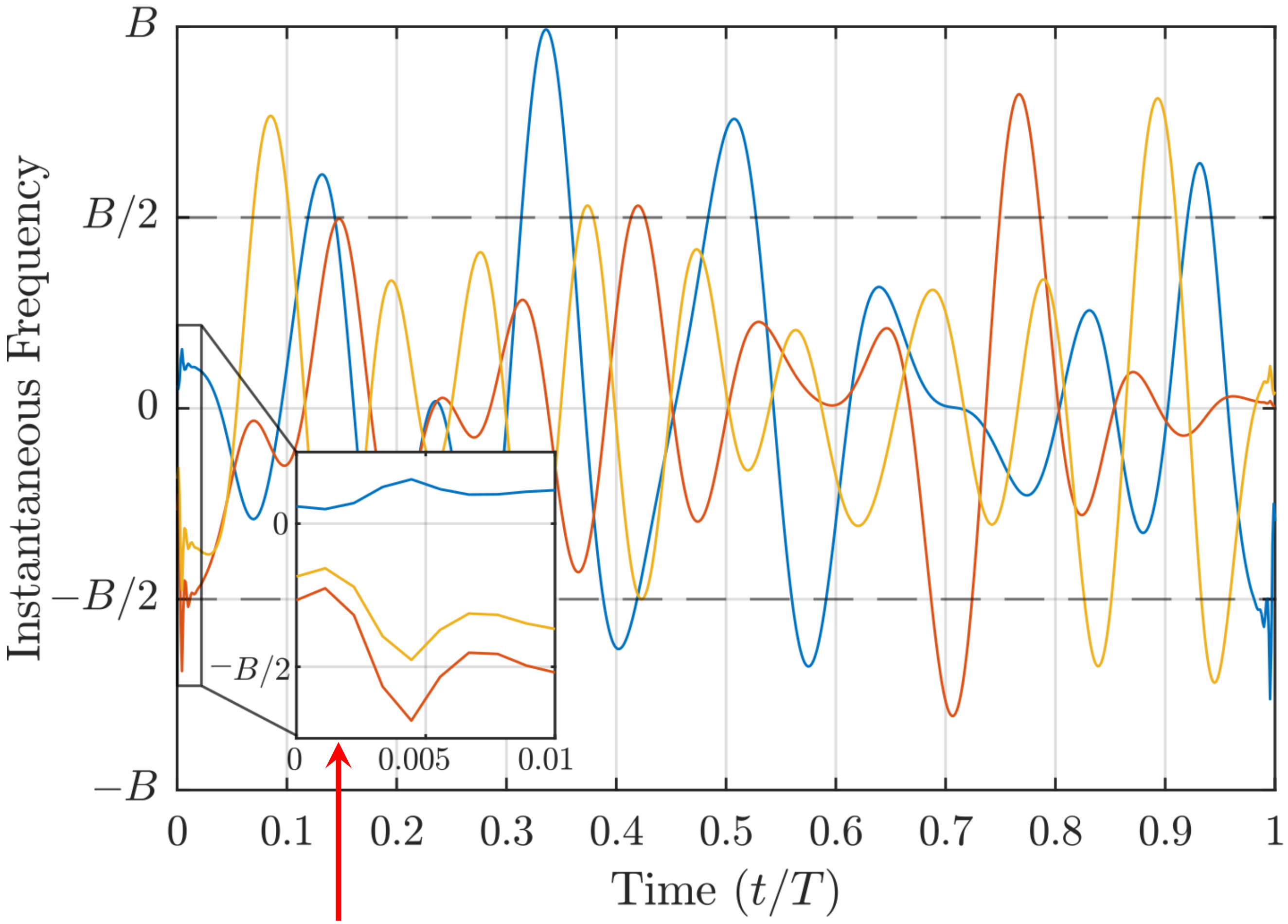
The unique basis function and degrees of correlation at the pulse edges appear to compensate for the extended spectrum of the rapid rising and falling edges

Demo: Some example StoWGe Waveforms

- Instantaneous frequency obtained via element-by-element difference for 3 unique StoWGe waveforms (generated from same **B** and **μ**)

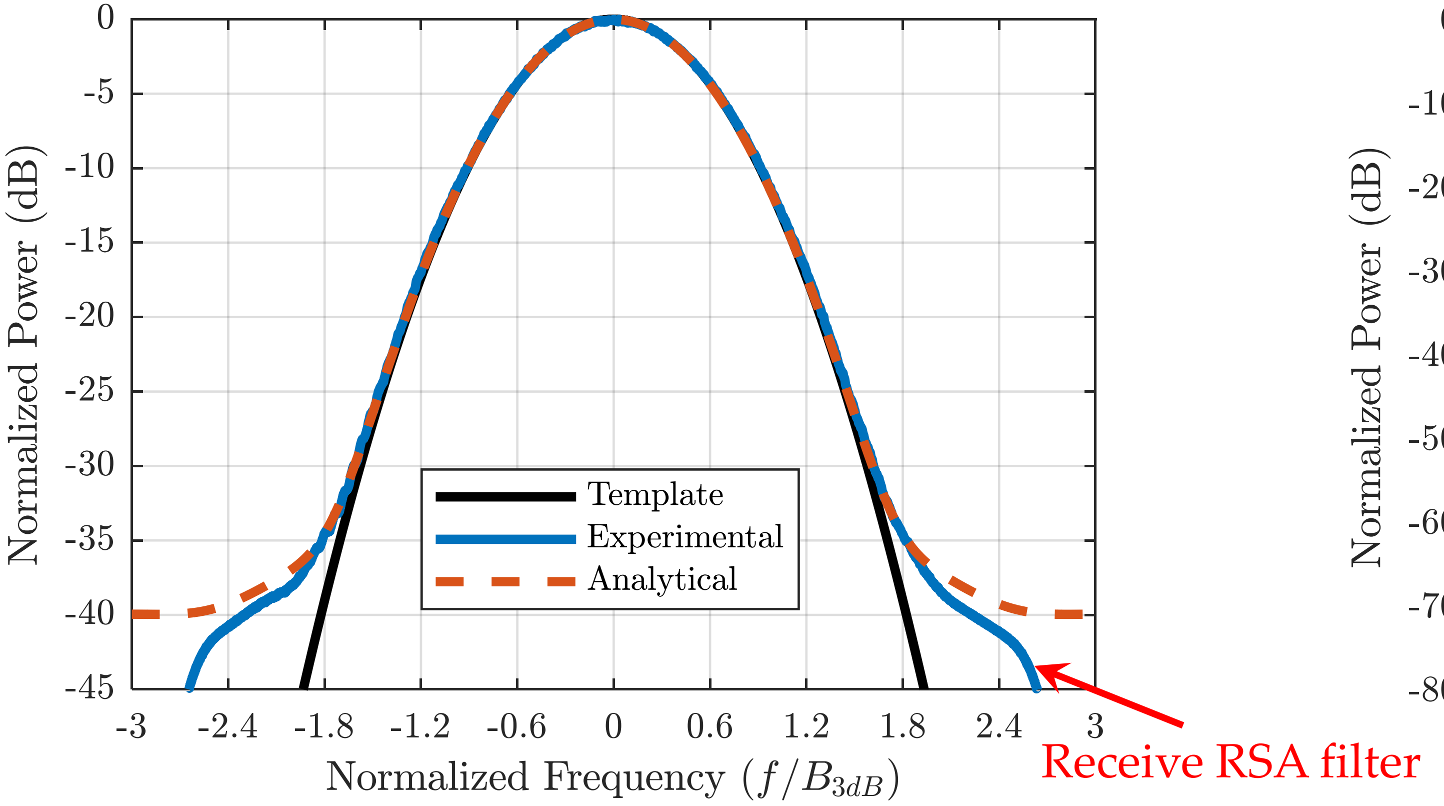
- In general, the waveforms exhibit smooth frequency functions that sometimes exceed the 3-dB bandwidth at $\pm B\backslash 2$

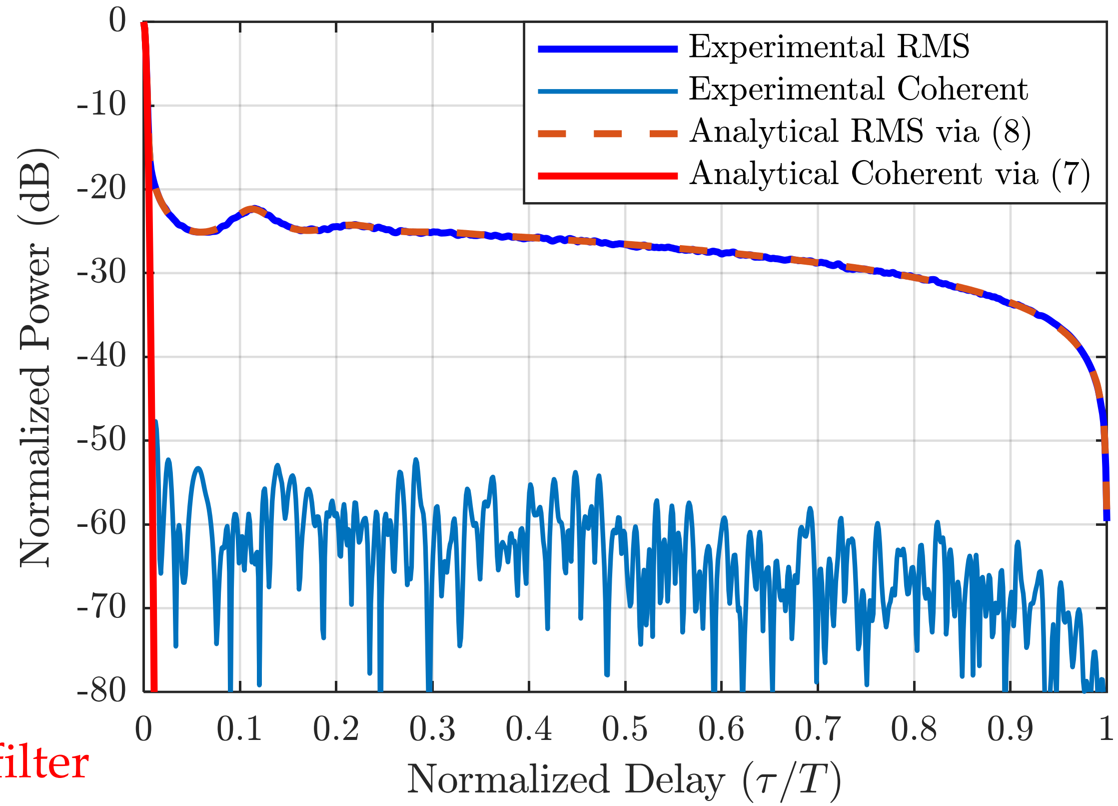
- At the pulse edges, more rapid frequency changes appear to compensate for the short rise and fall times



rapid frequency changes

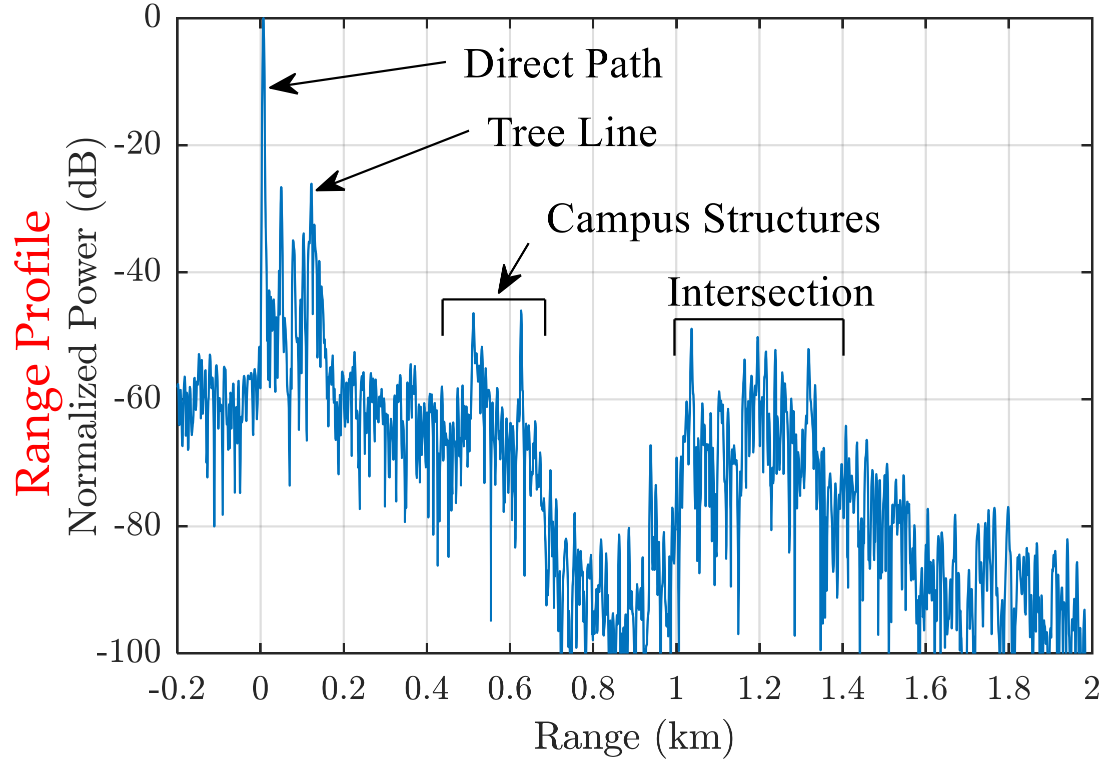
Demo: Loopback Experimental Results

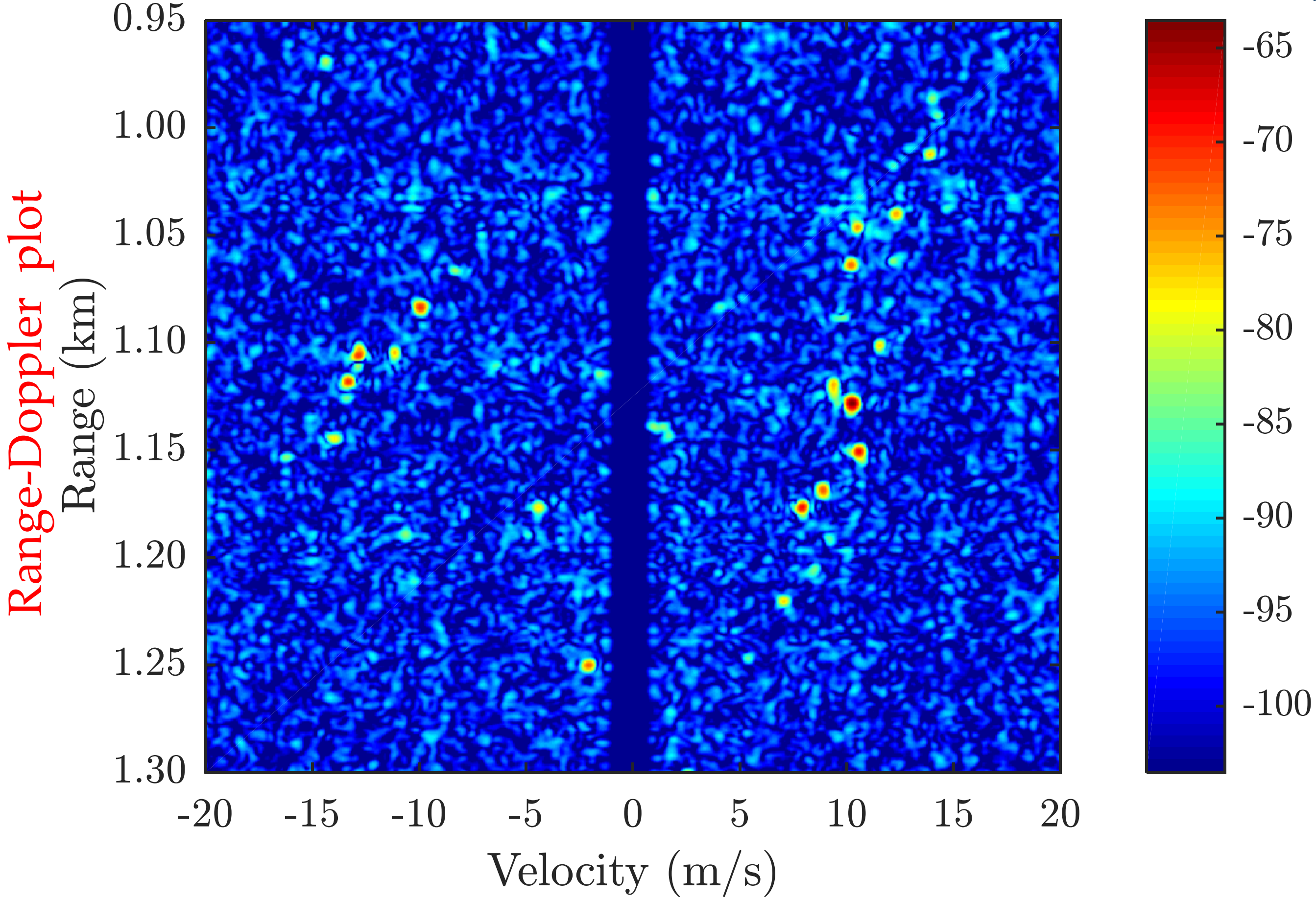




- The same 3000 waveforms were up converted to 3.55 GHz and implemented in hardware
- Loopback setup included amplifiers and attenuators to emulate a transmit/receive chain
- Loopback results are almost indistinguishable from the simulated results

Demo: Open Air Results





- The waveforms were transmitted in free-space on the University of Kansas campus
- The area of interest is an intersection at an approx. range of 1.1 km

- Projection-based clutter cancellation and a Hamming Doppler window were applied

- Many moving targets are easily identifiable

Conclusions

- The optimization of individual, random FM waveforms can be computationally expensive
- By modeling FM noise waveforms as a stochastic process, the optimization of individual waveforms can be replaced by designing a generating function.
- Thus, through the appropriate parameterization and optimization of this generating function, unique FM noise waveforms possessing desired spectral characteristics can be produced with negligible cost.