Alternative “Bases” for Gradient-Based Optimization of Parameterized FM Radar Waveforms

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Background

• Spectrally-shaped random FM (RFM) waveforms:
  – Are amendable to high power transmitters (constant amplitude & continuous phase)
  – Can be designed to achieve low sidelobes
  – Have been experimentally demonstrated for a variety of applications

• Parameterized version of RFM readily permit gradient-based optimization
  – Extensible to joint optimization across waveform sets
  – Preserve transmitter-amenable attributes (under certain conditions)
Motivation

• Here we consider other parameterized structures to examine the impact of choice in “basis”
  – Due to requirement for over-sampling (relative to 3-dB bandwidth) to ensure spectral containment, these structures are not true bases => hence, quasi-bases

• We specifically examine 2nd-order polyphase-coded FM (PCFM) and Fourier quasi-bases, along with 1st-order PCFM as a baseline [1]
  – These are chosen because we expect the ensuing RFM waveforms generated from each to possess different attributes

Parameterized FM Waveforms
Discretized FM Waveform Structure

• Define an arbitrary FM waveform (hence continuous) as

\[ s(t; x) = \exp(j\phi(t; x)) \]

in which [2]

\[ \phi(t; x) = \sum_{n=1}^{N} \alpha_n b_n(t) \]

\[ x = [\alpha_1 \alpha_2 \cdots \alpha_N]^T \]

permits a parameterized combining of continuous phase functions \( b_n(t) \)

• Properly discretizing these phase functions to ensure sufficient spectral containment then yields

\[ s = \exp(jBx) \]

\[ B = [b_1 b_2 \cdots b_N] \quad \rightarrow \quad M\times N \text{ for } M > N \]

1st-Order PCFM

- The 1st-order PCFM implementation conforms to this structure via

\[
\phi_1(t; x) = \int_0^t g(\tau) * \left[ \sum_{n=1}^{N} \alpha_n \delta(t - (n - 1)T_1) \right] d\tau
\]

The shaping filter (generally rectangular) is used to modulate the impulse train with separation \( T_1 \). So that the corresponding phase functions are

\[
b_n(t) = \int_0^t g(\tau - (n-1)T_1) d\tau = \begin{cases} 
0, & 0 \leq t < (n-1)T_1 \\
(t - (n - 1)T_1)/T_1, & (n-1)T_1 \leq t < nT_1 \\
1, & nT_1 \leq t \leq NT_1
\end{cases}
\]
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\[ b_n(t) = \int_0^t g(\tau - (n-1)T_1) \, d\tau = \begin{cases} 
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(t-(n-1)T_1)/T_1, & (n-1)T_1 \leq t < nT_1 \\
1, & nT_1 \leq t \leq NT_1 
\end{cases} \]
2nd-Order PCFM

• The 2nd-order PCFM implementation involves an additional integration so that

\[ \phi_2(t; x) = \bar{\omega}_2 t + \int_0^t \int_0^t \sum_{n=1}^{N} \alpha_n \ g(t - (n-1)T_1) \ d\tau' \ d\tau \]

results in the phase functions

\[ b_n(t) = \int_0^t \int_0^t g(\tau' - (n-1)T_1) \ d\tau' \ d\tau \]

\[ = \begin{cases} 
0, & 0 \leq t < (n-1)T_1 \\
\frac{t^2}{2T_1^2} - (n-1)t/T_1 + (n-1)^2/2, & (n-1)T_1 \leq t < nT_1 \\
t/T_1 + 1/2 - n & nT_1 \leq t \leq NT_1
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results in the phase functions

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\begin{cases} 
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\frac{t^2}{2T_1^2} - \frac{(n-1)t}{T_1} + \frac{(n-1)^2}{2}, & (n-1)T_1 \leq t < nT_1 \\
\frac{t}{T_1} + 1/2 - n, & nT_1 \leq t \leq NT_1
\end{cases}
\]
2\textsuperscript{nd}-Order PCFM

- The 2\textsuperscript{nd}-order PCFM implementation involves an additional integration so that

\[
\phi_2(t;\omega) = \bar{\omega}_2 t + \int_0^t \int_0^{N_n} \alpha_n g(t - (n-1)T_1) d\tau' d\tau
\]

results in the phase functions plus a frequency offset (becomes another column in B)

\[
b_n(t) = \int_0^t \int_0^{\tau} g(\tau' - (n-1)T_1) d\tau' d\tau
\]

\[
= \begin{cases} 
0, & t < (n-1)T_1 \\
\frac{t^2}{2}T_1^2 - (n-1)t/T_1 + (n-1)^2/2, & (n-1)T_1 \leq t < nT_1 \\
t/T_1 + 1/2 - n, & nT_1 \leq t \leq NT_1
\end{cases}
\]

linear ramp

quadratic

0 \leq t < (n-1)T_1

(n-1)T_1 \leq t < nT_1

nT_1 \leq t \leq NT_1
Finally, the **Fourier** implementation, which is what **CE-OFDM** employs, takes the form

\[
\phi_F (t; \mathbf{x}_F) = \Re \left\{ \sum_{n=1}^{N} \alpha_n \exp(j \omega_n t) \right\}
\]

\[
= \sum_{n=1}^{N} \Re \{ \alpha_n \} \cos(\omega_n t) + \Im \{ \alpha_n \} \sin(\omega_n t)
\]

\[
= \sum_{n=1}^{N} \alpha_{r,n} \cos(\omega_n t) + \alpha_{i,n} \sin(\omega_n t),
\]

so that the **phase functions** are clearly \( \cos(\omega_n t) \) and \( \sin(\omega_n t) \).

Because this implementation possesses **double the phase functions**, we shall set \( N \) to be half that of the other two implementations for fair comparison.
Gradient-Based Optimization
Cost Function

- **Discretizing waveform** \( s(t) \) as \( s \), the ensuing **autocorrelation** can be written as

\[
    r = A^H \left[ (A\bar{s}) \odot (A\bar{s})^* \right]
\]

for \( \bar{s} \) the zero-padded version, \( A \) the DFT, and \( A^H \) the inverse DFT.

- **By selecting the mainlobe** and **sidelobe** regions using \( w_{ML} \) and \( w_{SL} \), the **generalized integrated sidelobe level** (GISL) cost function is

\[
    J_p = \frac{\| w_{SL} \odot r \|_p^2}{\| w_{ML} \odot r \|_p^2}
\]

where \( p = 2 \) is the ISL metric and \( p \to \infty \) is the PSL metric.
Gradient-Based Optimization

- It was shown in [2] that the gradient of the GISL cost function is

\[
\nabla_x J_p = 4 J_p \bar{B}^T \mathcal{S} \left\{ \bar{s}^* \odot \left( A^H \left[ A \left( \frac{w_{SL}}{w_{SL}^T \lVert r \rVert^p} - \frac{w_{ML}}{w_{ML}^T \lVert r \rVert^p} \right) \odot \lVert r \rVert^{(p-2)} \odot r \right) \odot (A \bar{s}) \right) \right\}
\]

with \( \bar{B} \) having columns of zeros appended to agree with \( \bar{s} \).

- Gradient-descent update is then performed as

\[
x_i = x_{i-1} + \mu_i q_i \quad \text{for} \quad q_i = \begin{cases} -\nabla_x J_p(x_{i-1}) & \text{when } i = 0 \\ -\nabla_x J_p(x_{i-1}) + \beta q_{i-1} & \text{otherwise} \end{cases}
\]

Backtracking

Optimization Results
Initialization Mappings

• As noted in [2], the GISL cost function does not inherently address spectral containment.

• While a spectral constraint could be considered, we want to focus solely on the behaviors associated with these quasi-bases.

• Instead, we therefore use an initialization already possessing acceptable containment and rely on gradient-descent to stay in this local region.

• Hence, we initialize with PRO-FM waveforms [3] and then use least-squares (LS) in phase to map into the parameterization for each quasi-basis as

\[
\phi_{\text{PRO},k} \xrightarrow{\text{LS} \{ B_{1st} \}} x_{1st,k} \xrightarrow{\text{LS} \{ B_{2nd} \}} x_{2nd,k} \xrightarrow{\text{LS} \{ B_{F} \}} x_{F,k}
\]


Convergence Behavior Comparison

- **Fourier convergence**
  - drops ~9 dB in $10^3$ iterations, then flatlines

- **1\textsuperscript{st} order convergence**
  - drops ~6 dB in $10^3$ iterations, then gradually reduces another 4 dB until $10^5$

- **2\textsuperscript{nd} order convergence**
  - Takes nearly $10^4$ iterations before meaningful convergence takes hold, with only about 2 dB per decade of iterations thereafter
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[Graph showing convergence behavior]
Optimized Autocorrelation Comparison

- Initializing with $K = 3000$ unique PRO-FM waveforms having $TB = 200$ and $4 \times$ oversampling
- LS mapping into each quasi-basis, then 9000 gradient-descent iterations
- RMS (per-pulse average) and coherent slow-time combining yield similar outcomes despite differences in convergence
  - RMS: Fourier ~1 dB below 1\textsuperscript{st} order, which is ~4 dB below 2\textsuperscript{nd} order
• Close-up on the mainlobe shows **shoulder lobe** roll-off due to initial super-Gaussian shaping for PRO-FM

• **Fourier** case has largely **suppressed** the initial shoulder lobes, while 1\textsuperscript{st} and 2\textsuperscript{nd} order **preserve** them to different degrees

• Local minima associated with this effect may **explain differing convergence rates**
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• **Fourier** case has largely **suppressed** the initial shoulder lobes, while **1st** and **2nd** order **preserve** them to different degrees

• Local minima associated with this effect may **explain differing convergence rates**

• However, this attribute is also associated with better spectral containment
Power Spectral Density (PSD) Comparison

- Previous behavior is better understood after examining the resulting PSD for each case
  - Depicted for single waveform, RMS average, and PRO-FM initialization (average)

1\textsuperscript{st} order quasi-basis

2\textsuperscript{nd} order quasi-basis

Fourier quasi-basis
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best preserves good spectral containment
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Fourier quasi-basis

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tends toward Gaussian PSD, hence lower sidelobes
Power Spectral Density (PSD) Comparison

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  - Depicted for single waveform, RMS average, and PRO-FM initialization (average)

1st order quasi-basis: in between

2nd order quasi-basis: best preserves good spectral containment

Fourier quasi-basis: tends toward Gaussian PSD, hence lower sidelobes
**Instantaneous Phase** $\phi(t)$
- All are **continuous** since they are FM
- 1<sup>st</sup> order is **piece-wise linear**
- 2<sup>nd</sup> order could realize LFM perfectly
- Fourier: infinitely differentiable => **smooth phase**

**Instantaneous Frequency** $d\phi(t)/dt$
- 1<sup>st</sup> order now has **discontinuities**
- 2<sup>nd</sup> order is piece-wise linear
- Fourier does show higher freq. excursions that conform to more gradual roll-off
Experimental Results
Open-Air Measurements

• **Test Setup**
  – On roof of Nichols Hall on University of Kansas campus
  – Illuminating intersection of 23rd and Iowa streets
  – Interleaved waveforms to provide same scene for comparison

• **Test Parameters**
  – \( TB: 200 \)
  – \( PRI: 22 \mu s \)
  – \( CPI: 198 \text{ ms} \)
  – Center frequency: 3.45 GHz

• **Processing**
  – Simple projection-based clutter cancellation (since stationary platform)
  – Applied Taylor window (-35 dB) in Doppler
Range-Doppler Responses

- Different quasi-basis structures → Still physically realizable waveforms

Not surprising, but good to confirm

1st order quasi-basis

2nd order quasi-basis

Fourier quasi-basis
Conclusions

- Due to the need for **oversampling**, parameterized FM waveforms possess **quasi-bases** (not full bases) having different attributes
  - Suggests different selections for different uses (e.g. required spectral containment)
  - Almost certain to be other useful quasi-bases

- Of the 3 examined here, **Fourier** tends toward **lower sidelobes** at the cost of spectral spreading, and **vice-versa** for **2nd order**, with **1st order** in-between

- Other factors that can play a role for the particular waveforms generated are:
  - Nature of random initializations
  - Manner of imposing spectral containment
  - Alternative optimization methods/implementations
Questions?