Reading: Lecture Notes and Luenberger/Ye: Sections 7.5 - 7.9 (skip 7.6), Appendix A.3 - A.6, Sections 8.1 - 8.5

1. Luenberger/Ye, Problem 3, part (c) only, p. 213. FONC, SOSC.


3. A falling object on a nameless planet has been observed to have approximately the following heights $h_j$ at (mostly) one-second intervals $t_j$:

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_j$</td>
<td>0.0</td>
<td>0.5</td>
<td>1.5</td>
<td>2.5</td>
<td>3.5</td>
<td>4.5</td>
<td>5.5</td>
<td>6.5</td>
<td>7.5</td>
<td>8.5</td>
<td>9.5</td>
<td>10.0</td>
</tr>
<tr>
<td>$h_j$</td>
<td>100</td>
<td>97</td>
<td>87</td>
<td>78</td>
<td>67</td>
<td>58</td>
<td>49</td>
<td>39</td>
<td>29</td>
<td>17</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

According to the laws of physics on this planet, the height of the object at any time should be given by the formula:

$$h_j = a_0 - a_1 t_j - 0.5 a_2 t_j^2$$

where $a_0$ is the initial height, $a_1$ is the initial velocity, and $a_2$ is the acceleration due to gravity. But since the observations were not made exactly, there exists no choice of $a_0$, $a_1$, and $a_2$ that will cause all of the data to fit this formula exactly. Instead, we wish to estimate these three values by choosing them so as to minimize the "sum of squares":

$$\sum_{j=1}^{n} [h_j - (a_0 - a_1 t_j - 0.5 a_2 t_j^2)]^2$$

where $n$ is the number of observations. This expression is a measure of the error between the ideal formula and the observed behavior.

a. Show that the objective function for this problem is convex. Note that you can make use of the results of Problems 3 and 5 of Assignment 2.

b. Find optimal values for $a_0$, $a_1$, and $a_2$. Are these values unique? Justify your answer.

4. Does the sequence $x_k = 1/k!$ converge superlinearly? Does it converge with order 2.0?

5. Luenberger, Problem 10, p. 214. Closed Mappings. If your answer is yes, prove it. If your answer is no, prove it or give an example.

6. Refer for this problem to the lecture notes on the Fibonacci search. At step $k$, it was claimed that the point $y_{k-1}$ was guaranteed to be at a distance $\left(\frac{F_{N-k+1}}{F_{N-k+2}}\right) w_{k-1}$ from one of the endpoints $a_{k-1}$ or $b_{k-1}$ of the $(k-1)^{th}$ uncertainty interval. This claim allows us to choose point $x_k$ as the point that is at distance $\left(\frac{F_{N-k+1}}{F_{N-k+2}}\right) w_{k-1}$ from the other endpoint of the $(k-1)^{th}$ uncertainty interval. This claim was obviously true for $k = 2$ by the choice of $x_1 = y_1$. 

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Use an induction argument to show that this claim is true for all $k$ up to $k = N$. That is, assume that it is true at step $k - 1$ and then show that it must be true at step $k$. Since you have already shown that it is true for $k = 2$, this will show that it is true for all $k$ up to $k = N$.

7. Repeat the previous problem, replacing "Fibonacci" with "Golden Section", $\left(\frac{F_{N-k+1}}{F_{N-k+2}}\right) w_{k-1}$ with $\left(\frac{1}{\tau_1}\right) w_{k-1}$, and "$k = N$" with "$k = \infty$".

8. Tabulate the ratio of the final uncertainty width to the initial uncertainty width for Fibonacci search and Golden Section search (stopped after $N$ points have been evaluated) for the following values of $N$: 2, 3, 4, 5, 10, 15, 20, 30, 40, 50, 100.

9. Perform (a) Fibonacci search and (b) Golden Section search on the function $f(x) = e^{1.2x} - 6x$ in the interval $[1, 2]$ evaluating $N = 6$ points in each case. Your answer in each case will be the final uncertainty interval. Compare the final uncertainty widths; do they match your expectations?