11.1 Two-Port Power Gains

Reading Assignment: pp. 536-542

Specifying the gain of an amplifier is a bit more ambiguous than you may think. The problem is that there are so many ways to define power!

HO: THE POWERS THAT BE

HO: POWER GAIN

Q: The absorbed powers typically are less than the available powers. Isn’t there some way to better take advantage of the power available?

A: We know there is! The answer is matching networks.

HO: TURNING A GAIN ELEMENT INTO AN AMPLIFIER
The Powers that Be

To begin our discussion of amplifiers, we first must define and derive a number of quantities that describe the rate of energy flow (i.e., power).

Consider a source and a load that are connected together by some gain element:

\[ Z_g \]

\[ V_g \]

\[ |S_{21}| > 1 \]

The first power we consider is the available power from the source:

\[ P_{avS} \equiv \text{available power from the source} \]
We likewise consider the power $P_{in}$ **delivered** by the source; in other words the power **absorbed** by the input impedance of the gain element with a load attached:

![Circuit Diagram 1](image1.png)

On the output, we consider the power **available** from the **output** of the gain element:

![Circuit Diagram 2](image2.png)

$P_{av}$ = available power from the output port

And finally, we consider the power $P_L$ **delivered** by the output port— the power absorbed by load $Z_L$:
These four power quantities depend (at least in part) on the source parameters $V_g$ and $Z_g$, load $Z_L$, and the scattering parameters of $S_{11}, S_{21}, S_{22}, S_{12}$ the gain element.

Q: Yikes! How can we possibly determine the power values in terms of these circuit parameters?

A: Remember, the source, load and gain element (i.e. its scattering matrix) each are described by as set of equations. We simply need to solve these simultaneous equations!

Your text (pages 537-539) provides an algebraic solution. But you know me; I prefer to graphically solve the algebra using signal flow graphs!

Q: But there's a source in our circuit: How do we handle that in a signal flow graph?
A: Consider a simple source connected to a transmission line:

From KVL we know that:

\[ V_s = V_i + Z_s I_i \]

Whereas, from the telegraphers equations we know that:

\[ V_i = V(z = z_s) = V_0^+ e^{-j \beta z_s} + V_0^- e^{+j \beta z_s} \]

\[ I_i = I(z = z_s) = \frac{V_0^+}{Z_0} e^{-j \beta z_s} - \frac{V_0^-}{Z_0} e^{+j \beta z_s} \]

Substituting the definitions:

\[ a_s = V_0^- e^{+j \beta z_g} \] (complex amplitude of voltage wave incident on source)

\[ b_s = V_0^+ e^{-j \beta z_g} \] (complex amplitude of voltage wave exiting source)

we get:
\[ V_i = V(z = z_s) = b_s + a_s \]

\[ I_i = I(z = z_s) = \frac{b_s}{Z_0} - \frac{a_s}{Z_0} \]

And then our **KVL** equation can be written as:

\[ V_s = (b_s + a_s) + \frac{Z_s}{Z_0} (b_s - a_s) \]

And rearranging:

\[ b_g = \left( \frac{Z_0}{Z_g + Z_0} \right) V_g + \Gamma_g a_g \]

**Reluctantly** defining a “reflection coefficient”:

\[ \Gamma_s = \frac{Z_s - Z_0}{Z_s + Z_0} \quad \text{(Doh!)} \]

we find by rearranging:

\[ \frac{Z_0}{Z_0 + Z_s} = \frac{1 - \Gamma_s}{2} \]

and so:

\[ b_s = \left( \frac{1 - \Gamma_s}{2} \right) V_s + \Gamma_s a_s \]
We can express the above result graphically using a signal-flow graph:

$V_s \xrightarrow{1-\Gamma_s/2} b_s \xrightarrow{\Gamma_s} a_s$

$b_s = \left(\frac{1-\Gamma_s}{2}\right)V_s + \Gamma_s a_s$

Now, consider the case where we place a load (e.g., the input impedance of a two port network) at this source port:

We know from transmission line theory that:

$\Gamma_{in} = \frac{V_0^- e^{j\beta z_s}}{V_0^+ e^{-j\beta z_s}} = \frac{a_s}{b_s} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$

Thus, the relationship $a_s = \Gamma_{in} b_s$ can be added to the signal flow graph:

$V_s \xrightarrow{1-\Gamma_s/2} b_s \xrightarrow{\Gamma_s} a_s \xrightarrow{\Gamma_{in}}$
Using the splitting rule:

\[
\frac{1 - \Gamma_s}{2} V_s \quad b_s \quad \Gamma_{\text{in}} \quad a_s
\]

and then the self-loop rule:

\[
\frac{1 - \Gamma_s}{2} \frac{1}{(1 - \Gamma_s \Gamma_{\text{in}})}
\]

we can directly conclude that:

\[
b_s = V_s \frac{1 - \Gamma_s}{2} \frac{1}{1 - \Gamma_s \Gamma_{\text{in}}}
\]

\[
a_s = V_s \frac{1 - \Gamma_s}{2} \frac{\Gamma_{\text{in}}}{1 - \Gamma_s \Gamma_{\text{in}}}
\]

Note that the power incident on the load can now be determined:

\[
P_{\text{inc}} = \frac{|b_s|^2}{2Z_0} = \frac{|V_s|^2}{8Z_0} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_s \Gamma_{\text{in}}|^2}
\]
as well as the power reflected from the load:

\[
\begin{align*}
\rho_{\text{ref}} &= \frac{|a_s|^2}{2Z_0} = \frac{|V_s|^2}{8Z_0} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_s \Gamma_{\text{in}}|^2} |\Gamma_{\text{in}}|^2
\end{align*}
\]

so that the power absorbed by the load (i.e. the power delivered by the source) is:

\[
\begin{align*}
\rho_{\text{in}} &= \rho_{\text{inc}} - \rho_{\text{ref}} \\
&= \frac{|b_s|^2 - |a_s|^2}{2Z_0} \\
&= \frac{|V_s|^2}{8Z_0} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_s \Gamma_{\text{in}}|^2} \left(1 - |\Gamma_{\text{in}}|^2\right) \\
&= \frac{|V_s|^2}{2Z_0} \frac{Z_0}{Z_0 + Z_s} \frac{1 - |\Gamma_{\text{in}}|^2}{|1 - \Gamma_s \Gamma_{\text{in}}|^2}
\end{align*}
\]

It is evident from the result above that the amount of power delivered is dependent on the value of load impedance. To maximize this power, we must find the value \(\Gamma_{\text{in}}\) that maximizes the term:

\[
\frac{1 - |\Gamma_{\text{in}}|^2}{|1 - \Gamma_s \Gamma_{\text{in}}|^2}
\]
It can be shown that this term is maximized when $\Gamma_{in} = \Gamma_s^*$. No surprise here; the load must be conjugate matched to the source in order to maximize power transfer. This maximum value—resulting only when the load is conjugate matched to the source—is referred to as the available power of the source:

$$P_{avs} = P_{in} \bigg|_{\Gamma_{in}=\Gamma_s^*}$$

$$= \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{1-|\Gamma_s|^2}$$

$$= \frac{|V_s|^2}{2Z_0} \frac{Z_0}{Z_0 + Z_s} \left| \frac{1}{1-|\Gamma_s|^2} \right|^2$$

$$= \frac{1}{2} \frac{|V_s|^2}{4Re\{Z_s^*\}}$$

Now, consider the case where we connect some arbitrary two-port device to the source. We would like to determine the available power $P_{avn}$ from the output port of this two-port device.

![Two-port device diagram with available power symbol]
The signal-flow graph for this network:

We can reduce this signal-flow graph:
Now, for the purposes of determining the output power at port 2, we can ignore nodes $a_1$ and $b_1$ (in the final signal flowgraph above they are terminal nodes, no branches are leaving these nodes). Thus, the relevant portion of the reduced signal flow graph is:

$$\frac{1 - \Gamma_s}{2} \frac{s_{21}}{1 - \Gamma_s s_{11}}$$

Notice this signal flow graph has the same form as the source signal-flow graph:
To make this comparison more specific, we define variables:

\[ V_{\text{out}} = V_s \frac{1 - \Gamma_s}{1 - \Gamma_{\text{out}}} \frac{S_{21}}{1 - \Gamma_s S_{11}} \]

\[ \Gamma_{\text{out}} = S_{22} + \frac{\Gamma_s S_{12} S_{21}}{1 - \Gamma_s S_{11}} \]

And thus, using these definitions, our signal flow graph can be equivalently written as:

It is apparent that \( V_{\text{out}} \) and \( \Gamma_{\text{out}} \) define an equivalent source created when the original source is connected to a two-port device.
Thus, when some load is connected to the output of the two-port device, the signal-flow graph is:

\[ \frac{1 - \Gamma_{out}}{2} \]

Which has precisely the same form as:

\[ \frac{1 - \Gamma_s}{2} \]

As a result, the delivered power is precisely the same as the original case, with the exception that we use the equivalent values defined above:

\[ P_L = \frac{|b_2|^2 - |a_2|^2}{2Z_0} \]

\[ = \frac{|V_{out}|^2}{8Z_0} \frac{|1 - \Gamma_{out}|^2}{|1 - \Gamma_{out} \Gamma_L|^2} \left(1 - |\Gamma_L|^2\right) \]

\[ = \frac{|V_s|^2}{8Z_0} \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_{out} \Gamma_L|^2} \left(1 - |\Gamma_L|^2\right) \]
Likewise, the available power from port 2 is simply the maximum possible power absorbed by a load $\Gamma_L$. This again is found by maximizing the term:

$$\frac{1 - |\Gamma_L|^2}{|1 - \Gamma_{out} \Gamma_L|^2}$$

which again occurs when $\Gamma_L = \Gamma_{out}^*$. Thus, maximum power transfer occurs when the load is conjugate matched to the equivalent source impedance $Z_{out} (\Gamma_{out})$. As a result the available power from port 2 is:

$$P_{av} = P_L\big|_{\Gamma_L = \Gamma_{out}^*}$$

$$= \frac{|V_s|^2}{8Z_0} \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_{out} \Gamma_{out}^*|^2} \left(1 - |\Gamma_{out}|^2\right)^2$$

$$= \frac{|V_s|^2}{8Z_0} \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \frac{|1 - \Gamma_s|^2}{\left(1 - |\Gamma_{out}|^2\right)^2} \left(1 - |\Gamma_{out}|^2\right)$$

$$= \frac{|V_s|^2}{8Z_0} \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \frac{|1 - \Gamma_s|^2}{1 - |\Gamma_{out}|^2}$$
Two-Port Power Gains

There are three standard ways of defining amplifier gain:

1. Power Gain

Power gain is defined as:

\[ G = \frac{P_L}{P_{in}} \]

Thus, it describes the increase in delivered (i.e., absorbed) power from input to output. From our power definitions, we find that:

\[ G = \frac{P_L}{P_{in}} = \frac{|S_{21}|^2}{1 - |\Gamma_{in}|^2} \frac{1 - |\Gamma_L|^2}{1 - \Gamma_{out} \Gamma_L} \frac{1 - |\Gamma_s|^2}{\Gamma_{in}^2} \frac{1 - \Gamma_{out} \Gamma_L}{1 - \Gamma_{out} \Gamma_L} \]

\[ = \frac{|S_{21}|^2}{1 - |\Gamma_{in}|^2} \frac{1 - |\Gamma_L|^2}{1 - \Gamma_{out} \Gamma_L} \frac{1 - |\Gamma_s|^2}{\Gamma_{in}^2} \frac{1 - \Gamma_{out} \Gamma_L}{1 - \Gamma_{out} \Gamma_L} \]

\[ = \frac{|S_{21}|^2}{1 - |\Gamma_{in}|^2} \frac{1 - |\Gamma_L|^2}{1 - \Gamma_{out} \Gamma_L} \frac{1 - |\Gamma_s|^2}{\Gamma_{in}^2} \frac{1 - \Gamma_{out} \Gamma_L}{1 - \Gamma_{out} \Gamma_L} \]

\[ = \frac{|S_{21}|^2}{1 - |\Gamma_{in}|^2} \frac{1 - |\Gamma_L|^2}{1 - \Gamma_{out} \Gamma_L} \frac{1 - |\Gamma_s|^2}{\Gamma_{in}^2} \frac{1 - \Gamma_{out} \Gamma_L}{1 - \Gamma_{out} \Gamma_L} \]
Where we have used the fact (trust me!) that:

\[ |1 - \Gamma_s \Gamma_{in}|^2 = \frac{|1 - \Gamma_s S_{11}|^2 |1 - \Gamma_{out} \Gamma_L|^2}{|1 - S_{22} \Gamma_L|^2} \]

2. Available Gain

Available gain is defined as:

\[ G_A = \frac{P_{avn}}{P_{avs}} \]

Thus, it describes the increase in available power from input to output. From our power definitions, we find that:

\[ G_A = \frac{P_{avn}}{P_{avs}} = \frac{|S_{21}|^2 |1 - \Gamma_s|^2}{|1 - \Gamma_s \Gamma_{in}|^2} \frac{1 - |\Gamma_{out}|^2}{1 - |\Gamma_s|^2} \]

3. Transducer Gain

Transducer gain is defined as:

\[ G_T = \frac{P_L}{P_{avs}} \]
Thus, it relates the power available from the source to the power delivered to the load. It in effect describes how \textit{effectual} the amplifier was in extracting the available power from the source, increasing this power, and then delivering the power to the load.

\[
G_T = \frac{P_L}{P_{av_s}} = \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \frac{|1 - \Gamma_s|^2}{|1 - \Gamma_{out} \Gamma_L|^2} \left(1 - |\Gamma_L|^2\right) \frac{1}{|1 - \Gamma_{out} \Gamma_L|^2} \frac{1}{|1 - \Gamma_s S_{11}|^2}
\]

\[
= \frac{|S_{21}|^2}{1 - |\Gamma_s|^2} \left(1 - |\Gamma_L|^2\right) \frac{1}{|1 - \Gamma_{out} \Gamma_L|^2} \frac{1}{|1 - \Gamma_s S_{11}|^2}
\]

\[
= \frac{|S_{21}|^2}{1 - |\Gamma_s|^2} \left(1 - |\Gamma_L|^2\right) \frac{1}{|1 - \Gamma_s \Gamma_{in}|^2} \frac{1}{|1 - \Gamma_L S_{22}|^2}
\]

\[
= \frac{|S_{21}|^2}{1 - |\Gamma_s|^2} \left(1 - |\Gamma_L|^2\right) \frac{1}{|1 - \Gamma_s \Gamma_{in}|^2} \frac{1}{|1 - \Gamma_L S_{22}|^2}
\]

There are likewise a few \textbf{special cases} that we need to be aware of. If both the source and the load impedance are \(Z_0\), then we find \(\Gamma_s = \Gamma_L = 0\), and then not surprisingly:

\[
G_T = |S_{21}|^2
\]

Additionally, we often find that \(S_{12} = 0\) (or least approximately so), and as a result \(\Gamma_{in} = S_{11}\), so:
\[ G_T = \frac{|S_{21}|^2 \left(1 - |\Gamma_s|^2\right) \left(1 - |\Gamma_L|^2\right)}{|1 - \Gamma_s S_{11}|^2 |1 - \Gamma_L S_{22}|^2} \equiv G_{TU} \]

We call this gain the \textbf{unilateral transducer power gain} \( G_{TU} \).

\textbf{Q:} I’m so confused! \textbf{Which gain definitions should I use when specifying an amp? Which gain definition do amplifier vendors use to specify their performance?}

\textbf{A:} We find that for a \textbf{well-designed} amplifier, the three gain values generally do \textbf{not} provide significantly differing values. Your book (on page 539-540) provides a typically example, where \( G = 5.58 \), \( G_A = 5.85 \), and \( G_T = 5.49 \).

Most often then, microwave amplifier vendors do \textbf{not} explicitly specify the three values (for an assumed \( Z_0 \) source and load impedance). Instead, they provide a somewhat ambiguous value that they simply call \textbf{gain*}.

\* If you are inclined to be mischievous, ask an amplifier vendor if their gain spec. is actually \textbf{available gain} or \textbf{transducer gain}. 
Turning a Gain Element into an Amplifier

Say the design criteria for our amplifier is to maximize the power delivered to the load (i.e., maximize $P_L$). This power is maximized when:

1. The available power from the source is entirely delivered to the input of the gain element $P_{in} = P_{avgs}$.

2. The available power from the output of the gain element is entirely delivered to the load $P_L = P_{avn}$.

Recall this happy occurrence results when $\Gamma_{in} = \Gamma_s^*$ and $\Gamma_L = \Gamma_{out}^*$.

Q: But what if this is not the case? What if our gain element is not matched to our source, or to our load? Must we simply accept inferior power transfer?

A: Nope! Remember, we can always build lossless matching networks to efficiently transfer power between mismatched sources and loads.

Q: I see! We need to modify the source impedance $Z_s$ and modify the output impedance $Z_{out}$ such that $Z_s = Z_{in}^*$ and $Z_{out} = Z_L^*$. Right?
A: Not exactly.

Remember, it is true that a lossless matching network can change the source impedance to match a specific load. But the lossless matching network likewise alters the source voltage $V_s$ such that the available power is preserved!

Messing around directly with the source impedance will undoubtedly reduce the available power of the source (this is bad!).

For example, consider this simple problem. Say we have this source, with a robust available power of 1.25 W:

\[
Z_s = 10 \Omega \\
V_s = 10
\]

and wish to deliver this power to an impedance of $Z_{in} = 50 \Omega$:

\[
P_{\text{avg}} = \frac{|V_s|^2}{8 \text{Re}\{Z_s\}} \\
= \frac{10^2}{8(10)} \\
= 1.25 \text{ W}
\]
Although increasing the source impedance by 40 Ω would result in a conjugate match, it would likewise reduce the available power to a measly 0.25 Watts.

\[ Z_s = 50 \Omega \]

\[ P_{\text{avg}} = \frac{|V_s|^2}{8 \Re\{Z_s\}} \]

\[ = \frac{10^2}{8(50)} \]

\[ = 0.25 \, W \]

Thus, although finagling the source impedance does result in extracting all the available power from the resulting source, it likewise decreases this available power by 80%!

Moreover, we find that the delivered power to would be greater if we simply left the darn thing alone!

\[ P_L = \frac{1}{2} \frac{|V_g|^2 \Re\{Z_L\}}{|Z_g + Z_L|^2} \]

\[ = \frac{10^2}{2} \frac{50}{(20 + 50)^2} \]

\[ = \frac{50^2}{70^2} \]

\[ = 0.51 \, W \]
In contrast, a properly designed matching network will transform the source impedance to a matched value of 50 Ω, but it likewise transforms the source voltage such that the absorbed power remains the same—1.25 Watts is delivered to the 50 Ω load!

We have our cake. We eat it too.

So, to maximize the power delivered to a load, we need to insert lossless matching networks between the source and gain element, and between the gain element and the load:

The three stages together—input matching network, gain element, and output matching network—form a microwave amplifier!
Of course, the impedance of both the source and the load connected to this amp will most certainly be that of transmission line characteristic impedance $Z_0$. Thus, our amplifier circuit is typically:

![A Microwave Amplifier Diagram](image-url)
The input network is thus required to match $Z_0$ to the gain element input impedance $Z_{in}$. For the purposes of amplifier design, we view the input matching network as one that transforms the source impedance $Z_0$ into a new source impedance $Z_s$, one that is conjugate matched to the gain element input impedance $Z_{in}$:

If our input matching network is properly designed, we then find:

$$Z_s = Z_{in}^*$$

and so

$$\Gamma_s = \Gamma_{in}^*$$

Likewise, the output matching network is used to match $Z_0$ to the gain element output impedance $Z_{out}$. For the purposes of amplifier design, we view the output matching network as one that transforms the load impedance $Z_0$ into a new load impedance $Z_L$, one that is conjugate matched to the gain element output impedance $Z_{out}$:
Thus, if our input matching network is properly designed, then we find:

\[ Z_L = Z_{out}^* \quad \text{and so} \quad \Gamma_L = \Gamma_{out}^* \]

And so, our amplifier design problem can be described as:

where the values of \( \Gamma_s \) and \( \Gamma_L \) depend on the input and output matching networks.

Q: Alright, we get it. We know how to make matching networks. Can’t we move on to something else?

A: Not so fast! There’s one little problem that makes this procedure more difficult than it otherwise might appear.

Note that the value of \( \Gamma_{out} \) depends on the value of \( Z_s \) (i.e., depends on \( \Gamma_s \)).
Likewise, the value of $\Gamma_{in}$ depends on the value of $Z_L$ (i.e., depends on $\Gamma_L$).

\[
S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = \Gamma_{in}
\]

It's a classic \textbf{chicken and egg}!

1. We can't design the input matching network until we determine $\Gamma_{in}$.

2. We can't determine $\Gamma_{in}$ until we design the output matching network.

3. We can't determine the output matching network until we determine $\Gamma_{out}$.

4. We can't determine $\Gamma_{out}$ until we design the input matching network.

5. But we can't design the input matching network until we determine $\Gamma_{in}$!

Our matching network design problems are thus \textbf{coupled}. The solution to this coupled problem is provided in your textbook.
on page 550, and provides simultaneous solutions for \( \Gamma_s = \Gamma_{in}^* \) and \( \Gamma_L = \Gamma_{out}^* \).

Now for some good news!

Recall that for many gain elements, the value of \( S_{12} \) is exceedingly small. Often it is so small that we can approximate as zero.

**Q:** So?

**A:** Look at what this does to the value of \( \Gamma_{in} \) and \( \Gamma_{out} \):

\[
\Gamma_{out} \big|_{S_{12} = 0} = S_{22} + \frac{S_{12} S_{21} \Gamma_s}{1 - S_{11} \Gamma_s} \bigg|_{S_{12} = 0} = S_{22}
\]

\[
\Gamma_{in} \big|_{S_{12} = 0} = S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L} \bigg|_{S_{12} = 0} = S_{11}
\]

Thus, for this unilateral gain element, the matching network design problem decouples, and our matching network design simplifies to these two independent equations:

\[
\Gamma_s = \Gamma_{in}^* \quad \text{and} \quad \Gamma_L = \Gamma_{out}^*
\]