Equivalent-circuit formulas for metal grid reflectors at a dielectric boundary

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This paper points out a number of errors that have appeared in the literature concerning transmission-line models for metal grid reflectors (strip gratings and meshes) especially in regard to the design of laser mirrors and filter elements for use at submillimeter wavelengths. General formulas are given for the transmittance of lossy grids and for the equivalent circuit impedances to be used in these formulas for strip gratings and meshes at a plane boundary between two lossless dielectrics. The results apply for normal incidence and for wavelengths in both dielectrics greater than the grid period. Limitations of the transmission-line models for meshes at dielectric boundaries are discussed.

One-dimensional strip gratings and 2-D metal meshes, as shown in Fig. 1, are commonly used as partial reflectors and filter elements at submillimeter wavelengths. For wavelengths longer than the grid period \( g \) (Fig. 1) such structures reflect and transmit in zero order only, and their reflectance and transmittance can be calculated approximately by means of equivalent circuit transmission-line models. The model for strip gratings was first published by Marcuvitz,\(^1\) and this was subsequently extended semiempirically by Ulrich\(^2\) to treat meshes. Ulrich’s paper,\(^2\) which deals with grids in free space (or in a uniform dielectric), contains one important error\(^3\) which has been reproduced in a number of other references.\(^4-8\) Recent papers\(^9-13\) dealing with the extension of these models to allow for the effect of a thick dielectric substrate on a grid contain additional errors.\(^14\) We summarize here a general set of equivalent circuit transmittance formulas for normal incidence on grids at an arbitrary dielectric boundary. Sufficient detail is given to allow the present results to be checked and the errors indicated above\(^3,8,14\) to be corrected. These results apply only for thin grids (thickness \( t \ll \lambda \)) except in one special case mentioned below [Eqs. (13) and (14)].

We consider grids situated on plane boundaries between dielectrics of refractive indices \( n_1 \) and \( n_2 \) and assume, without loss of generality, that \( n_2 \geq n_1 \). For the four types of grid shown in Fig. 1 the equivalent transmission-line circuits are shown in Fig. 2.\(^2\) A lossless dielectric of refractive index \( n_i \) is represented by a transmission line of characteristic impedance \( Z_i/n_i \), where \( Z_i \) is the impedance of free space. We consider first the case of lossless grids \((R_0 = 0 \text{ in Fig. 2})\). For a general reactance \( X \) shunting the lines at their junction it can be shown readily from transmission-line theory that the power transmittance \( T \) through the shunted junction is given by

\[
T(n_1,n_2) = \frac{4n_1n_2(X/Z_s)^2}{1 + (n_2 + n_1)^2(X/Z_s)^2}.
\]  

(1)

It follows from Eq. (1) that, for the special case of a grid whose reactance \( X \) is independent of \( n_1 \) and \( n_2 \), the general transmittance \( T(n_1,n_2) \) is related to the free-space transmittance \( T(1,1) \) by

\[
T(n_1,n_2) = \frac{n_1n_2T(1,1)}{1 + T(1,1)[(n_1 + n_2)^2/4 - 1]}.
\]  

(2)

This result, or its equivalent, has been widely reported.\(^5,7,9,10,15\) It should be stressed that it is true only for grids whose equivalent circuits contain no capacitors, such as inductive strip gratings or wire grids, because capacitance in general is a function of \( n_1 \) and \( n_2 \).

For an inductive strip grating [Fig. 1(a)] the equivalent circuit is an inductor [Fig. 2(a)] of reactance \( X_l \) given by\(^1,2\)

\[
\frac{X_l}{Z_i} = \frac{g}{\lambda} \ln \csc \frac{\pi a}{g},
\]  

(3)

where \( \lambda \) is the free-space wavelength, and the grid parameters \( 2a \) and \( g \) are as shown in Fig. 1(a). \( X_l \) is independent of \( n_1 \) and \( n_2 \), because, as mentioned above,
the transmittance of a strip grating plus the transmittance of its complement (rotated by 90° in its own plane) equals unity, as predicted by Babinet’s principle. The condition relating \( X_I \) and \( X_c \) for complementary strip gratings can then be derived from Eqs. (3) and (4) to be

\[
X_I X_c = -Z_0^2/4n^2.
\]  

Babinet’s principle does not hold if \( n_1 \neq n_2 \), but a modified version for strip gratings at a dielectric boundary has been deduced elsewhere.\(^6\)

The transmission-line model for meshes [Figs. 1(c) and 1(d)] in free space was developed semiempirically by Ulrich,\(^2\) who noted that inductive and capacitive meshes behave at long wavelengths (\( \lambda \gg g \)), like strip gratings with the same parameters (\( g \) and 2\( a \)), but have a resonance near \( \lambda = g \) (typically at 1.0 \( g \) to 1.1 \( g \)) at which inductive meshes are totally transmitting, and capacitive meshes are totally reflecting. To duplicate this behavior in the transmission-line model Ulrich added a capacitor to the equivalent circuit of an inductive strip grating and an inductor to the equivalent circuit of a capacitive strip grating to obtain the corresponding mesh equivalent circuits shown in Fig. 2. The values of these added components are chosen to give resonance at a specified wavelength\(^2\) near \( \lambda = g \). A convenient notation introduced by Ulrich\(^2\) is to work in the domain of normalized frequency \( \omega = g/\lambda \). The resonant frequency \( \omega_0 \) is then close to unity. The procedure outlined above can be easily shown to give the results presented in Table 3 of Ulrich’s paper\(^2\) provided that the definition of his \( Z_0 \) is modified as follows: for inductive meshes \( Z_0 = 2\omega_0 \ln \csc(\pi a/g) \) and for capacitive meshes \( Z_0 = (2\omega_0 \ln \csc(\pi a/g))^2 \). Apart from corrections by factors of 2 these expressions now include the extra factor \( \omega_0 \), which has been absent from all previous treatments that we are aware of. The need for this factor is discussed below.

To treat meshes on dielectric boundaries we take the capacitive parts of their equivalent circuits, derived as described above, and multiply these by \((n_1^2 + n_2^2)/2\), just as for capacitive strip gratings. The resulting mesh reactances are

\[
X_I = -\left( \frac{\omega_0}{Z_0} \ln \csc \frac{\pi a}{g} \right) \left( \frac{\omega^2 - \omega_0^2}{\omega} \right),
\]

\[
X_c = \frac{2}{n_1 + n_2} \left( 4\omega_0 \ln \csc \frac{\pi a}{g} \right) \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right).
\]

Here \( \omega_0 \) is the resonant frequency of the mesh at the dielectric interface, which is related to the resonant frequency of the same mesh in free space \( \omega_0 \) by

\[
\omega_0 = \omega_0 \sqrt{\frac{2}{n_1 + n_2}}.
\]

Equations (6–8) are new results which reduce to the corrected results of Ulrich\(^2,3\) when \( n_1 = n_2 = 1 \). In this case \( \omega_0 \) becomes \( \omega_0 \) and it is clear that this factor is needed in the first brackets of Eqs. (6) and (7) so that at low frequencies (\( \omega/\omega_0 \to 0 \)) Eq. (6) reduces to Eq. (3) and Eq. (7) reduces to Eq. (4). For \( n_1 = n_2 = n \) Eqs. (6) and (7) satisfy Eq. (5), showing that complementary...
meshes in a uniform dielectric obey Babinet’s principle in the same way as complementary strip gratings. Of course, rotation of the complementary structure through 90° is redundant in this case because of the square symmetry of meshes.

To ensure that energy can be reflected and transmitted only in zero order it is necessary that the wavelength in the two dielectrics, λ/n₁ and λ/n₂, should both be greater than the grid period g. In fact, Eqs. (1)–(8) should be used only for small values of 2a/g and for λ/g somewhat greater than n₁/n₂ ≥ n₁.1,2 Rigorous numerical calculations16,17 show that Eqs. (3) and (4) can be very inaccurate for λ/g in the neighborhood of n₂.

For meshes in free space the resonance at or near λ = g is clearly associated with the onset of diffraction of an incident plane wave into reflection and transmission orders higher than zero, which can in principle occur for λ ≤ g, or ω(= g/λ) ≥ 1. Generally, therefore, ω₀ can be assumed equal to unity as a first approximation, but small adjustments to ω₀ may sometimes lead to better agreement between the measured and calculated properties of meshes, as shown by Ulrich.2 For meshes at a dielectric interface the transmission-line model has a fundamental limitation near the resonance. Once a given mesh in free space is characterized by its equivalent circuit inductance and capacitance, it is reasonable to expect that, for the same mesh at a dielectric boundary, the inductance would remain unchanged and the capacitance would be increased by the factor16 (n₁² + n₂³)/2. Thus a free-space resonant frequency ω₀ near unity would transform for a mesh at a boundary to a resonance frequency ω₀ near √(2/(n₁² + n₂³)) as given by Eq. (8). However, it might be expected intuitively that the resonance would be near the onset of diffraction, at ω = 1/n₂, and experimental results18 show that this is the case. The origin of this contradiction probably lies in the phenomenological basis of the transmission-line models for meshes. It will be less of a problem when there is a greater body of experimental data giving the resonant frequencies of meshes on dielectric boundaries. At present, therefore, near resonance, the transmission-line model for meshes at dielectric boundaries must be used with caution.

All the results given above refer to lossless grids, for which the reflectance R is, of course, given by R = 1 – T. Ulrich2 has incorporated a loss term in his transmission-line models for meshes, which can be generalized to give the following results for a mesh at a dielectric boundary. The transmittance T, reflectance R₁₂, and absorptance A₁₂ (for incidence from medium 1 to medium 2) are then given by

\[ T = \frac{4n₁n₂[R₀/Z_₀]² + (X/Z_₂)²}{[1 + (n₂ + n₁)R₀/Z₀]² + [n₁ + n₂]²(X/Z₂)²}, \]

\[ R₁₂ = \frac{1 + (n₂ - n₁)R₀/Z_₀]² + [n₁ + n₂]²(X/Z₂)²}{[1 + (n₂ + n₁)R₀/Z₀]² + [n₁ + n₂]²(X/Z₂)²}, \]

\[ A₁₂ = \frac{4n₁R₀Z₀}{[1 + (n₂ + n₁)R₀/Z₀]² + [n₁ + n₂]²(X/Z₂)²}, \]

where now a loss resistance R₀ in series with the mesh reactance X shunts the transmission lines at their junction as shown in Figs. 2(c) and (d). For meshes in free space (n₁ = n₂ = 1) Ulrich2 finds that the above formulas give good agreement with experiment when

\[ \frac{R₀}{Z₀} = \sqrt{\frac{4\pi ε_0 c}{\lambda σ}} \],

where ε₀ is the permittivity of free space, c is the velocity of light, and σ is taken as one quarter of the bulk dc conductivity of the mesh metal. η is a form factor given by g/2a for inductive meshes and 1/(1 – 2a/g) for capacitive meshes.2 Equations (9)–(11) are equally applicable for all grids (strip gratings and meshes) for suitable values of R₀/Z₀ and form factor. Equation (9) shows that the transmittance of any grid at a dielectric boundary is the same in both directions. However, reflectance and absorptance are direction dependent for n₁ ≠ n₂, as shown by Eqs. (10) and (11). For a given grid on a dielectric boundary the absorptance is smaller, and reflectance correspondingly larger, for incidence from the medium of lower refractive index.

For n₁ = n₂ = 1 Eqs. (1)–(12) describe the transmission properties of thin grids (thickness t ≤ 10⁻⁴g, say16) in free space. For inductive meshes in free space Ulrich et al.18 give an effective strip width 2a_eff that can be used instead of 2a in these equations to allow for finite thickness t (<<2a) and refer to Marcuvitz1 for its derivation. They give a_eff as

\[ a_{eff} = a + \frac{t}{2π} \ln \left( \frac{8πa}{t} + 1 \right). \]

We have been able to derive this result from the treatment of Marcuvitz1 for thick strip gratings (pp. 285 and 264) but only for g/λ ≪ 1 and for values of a/g small enough that \( \csc(πa/g) \approx g/πa \). However, Ulrich et al.18 suggest that the result has been tested experimentally for inductive meshes, and we would expect it to be equally applicable for inductive strip gratings. The corresponding result for capacitive strip gratings in free space can be derived from Marcuvitz1 (pp. 285 and 251) to be

\[ a_{eff} = a - \frac{t}{2π} \ln \left( \frac{4π(g - 2a)}{t} + 1 \right). \]

Equation (14) is applicable for g/λ ≪ 1, t ≪ (g - 2a), and 2a/g > 0.5. As far as we know this result has been neither presented elsewhere nor tested experimentally.

Finally it should be mentioned that for strip gratings on a dielectric boundary simple formulas have recently been developed to calculate transmission properties quite accurately for all g/λ < 1/n₂.17 There is a need for similar formulas for meshes on dielectric substrates. For meshes in free space the transmission-line model may be used close to resonance with reasonable confidence thanks to the body of empirical work that has been done to determine the resonant frequency.2,7

References


3. For the results in Table 3 of Ref. 2 to be correct $Z_0$ must be taken as $2\omega_0 \ln \csc(\pi a/g)$ for inductive meshes and the reciprocal of this expression for capacitive meshes. Equation (13) is, therefore, incorrect.


8. In Ref. 4 Eq. (3) should be $Z_0 = [2\omega_0 \ln \csc(\pi a/g)]^{-1}$. In Ref. 5 Table 3.2 is correct for $Z_\parallel = \omega_0 \ln \csc(\pi a/g)$ and $Z_\perp = [4\omega_0 \ln \csc(\pi a/g)]^{-1}$, contradicting Eq. (3.31). In Eq. (3.33) $\sigma$ must be taken as one quarter of the dc bulk conductivity of the mesh metal. In Ref. 6 Eq. (50) should be $Z_0 = [2\omega_0 \ln \csc(\pi a/g)]^{-1}$. In Ref. 7 Table II-B is correct for $Z_\parallel = Z_\parallel$ and $Z_\perp = Z_\perp$ as given above, contradicting Eq. (25). In Eq. (31) $\sigma$ is one quarter of the dc bulk conductivity.


14. In Ref. 9 Eqs. (13)-(15) treat an inductive strip grating, not a mesh. The applicability of Eqs. (16) and (17) is discussed in this paper. In Ref. 10 the first of the equations on the bottom of p. 2646 is not applicable for meshes, as discussed in this paper [note the misprint in Eq. (2): the square bracket should be squared]. In Ref. 11 the curves in Fig. 1 cannot be correct because they show no shift in mesh resonance caused by the dielectric substrate [see discussion following Eqs. (6)-(8), this paper]. In Ref. 12, Fig. 1, the equations are correct for $\omega_0 = 2\pi f$. (Note that the definition of $\omega_0 $ and $\omega = 2\pi f/\lambda$ differ from those used here.) However, $\omega_0 = 2\pi f/\lambda$ is not reasonable for a mesh at a dielectric boundary [see discussion following Eqs. (6)-(8) and Ref. 12, p. 1357]. For $\omega_0 = k \times 2\pi f$, $\lambda_f$ should be $2h [(n_1^2 + n_2^2) / \csc(\pi a/g)]$. In Ref. 13 Eqs. (5)-(11) are not correct, see Ref. 16.


