1. (c) It is enough to show that $f$ is convex on $E^3$ (also convex). By Theorem 2 on p.197, if $F$ is positive semi-definite throughout $E^3$, then $F$ is constant throughout $E^3$ and positive definite (stronger than pos. semi-def.). So, $f$ is convex and since $\nabla f(1,2,1,2,3,4) = (0,0,0)$, then the point is a global minimum.

2. I believe we only need $f \in C^1$ (not $f \in C^2$). Let $f \in C^1$ on $\Omega \subset E^3 \times$, $x^*$ interior point of $\Omega$.

Now if $\nabla f(x^*) = 0$, we have $f(y) \geq f(x^*)$, that is, $x^*$ is a relative min point of $f$.

3. Suppose $\exists x_k \rightarrow x$ with $x_k, x \in E^n$ and $\forall y \rightarrow y$ with $y_k \in A(x_k)$, i.e. $y_k^T x_k = b$, but $y_k \notin A(x)$, i.e. $y_k^T x > b$. Then $y_k^T x - y_k^T x_k > 0 \Rightarrow \exists \delta > 0$ s.t. $y_k^T x - y_k^T x_k = \delta \forall k$.

5. Take $\varepsilon_1 = \frac{\varepsilon_2}{\sqrt{2}}$. Let $\varepsilon_2 = \frac{\max_{k \geq N_2}}{y_k^T x_k}$. Then:

$\frac{1}{\varepsilon_2} > \frac{1}{\varepsilon_1} \cdot \frac{1}{\|y_k - x_k\|} \geq \frac{1}{\varepsilon_1 \cdot \|y_k - x_k\|} = \frac{1}{\varepsilon_1 \cdot \|y_k - x_k\|}$

$\frac{1}{\varepsilon_2} > \frac{1}{\varepsilon_1} \cdot \|y_k - x_k\| \geq \frac{1}{\varepsilon_1} \cdot (y_k - x_k)^T x_k \geq (y_k - x_k)^T x_k$

$\frac{1}{\varepsilon_2} > \frac{1}{\varepsilon_1} \cdot (y_k - x_k)^T x_k = y_k^T x - y_k^T x_k + (y_k - x_k)^T x_k - y_k^T x_k = y_k^T x - y_k^T x_k - y_k^T x_k = y_k^T x - y_k^T x_k$ CONTRADICTION

Hence we must have $y \in A(x)$, so $A$ is closed.
3. \[ f(a) = \sum_{j=1}^{m} \left[ h_j - (a_0 - a_1 j - 0.5 a_2 j^2) \right]^2. \]

This is similar to problem 3 of HW2, but generalized slightly: using notation from that problem, here we have:

\[ f(a) = \sum_{k=1}^{m} \left[ g(t_k) - (z_0 a_0 + z_1 a_1 t_k + z_2 a_2 t_k^2) \right]^2 \]

where \( m = 12 \), \( g(t_k) = h_k \), \( z_0 = 1 \), \( z_1 = -1 \) and \( z_2 = -\frac{1}{2} \).

The result is still that \( f(a) = a^T Q a - 2 b^T a + c \), but the expressions for \( g(t_k) \) are different from those used in HW2.

In particular, \( g(t_k) = \sum_{k=1}^{m} (t_k)^2 \) with \( t_k \) running \( 0 \) to \( 2 \).

a) To show convexity, it is sufficient to show that \( F \) is positive semi-definite, and from problem 5 of HW2,

\[ F = 2Q \] for a quadratic function of this form.

Letting \( T = \sum_{k=1}^{m} t_k \), we have

\[
\begin{pmatrix}
2T(0) & -2T(1) & -T(2) \\
-2T(0) & 2T(1) & T(2) \\
-T(2) & T(2) & \frac{1}{2} T(4)
\end{pmatrix}
\]

With the given \( t_k \) values,

\[ F = \begin{pmatrix}
24 & -120 & -432.5 \\
-120 & 865 & 3487.5 \\
-432.5 & 3487.5 & 14917
\end{pmatrix} \]
3a) continued Now check determinant of each principal sub-matrix:

\[ |124| = 24 \geq 0 \, \checkmark \]
\[ \begin{vmatrix} 24 & -120 \\ -120 & 865 \end{vmatrix} = 6340 \geq 0 \, \checkmark \]
\[ |11| = 2.8 \times 10^{10} \geq 0 \, \checkmark \]

b) Since \( f \) is convex, FOC is become sufficient. Again from problem 5 of HW2, we have:

\[ \nabla f(a) = 2a^T \mathbf{Q} - 2b^T \]
so, solve \( 2a^T \mathbf{Q} = 2b^T \Rightarrow a^T = b^T \mathbf{Q}^{-1} \)

\[ b_j = \frac{12}{k!} \sum_{k=1}^{j} h_k \Rightarrow b_0 = \sum_k h_k = 626 \]

\[ b_1 = -\sum_k h_k + k = -1802 \quad b_2 = -\frac{1}{2} \sum_k h_k + k^2 = -4572 \]

Solution is then \( a_0 = 100.3 \quad a_1 = 8.44 \quad a_2 = 0.277 \)

4. \( x_k = \frac{1}{k!} \) convergence clearly \( x_k \to 0 \) and \( x_k \geq 0 \) \( \forall k \).

\[
\frac{|x_{k+1} - x^*|}{|x_k - x^*|} = \frac{x_{k+1} - (k+1)!}{x_k - k!} \frac{k!}{(k+1)!} = \frac{1}{k+1} \to 0 \quad \text{as } k \to \infty.
\]

\[ \frac{|x_{k+1} - x^*|^2}{|x_k - x^*|^2} = \frac{k!}{(k+1)!} \frac{(k+1)!}{k!} = \frac{k!}{k+1} \to \infty \quad \text{not quadratic (order 2) conv.} \]

\[ \text{super-linear convergence.} \]
6. Assume \( y_{k-1} \) is at distance \( \left( \frac{F_{N-k+1}}{F_{N-k+2}} \right) w_{k-1} \) from \( a_{k-1} \).

\[
\begin{align*}
& a_{k-1} = a_k \quad x_k = y_k \quad y_{k-1} = b_k \quad b_{k-1} \hfill \\
\hline
& W = \left( \frac{F_{N-k+1}}{F_{N-k+2}} \right) w_{k-1} \quad \text{by algorithm choice of } x_k.
\end{align*}
\]

Assume \( f(x_k) < f(y_{k-1}) \) \( \Rightarrow \) \( a_k = a_{k-1}, \ b_k = y_{k-1}, \ y_k = x_k \).

Let \( w = \text{width from } y_k \text{ to } a_k \). Show \( w = \left( \frac{F_{N-k}}{F_{N-k+1}} \right) w_{k-1} \).

\[
\begin{align*}
W &= W_{k-1} - \left( \frac{F_{N-k+1}}{F_{N-k+2}} \right) W_{k-1} = \left( 1 - \frac{F_{N-k+1}}{F_{N-k+2}} \right) W_{k-1} = \frac{F_{N-k}}{F_{N-k+2}} W_{k-1} \\
\text{Also, } W_k &= \left( \frac{F_{N-k+1}}{F_{N-k+2}} \right) W_{k-1} \quad \Rightarrow \quad W_{k-1} = \left( \frac{F_{N-k+2}}{F_{N-k+1}} \right) W_k. \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow w &= \left( \frac{F_{N-k}}{F_{N-k+2}} \right) \left( \frac{F_{N-k+2}}{F_{N-k+1}} \right) W_k = \left( \frac{F_{N-k}}{F_{N-k+1}} \right) W_k \quad \text{Done.}
\end{align*}
\]

This is one of four cases: \( y_{k-1} \) could be at given distance from either \( a_{k-1} \) or \( b_{k-1} \), \( f(x_k) < \) or \( > f(y_{k-1}) \).

Same result for other 3 cases.

Since true for \( k = 2 \) by choice of \( x_i = y_i \), induction is complete.
7. Assume \( y_{k-1} \) is at distance \( \left( \frac{1}{\epsilon_1} \right) w_{k-1} \) from \( a_k \).

\[
\begin{align*}
& a_k = x_k \\
& x_k = y_k \\
& y_k = -b_k \\
& w = \frac{1}{\epsilon_1} w_{k-1} \quad \text{by algorithm choice of } x_k
\end{align*}
\]

Assume \( f(x_k) < f(y_{k-1}) \Rightarrow a_k = a_{k-1}, \ b_k = y_{k-1}, \ y_k = x_k \)

\( w \) = width from \( y_k \) to \( a_k \)

Show that \( w = \left( \frac{1}{\epsilon_1} \right) w_k \). Note that \( w_k = b_k - a_k = \left( \frac{1}{\epsilon_1} \right) w_{k-1} \).

So we can show that \( w = \left( \frac{1}{\epsilon_1} \right)^2 w_{k-1} \). \quad \text{def of } \frac{1}{\epsilon_1} \in \mathbb{R}

\[
\begin{align*}
w &= w_{k-1} - \left( \frac{1}{\epsilon_1} \right) w_{k-1} \\
&= \left( 1 - \frac{1}{\epsilon_1} \right) w_{k-1} = \left( 1 + \epsilon_2 \right) w_{k-1} = \left( \frac{1}{1 - \epsilon_2} \right) w_{k-1}
\end{align*}
\]

So \( w = \left( \frac{1}{\epsilon_1} \right)^2 w_{k-1} = \left( \frac{1}{\epsilon_1} \right)^2 w_{k-1} \). \quad \text{Done.}

Again, this is one of 4 cases, with same result for other 3.

True for \( k = 2 \) by choice of \( x_1 = y_1 \), so induction is complete.
Problem 8.

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Problem 9.

Solution to min exp(1.2x) - 6x from FONC and SOSC: x = [ln(5)]/1.2 = 1.3412

Fibonacci Search

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