Circuit Symmetry

One of the most powerful concepts in for evaluating circuits is that of symmetry. Normal humans have a conceptual understanding of symmetry, based on an aesthetic perception of structures and figures.

On the other hand, mathematicians (as they are wont to do) have defined symmetry in a very precise and unambiguous way. Using a branch of mathematics called Group Theory, first developed by the young genius Évariste Galois (1811-1832), symmetry is defined by a set of operations (a group) that leaves an object unchanged.

Initially, the symmetric “objects” under consideration by Galois were polynomial functions, but group theory can likewise be applied to evaluate the symmetry of structures.

For example, consider an ordinary equilateral triangle; we find that it is a highly symmetric object!
Q: *Obviously this is true. We don’t need a mathematician to tell us that!*

A: Yes, but **how** symmetric is it? How does the symmetry of an equilateral triangle **compare** to that of an isosceles triangle, a rectangle, or a square?

To determine its level of symmetry, let’s first label each corner as corner 1, corner 2, and corner 3.

First, we note that the triangle exhibits a plane of **reflection symmetry**:
Thus, if we “reflect” the triangle across this plane we get:

Note that although corners 1 and 3 have changed places, the triangle itself remains unchanged—that is, it has the same shape, same size, and same orientation after reflecting across the symmetric plane!

Mathematicians say that these two triangles are congruent.

Note that we can write this reflection operation as a permutation (an exchange of position) of the corners, defined as:

\[
1 \rightarrow 3 \\
2 \rightarrow 2 \\
3 \rightarrow 1
\]

**Q:** But wait! Isn’t there is more than just one plane of reflection symmetry?

**A:** Definitely! There are two more:
In addition, an equilateral triangle exhibits rotation symmetry!

Rotating the triangle 120° clockwise also results in a congruent triangle:

Likewise, rotating the triangle 120° counter-clockwise results in a congruent triangle:
Additionally, there is one more operation that will result in a congruent triangle—do nothing!

This seemingly trivial operation is known as the identity operation, and is an element of every symmetry group.

These 6 operations form the dihedral symmetry group $D_3$ which has order six (i.e., it consists of six operations). An object that remains congruent when operated on by any and all of these six operations is said to have $D_3$ symmetry.

An equilateral triangle has $D_3$ symmetry!

By applying a similar analysis to a isosceles triangle, rectangle, and square, we find that:
An isosceles trapezoid has $D_1$ symmetry, a dihedral group of order 2.

A rectangle has $D_2$ symmetry, a dihedral group of order 4.

A square has $D_4$ symmetry, a dihedral group of order 8.

Thus, a square is the most symmetric object of the four we have discussed; the isosceles trapezoid is the least.

Q: Well that’s all just fascinating—but just what the heck does this have to do with microwave circuits?!

A: Plenty! Useful circuits often display high levels of symmetry.

For example consider these $D_1$ symmetric multi-port circuits:
Or this circuit with $D_2$ symmetry:

which is congruent under these permutations:

1 → 3   1 → 2   1 → 4
2 → 4   2 → 1   2 → 3
3 → 1   3 → 4   3 → 2
4 → 2   4 → 3   4 → 1
Or this circuit with $D_4$ symmetry:

\[ \begin{align*}
\text{Port 1} & & \text{Port 2} \\
\downarrow & & \downarrow \\
50 \Omega & & 50 \Omega \\
\text{Port 3} & & \text{Port 4} \\
\downarrow & & \downarrow \\
50 \Omega & & 50 \Omega \\
\end{align*} \]

which is congruent under these permutations:

\[
\begin{align*}
1 \rightarrow 3 & & 1 \rightarrow 2 & & 1 \rightarrow 4 & & 1 \rightarrow 4 & & 1 \rightarrow 1 \\
2 \rightarrow 4 & & 2 \rightarrow 1 & & 2 \rightarrow 3 & & 2 \rightarrow 2 & & 2 \rightarrow 3 \\
3 \rightarrow 1 & & 3 \rightarrow 4 & & 3 \rightarrow 2 & & 3 \rightarrow 3 & & 3 \rightarrow 2 \\
4 \rightarrow 2 & & 4 \rightarrow 3 & & 4 \rightarrow 1 & & 4 \rightarrow 1 & & 4 \rightarrow 4 \\
\end{align*}
\]

The importance of this can be seen when considering the scattering matrix, impedance matrix, or admittance matrix of these networks.

For example, consider again this symmetric circuit:

\[ \begin{align*}
\text{Port 1} & & \text{Port 2} \\
\downarrow & & \downarrow \\
50 \Omega & & 200 \Omega \\
\text{Port 3} & & \text{Port 4} \\
\downarrow & & \downarrow \\
200 \Omega & & 100 \Omega \\
\end{align*} \]
This four-port network has a single plane of reflection symmetry (i.e., $D_1$ symmetry), and thus is congruent under the permutation:

\[
\begin{align*}
1 & \rightarrow 2 \\
2 & \rightarrow 1 \\
3 & \rightarrow 4 \\
4 & \rightarrow 3
\end{align*}
\]

So, since (for example) $1 \rightarrow 2$, we find that for this circuit:

\[
S_{11} = S_{22} \quad Z_{11} = Z_{22} \quad Y_{11} = Y_{22}
\]

must be true!

Or, since $1 \rightarrow 2$ and $3 \rightarrow 4$ we find:

\[
S_{13} = S_{24} \quad Z_{13} = Z_{24} \quad Y_{13} = Y_{24}
\]

\[
S_{31} = S_{42} \quad Z_{31} = Z_{42} \quad Y_{31} = Y_{42}
\]

Continuing for all elements of the permutation, we find that for this symmetric circuit, the scattering matrix must have this form:

\[
S = \begin{bmatrix}
S_{11} & S_{21} & S_{13} & S_{14} \\
S_{21} & S_{11} & S_{14} & S_{13} \\
S_{31} & S_{41} & S_{33} & S_{43} \\
S_{41} & S_{31} & S_{43} & S_{33}
\end{bmatrix}
\]
and the **impedance** and **admittance** matrices would likewise have this same form.

Note there are just 8 independent elements in this matrix. If we also consider **reciprocity** (a constraint independent of symmetry) we find that \( s_{31} = s_{13} \) and \( s_{41} = s_{14} \), and the matrix reduces further to one with just 6 independent elements:

\[
S = \begin{bmatrix}
  s_{11} & s_{21} & s_{31} & s_{41} \\
  s_{21} & s_{11} & s_{41} & s_{31} \\
  s_{31} & s_{41} & s_{33} & s_{43} \\
  s_{41} & s_{31} & s_{43} & s_{33}
\end{bmatrix}
\]

Or, for circuits with this \( \mathbf{D}_1 \) symmetry:

**Q:** *Interesting. But why do we care?*
**A:** This will greatly **simplify** the analysis of this symmetric circuit, as we need to determine **only** six matrix elements!

For a circuit with \(D_2\) symmetry:

\[
\begin{bmatrix}
Z_{11} & Z_{21} & Z_{31} & Z_{41} \\
Z_{21} & Z_{11} & Z_{41} & Z_{31} \\
Z_{31} & Z_{41} & Z_{11} & Z_{21} \\
Z_{41} & Z_{31} & Z_{21} & Z_{11}
\end{bmatrix}
\]

we find that the impedance (or scattering, or admittance) matrix has the form:

Note here that there are just **four** independent values!
For a circuit with $D_4$ symmetry:

\[
\begin{bmatrix}
Y_{11} & Y_{21} & Y_{21} & Y_{41} \\
Y_{21} & Y_{11} & Y_{41} & Y_{21} \\
Y_{21} & Y_{41} & Y_{11} & Y_{21} \\
Y_{41} & Y_{21} & Y_{21} & Y_{11}
\end{bmatrix}
\]

we find that the admittance (or scattering, or impedance) matrix has the form:

Note here that there are just three independent values!

One more interesting thing (yet another one!); recall that we earlier found that a matched, lossless, reciprocal 4-port device must have a scattering matrix with one of two forms:
\[ S = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix} \]  

The “symmetric” solution

\[ S = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix} \]  

The “anti-symmetric” solution

*Compare* these to the matrix forms above. The “symmetric solution” has the *same form* as the scattering matrix of a circuit with \( D_2 \) symmetry!

\[ S = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix} \]

**Q:** Does this mean that a matched, lossless, reciprocal four-port device with the “symmetric” scattering matrix *must* exhibit \( D_2 \) symmetry?

**A:** That’s *exactly* what it means!
Not only can we determine from the form of the scattering matrix whether a particular design is possible (e.g., a matched, lossless, reciprocal 3-port device is impossible), we can also determine the general structure of a possible solutions (e.g., the circuit must have $D_2$ symmetry).

Likewise, the “anti-symmetric” matched, lossless, reciprocal four-port network must exhibit $D_1$ symmetry!

$$S = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

We’ll see just what these symmetric, matched, lossless, reciprocal four-port circuits actually are later in the course!