EECS 723-Microwave Engineering

Teacher: *"Bart, do you even **know** your multiplication tables?"*

Bart: *" Well, I know **of** them".*

Like Bart and his multiplication tables, many electrical engineers know **of** the concepts of microwave engineering.

Concepts such as characteristic impedance, scattering parameters, Smith Charts and the like are familiar, but often we find that a **complete**, **thorough** and **unambiguous** understanding of these concepts can be somewhat lacking.

Thus, the goals of this class are for **you** to:

1. Obtain a complete, thorough, and unambiguous understanding of the fundamental concepts on microwave engineering.

2. Apply these concepts to the **design** and **analysis** of useful microwave devices.

Almost all the devices we study will be both **linear** and **time-invariant**. Thus, almost all our analysis will have at it root the **mathematics** of linear, time-invariant systems.

Certainly, **all** electrical engineers know of linear systems theory. But, it is helpful to first **review** these concepts to make sure that we all understand **what** this theory is, **why** it works, and **how** it is useful.

First, we must carefully **define** a linear-time invariant system.

HO: THE LINEAR, TIME-INVARIANT SYSTEM

Linear systems theory is useful for microwave engineers because most **microwave devices and systems are linear** (at least approximately).

HO: LINEAR CIRCUIT ELEMENTS

The most powerful tool for analyzing linear systems is its **eigen function**.

HO: THE EIGEN FUNCTION OF LINEAR SYSTEMS

Complex votages and currents at times cause much **head scratching**; let's make sure we know what these complex values and functions **physically** mean.

HO: A COMPLEX REPRESENTATION OF SINUSOIDAL FUNCTIONS

Signals may **not** have the explicit form of an eigen function, **but** our linear systems theory allows us to (relatively) easily analyze this case as well.

HO: Analysis of Circuits Driven by Arbitrary Functions

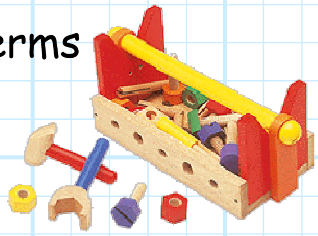
If our linear system is a linear **circuit**, we can apply **basic** circuit analysis to determine all its **eigen values**!

HO: The Eigen Spectrum of Linear Circuits

The Linear, Time-Invariant System

Most of the microwave devices and networks that we will study in this course are both **linear** and **time invariant** (or approximately so).

Let's make sure that we understand what these terms **mean**, as linear, time-invariant systems allow us to apply a large and helpful **mathematical** toolbox!



Linearity

Mathematicians often speak of **operators**, which is "mathspeak" for any mathematical operation that can be applied to a single **element** (e.g., value, variable, vector, matrix, or function).

...operators, operators, operators!!

For example, a **function** $f(x)$ describes an operation **on** variable x (i.e., $f(x)$ is operator on x). E.G.:

$$f(y) = y^2 - 3 \qquad g(t) = 2t \qquad y(x) = |x|$$

Moreover, we find that functions can likewise be operated **on**! For example, **integration** and **differentiation** are likewise mathematical operations—operators that operate **on functions**. E.G.,:

$$\int f(y)\,dy \qquad \frac{d\,g(t)}{dt} \qquad \int_{-\infty}^{\infty} |y(x)|\,dx$$

A special and very important class of operators are **linear operators**.

Linear operators are **denoted** as $\mathcal{L}[y]$, where:

* $\mathcal{L}$ symbolically denotes the mathematical **operation**;

* And y denotes the **element** (e.g., function, variable, vector) being **operated on**.

A **linear** operator is any operator that satisfies the following **two** statements for any and **all** y:

1. $\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2]$

2. $\mathcal{L}[a\,y] = a\,\mathcal{L}[y]$, where a is any **constant**.

From these two statements we can **likewise** conclude that a linear operator has the property:

$$\mathcal{L}\left[a\, y_1 + b\, y_2\right] = a\,\mathcal{L}\left[y_1\right] + b\,\mathcal{L}\left[y_2\right]$$

where both a and b are constants.

Essentially, a linear operator has the property that any weighted sum of solutions is **also** a solution!

For **example**, consider the function:

$$\mathcal{L}[t] = g(t) = 2t$$

At $t = 1$:

$$g(t=1) = 2(1) = 2$$

and at $t = 2$:

$$g(t=2) = 2(2) = 4$$

Now at $t = 1 + 2 = 3$ we find:

$$\begin{aligned} g(1+2) &= 2(3) \\ &= 6 \\ &= 2 + 4 \\ &= g(1) + g(2) \end{aligned}$$

More generally, we find that:

$$\begin{aligned} g(t_1+t_2) &= 2(t_1+t_2) \\ &= 2t_1+2t_2 \\ &= g(t_1)+g(t_2) \end{aligned}$$

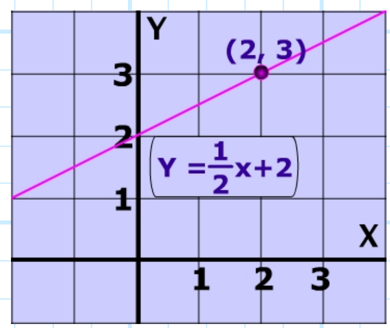
and

$$\begin{aligned} g(at) &= 2at \\ &= a\,2t \\ &= a\,g(t) \end{aligned}$$

Thus, we conclude that the function $g(t)=2t$ is **indeed** a **linear** function!

Now consider **this** function:

$$y(x)=mx+b$$



Q: *But that's the equation of a **line**! That **must** be a linear function, right?*

A: I'm **not sure**—let's find out!

We find that:

$$\begin{aligned} y(ax) &= m(ax)+b \\ &= a\,mx+b \end{aligned}$$

but:

$$\begin{aligned} a\,y(x) &= a(mx+b) \\ &= a\,mx+ab \end{aligned}$$

therefore:

$$y(a\,x) \neq a\,y(x) \quad !!!$$

Likewise:

$$\begin{aligned} y(x_1 + x_2) &= m(x_1 + x_2) + b \\ &= m\,x_1 + m\,x_2 + b \end{aligned}$$

but:

$$\begin{aligned} y(x_1) + y(x_2) &= (m\,x_1 + b) + (m\,x_2 + b) \\ &= m\,x_1 + m\,x_2 + 2b \end{aligned}$$

therefore:

$$y(x_1 + x_2) \neq y(x_1) + y(x_2) \quad !!!$$

→ The equation of a line is **not** a linear function!

Moreover, **you** can show that the functions:

$$f(y) = y^2 - 3 \qquad\qquad y(x) = |x|$$

are likewise **non-linear**.

Remember, linear operators need **not** be **functions**. Consider the derivative operator, which operates **on** functions.

$$\frac{d\,f(x)}{dx}$$

Note that:

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d\,f(x)}{dx}+\frac{d\,g(x)}{dx}$$

and also:

$$\frac{d}{dx}\left[a\,f(x)\right]=a\frac{d\,f(x)}{dx}$$

We thus can conclude that the **derivative** operation is a **linear** operator **on function** $f(x)$:

$$\frac{d\,f(x)}{dx}=\mathcal{L}\left[f(x)\right]$$

You can likewise show that the **integration** operation is likewise a **linear operator**:

$$\int f(y)\,dy=\mathcal{L}\left[f(y)\right]$$

But, you will find that operations such as:

$$\frac{d\,g^2(t)}{dt} \qquad \int_{-\infty}^{\infty}\left|y(x)\right|dx$$

are **not** linear operators (i.e., they are **non-linear** operators).

We find that **most** mathematical operations are in fact **non-linear**! Linear operators are thus form a small **subset** of all possible mathematical operations.

Q: *Yikes! If linear operators are so **rare**, we are we **wasting** our time learning about them??*

A: **Two** reasons!

Reason 1: In electrical engineering, the behavior of most of our fundamental **circuit elements** are described by **linear operators**—linear operations are prevalent in **circuit analysis**!

Reason 2: To our great relief, the two characteristics of linear operators allow us to **perform** these mathematical operations with **relative ease**!

Q: ***How** is performing a **linear** operation easier than performing a **non-linear** one??*

A: The "secret" lies is the result:

$$\mathcal{L}\left[a\, y_1 + b\, y_2\right] = a\,\mathcal{L}\left[y_1\right] + b\,\mathcal{L}\left[y_2\right]$$

Note here that the linear operation performed on a relatively **complex** element $a\, y_1 + b\, y_2$ can be determined immediately from the result of operating on the **"simple"** elements y_1 and y_2.

To see how this might work, let's consider some **arbitrary** function of **time** $v(t)$, a function that exists over some **finite** amount of time T (i.e., $v(t)=0$ for $t<0$ and $t>T$).

Say we wish to perform some **linear** operation **on this function**:

$$\mathcal{L}\left[v(t)\right]=??$$

Depending on the **difficulty** of the operation $\mathcal{L}$, and/or the **complexity** of the function $v(t)$, directly performing this operation could be very **painful** (i.e., approaching impossible).

Instead, we find that we can often **expand** a very complex and **stressful** function in the following way:

$$v(t)=a_0\,\psi_0(t)+a_1\,\psi_1(t)+a_2\,\psi_2(t)+\cdots=\sum_{n=-\infty}^{\infty}a_n\,\psi_n(t)$$

where the values a_n are **constants** (i.e., coefficients), and the functions $\psi_n(t)$ are known as **basis functions**.

For example, we could **choose** the basis functions:

$$\psi_n(t)=t^n \quad \text{for} \quad n\geq 0$$

Resulting in a **polynomial** of variable t.

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots = \sum_{n=0}^{\infty} a_n t^n$$

This signal expansion is of course know as the **Taylor Series** expansion. However, there are **many other** useful expansions (i.e., many other useful basis $\psi_n(t)$).

* The key thing is that the basis functions $\psi_n(t)$ are **independent** of the function $v(t)$. That is to say, the basis functions are **selected** by the engineer (i.e., **you**) doing the analysis.

* The set of selected basis functions form what's known as a **basis.** With this basis we can **analyze** the function $v(t)$.

* The **result** of this analysis provides the **coefficients** a_n of the signal expansion. Thus, the coefficients **are** directly dependent on the form of function $v(t)$ (as well as the basis used for the analysis). As a result, the set of coefficients $\{a_1, a_2, a_3, \cdots\}$ **completely describe** the function $v(t)$!

Q: *I don't see why this "expansion" of function of $v(t)$ is helpful, it just looks like a lot more **work** to me.*

A: Consider what happens when we wish to perform a **linear** operation on this function:

$$\mathcal{L}\left[v(t)\right]=\mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n\,\psi_n(t)\right]=\sum_{n=-\infty}^{\infty} a_n\,\mathcal{L}\left[\psi_n(t)\right]$$

Look what happened! **Instead** of performing the linear operation on the arbitrary and **difficult** function $v(t)$, we can apply the operation to **each** of the individual basis functions $\psi_n(t)$.

Q: *And that's supposed to be* ***easier****??*

A: It **depends** on the linear operation and on the basis functions $\psi_n(t)$. **Hopefully**, the operation $\mathcal{L}[\psi_n(t)]$ is **simple** and straightforward. **Ideally**, the solution to $\mathcal{L}[\psi_n(t)]$ is **already known**!

Q: *Oh yeah, like I'm going to get so* ***lucky****. I'm sure in all my circuit analysis problems evaluating $\mathcal{L}[\psi_n(t)]$ will be long, frustrating, and* ***painful****.*

A: Remember, **you** get to choose the **basis** over which the function $v(t)$ is analyzed. A **smart** engineer will **choose** a basis for which the operations $\mathcal{L}[\psi_n(t)]$ are simple and **straightforward**!

Q: *But I'm* ***still*** *confused.* ***How*** *do I choose what basis $\psi_n(t)$ to use, and* ***how*** *do I analyze the function $v(t)$ to determine the coefficients a_n??*

A: Perhaps an **example** would help.

Among the **most popular** basis is this one:

$$\psi_n = \begin{cases} e^{j\left(\frac{2\pi n}{T}\right)t} & 0 \le t \le T \\ 0 & t \le 0, t \ge T \end{cases}$$

and:

$$a_n = \frac{1}{T}\int_0^T v(t)\,\psi_n^*(t)\,dt = \frac{1}{T}\int_0^T v(t)\,e^{-j\left(\frac{2\pi n}{T}\right)t}\,dt$$

So therefore:

$$v(t) = \sum_{n=-\infty}^{\infty} a_n\, e^{j\left(\frac{2\pi n}{T}\right)t} \qquad \text{for } 0 \le t \le T$$

The **astute** among you will recognize this signal expansion as the **Fourier Series**!

Q: *Yes, just why is Fourier analysis* ***so*** *prevalent?*

A: The answer reveals itself when we apply a **linear operator** to the signal expansion:

$$\mathcal{L}\left[v(t)\right] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n\, e^{-j\left(\frac{2\pi n}{T}\right)t}\right] = \sum_{n=-\infty}^{\infty} a_n\, \mathcal{L}\left[e^{-j\left(\frac{2\pi n}{T}\right)t}\right]$$

Note then that we must **simply** evaluate:

$$\mathcal{L}\left[e^{-j\left(\frac{2\pi n}{T}\right)t} \right]$$

for all *n*.

We will find that **performing** almost any linear operation $\mathcal{L}$ on basis functions of this type to be exceeding **simple** (more on this later)!

Time Invariance

Q: *That's right! You said that most of the microwave devices that we will study are (approximately) linear,* ***time-invariant*** *devices. What does time invariance* ***mean****?*

A: From the standpoint of a linear operator, it means that that the operation is **independent of time**—the result does not depend on when the operation is applied. I.E., if:

$$\mathcal{L}\left[x(t) \right] = y(t)$$

then:

$$\mathcal{L}\left[x(t-\tau) \right] = y(t-\tau)$$

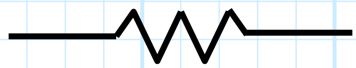
where τ is a **delay** of any value.

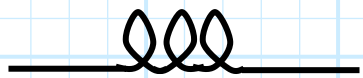
The devices and networks that you are about to study in EECS 723 are in fact **fixed** and **unchanging** with respect to time (or at least approximately so).

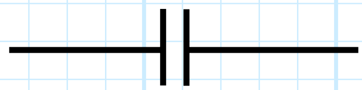
As a result, the mathematical operations that describe most (but not all!) of our circuit devices are **both** linear and time-invariant operators. We therefore refer to these devices and networks as **linear, time-invariant systems**.

Linear Circuit Elements

Most microwave devices can be described or modeled in terms of the **three** standard circuit elements:

1. RESISTANCE (R) 

2. INDUCTANCE (L) 

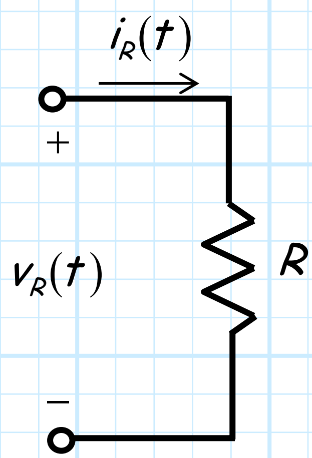
3. CAPACITANCE (C) 

For the purposes of circuit analysis, each of these three elements are **defined** in terms of the **mathematical** relationship between the difference in electric potential $v(t)$ between the two terminals of the device (i.e., the **voltage** across the device), and the **current** $i(t)$ flowing through the device.

We find that for these three circuit elements, the relationship between $v(t)$ and $i(t)$ can be expressed as a linear operator!

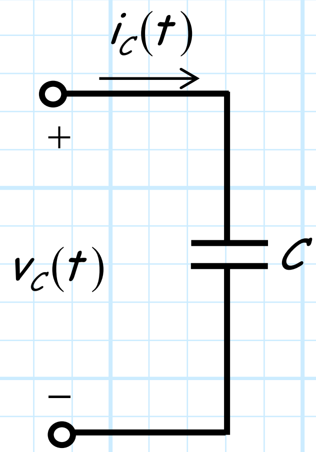
$$\mathcal{L}_{\mathcal{Y}}^{R}\left[v_R(t)\right] = i_R(t) = \frac{v_R(t)}{R}$$

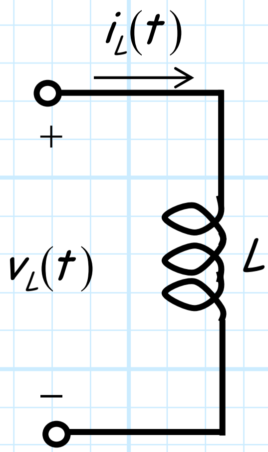
$$\mathcal{L}_{\mathcal{Z}}^{R}\left[i_R(t)\right] = v_R(t) = R\, i_R(t)$$



$$\mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] = i_C(t) = C\,\frac{d\,v_C(t)}{d\,t}$$

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[i_C(t)\right] = v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\,dt'$$





$$\mathcal{L}_{\mathcal{Y}}^{L}\left[v_L(t)\right] = i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t')\,dt'$$

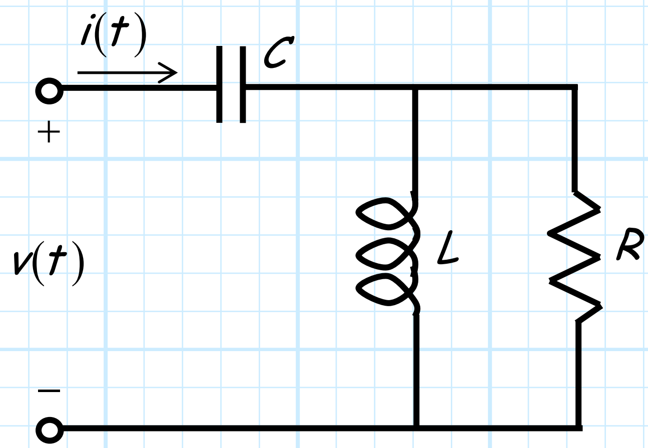
$$\mathcal{L}_{\mathcal{Z}}^{L}\left[i_L(t)\right] = v_L(t) = L\,\frac{d\,i_L(t)}{d\,t}$$

Since the circuit behavior of these devices can be expressed with **linear** operators, these devices are referred to as **linear circuit elements**.

Q: *Well, that's simple enough, but what about an element formed from a **composite** of these fundamental elements?*

*For **example**, for example, how are $v(t)$ and $i(t)$ related in the circuit below??*

$$\mathcal{L}_{\mathcal{Z}}\left[i(t)\right] = v(t) = ???$$



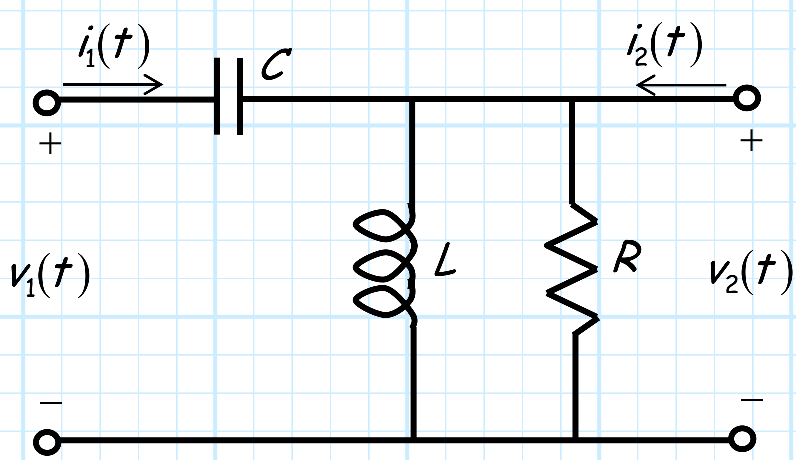
A: It turns out that **any** circuit constructed **entirely** with linear circuit elements is **likewise** a linear system (i.e., a linear circuit).

As a result, we know that that there **must** be some linear operator that relates $v(t)$ and $i(t)$ in your example!

$$\mathcal{L}_{\mathcal{Z}}\left[i(t)\right] = v(t)$$

The circuit above provides a good example of a **single-port** (a.k.a. **one-port**) network.

We can of course construct networks with **two or more** ports; an example of a **two-port network** is shown below:



Since this circuit is **linear**, the relationship between **all** voltages and currents can likewise be expressed as **linear operators, e.g.**:

$$\mathcal{L}_{21}\left[v_1(t)\right] = v_2(t)$$

$$\mathcal{L}_{\mathcal{Z}21}\left[i_1(t)\right] = v_2(t)$$

$$\mathcal{L}_{\mathcal{Z}22}\left[i_2(t)\right] = v_2(t)$$

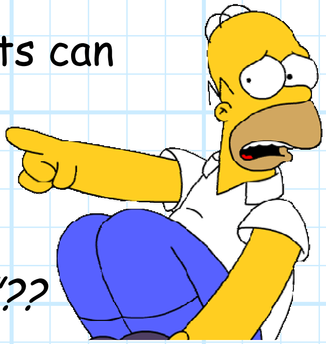
Q: *Yikes! What would these linear operators for this circuit **be**? How can we **determine** them?*

A: It turns out that linear operators for **all** linear circuits can all be expressed in precisely the **same** form! For example, the linear operators of a single-port network are:

$$v(t) = \mathcal{L}_{\mathcal{Z}}\left[i(t)\right] = \int_{-\infty}^{t} g_{\mathcal{Z}}(t-t')\ i(t')\,dt'$$

$$i(t) = \mathcal{L}_{\mathcal{Y}}\left[v(t)\right] = \int_{-\infty}^{t} g_{\mathcal{Y}}(t-t')\ v(t')\,dt'$$

In other words, the linear operator of linear circuits can always be expressed as a **convolution** integral—a convolution with a **circuit impulse function** $g(t)$.



Q: *But just what **is** this "circuit impulse response"??*

A: An impulse response is simply the **response** of one circuit function (i.e., $i(t)$ or $v(t)$) due to a **specific** stimulus by another.

That specific stimulus is the **impulse function** $\delta(t)$.

The impulse function **can** be defined as:

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} \frac{\sin\left(\frac{\pi t}{\tau}\right)}{\left(\frac{\pi t}{\tau}\right)}$$

Such that is has the following two **properties**:

1. $\delta(t) = 0 \quad \text{for} \quad t \neq 0$

2. $\int_{-\infty}^{\infty} \delta(t)\, dt = 1.0$

The impulse responses of the **one-port example** are therefore defined as:

$$g_z(t) \doteq v(t)\big|_{i(t)=\delta(t)}$$

and:

$$g_y(t) \doteq i(t)\big|_{v(t)=\delta(t)}$$

Meaning simply that $g_z(t)$ is equal to the **voltage** function $v(t)$ when the circuit is "thumped" with a **impulse current** (i.e., $i(t)=\delta(t)$), and $g_y(t)$ is equal to the **current** $i(t)$ when the circuit is "thumped" with a **impulse voltage** (i.e., $v(t)=\delta(t)$).

Similarly, the relationship between the **input** and the **output** of a **two-port** network can be expressed as:

$$v_2(t) = \mathcal{L}_{21}\left[v_1(t)\right] = \int_{-\infty}^{t} g(t-t')\, v_1(t')\, dt'$$

where:

$$g(t) \doteq v_2(t)\Big|_{v_1(t)=\delta(t)}$$

Note that the circuit impulse response must be **causal** (nothing can occur at the output **until** something occurs at the input), so that:

$$g(t)=0 \quad \text{for} \quad t<0$$

Q: *Yikes! I recall evaluating convolution integrals to be messy, difficult and **stressful**. Surely there is an **easier** way to describe linear circuits!?!*

A: Nope! The convolution integral is **all** there is. **However**, we can use our linear systems theory toolbox to greatly **simplify the evaluation** of a convolution integral!

The Eigen Function of Linear, Time-Invariant Systems

Recall that that we can express (**expand**) a time-limited signal with a weighted summation of **basis functions:**

$$v(t) = \sum_n a_n \psi_n(t)$$

where $v(t) = 0$ for $t < 0$ and $t > T$.

Say now that we **convolve** this signal with some system **impulse function** $g(t)$:

$$\begin{aligned}\mathcal{L}\left[v(t)\right] &= \int_{-\infty}^{t} g(t-t')\, v(t')\, dt' \\ &= \int_{-\infty}^{t} g(t-t') \sum_n a_n\, \psi_n(t')\, dt' \\ &= \sum_n a_n \int_{-\infty}^{t} g(t-t')\, \psi_n(t')\, dt'\end{aligned}$$

Look what happened!

Instead of convolving the general function $v(t)$, we now find that we must simply convolve with the set of basis functions $\psi_n(t)$.

Q: *Huh? You say we must* ***"simply"*** *convolve the set of basis functions $\psi_n(t)$.* ***Why*** *would this be any simpler?*

A: Remember, **you** get to **choose** the basis $\psi_n(t)$. If you're **smart**, you'll choose a set that makes the convolution integral **"simple"** to perform!

Q: *But don't I first need to* ***know*** *the explicit form of $g(t)$* ***before*** *I intelligently choose $\psi_n(t)$??*

A: Not necessarily!

The **key** here is that the convolution integral:

$$\mathcal{L}\left[\psi_n(t)\right] = \int_{-\infty}^{t} g(t-t')\,\psi_n(t')\,dt'$$

is a **linear, time-invariant** operator. Because of this, there exists one **basis** with an **astonishing** property!

These **special** basis functions are:

$$\psi_n(t) = \begin{cases} e^{j\omega_n t} & \text{for} \quad 0 \le t \le T \\ 0 & \text{for} \quad t < 0, t > T \end{cases} \qquad \text{where} \qquad \omega_n = n\left(\frac{2\pi}{T}\right)$$

Now, inserting this function (get ready, here comes the **astonishing** part!) into the convolution integral:

$$\mathcal{L}\left[e^{j\omega_n t}\right] = \int_{-\infty}^{t} g(t-t')\, e^{j\omega_n t'}\, dt'$$

and using the substitution $u = t - t'$, we get:

$$\int_{-\infty}^{t} g(t-t')\, e^{j\omega_n t}\, dt' = \int_{t-(-\infty)}^{t-t} g(u)\, e^{j\omega_n (t-u)}\,(-du)$$

$$= e^{j\omega_n t} \int_{+\infty}^{0} g(u)\, e^{-j\omega_n u}\,(-du)$$

$$= e^{j\omega_n t} \int_{0}^{\infty} g(u)\, e^{-j\omega_n u}\, du$$

See! Doesn't **that** astonish!

Q: *I'm astonished only by how **lame** you are. How is this result any **more** "astonishing" than any of the **other** supposedly "useful" things you've been telling us?*

A: Note that the integration in this **result** is **not** a convolution—the integral is simply a **value** that depends on n (but **not** time t):

$$G(\omega_n) \doteq \int_{0}^{\infty} g(t)\, e^{-j\omega_n t}\, dt$$

As a result, convolution with this "special" set of basis functions can **always** be expressed as:

$$\int_{-\infty}^{t} g(t-t')\, e^{j\omega_n t'}\, dt' = \mathcal{L}\left[e^{j\omega_n t}\right] = G(\omega_n)\, e^{j\omega_n t}$$

The **remarkable** thing about this result is that the linear operation on function $\psi_n(t) = exp[j\omega_n t]$ results in precisely the **same** function of **time** t (save the **complex** multiplier $G(\omega_n)$)! I.E.:

$$\mathcal{L}\left[\psi_n(t)\right] = G(\omega_n)\ \psi_n(t)$$

Convolution with $\psi_n(t) = exp[j\omega_n t]$ is accomplished by simply **multiplying** the function by the **complex** number $G(\omega_n)$!

Note this is true **regardless** of the impulse response $g(t)$ (the function $g(t)$ affects the **value** of $G(\omega_n)$ **only**)!

Q: *Big deal! Aren't there lots of **other** functions that would satisfy the equation above equation?*

A: Nope. The **only** function where this is true is:

$$\psi_n(t) = e^{j\omega_n t}$$

This function is thus **very** special. We call this function the **eigen function** of linear, time-invariant systems.

Q: *Are you **sure** that there are no other eigen functions??*

A: Well, sort of.

Recall from **Euler's equation** that:

$$e^{j\omega_n t} = \cos \omega_n t + j \sin \omega_n t$$

It can be shown that the sinusoidal functions $\cos \omega_n t$ and $\sin \omega_n t$ are **likewise** eigen functions of linear, time-invariant systems.

The real and imaginary components of eigen function $\exp[j\omega_n t]$ are **also** eigen functions.

Q: *What about the set of **values** $G(\omega_n)$?? Do they have any significance or **importance**??*

A: Absolutely!

Recall the values $G(\omega_n)$ (one for each *n*) depend on the **impulse response** of the system (e.g., circuit) **only**:

$$G(\omega_n) \doteq \int_0^{\infty} g(t)\, e^{-j\omega_n t}\, dt$$

Thus, the set of values $G(\omega_n)$ completely **characterizes** a linear time-invariant **circuit** over time $0 \le t \le T$.

We call the values $G(\omega_n)$ the **eigen values** of the linear, time-invariant circuit.

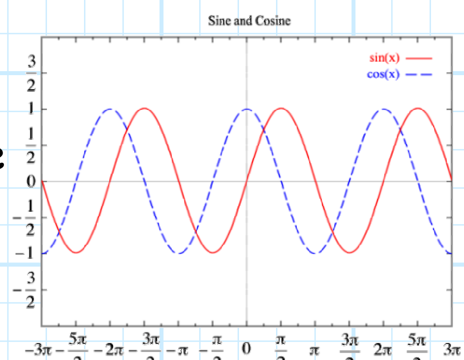
Q: *OK Poindexter, all **eigen** stuff this **might** be interesting if you're a mathematician, but is it at all **useful** to us **electrical engineers?***

A: It is **unfathomably** useful to us electrical engineers!

Say a linear, time-invariant circuit is excited (only) by a **sinusoidal** source (e.g., $v_s(t) = \cos \omega_o t$). Since the source function is the **eigen function** of the circuit, we will find that at **every** point in the circuit, **both** the current and voltage will have the **same functional form**.

That is, every current and voltage in the circuit will likewise be a **perfect sinusoid** with frequency ω_o!!

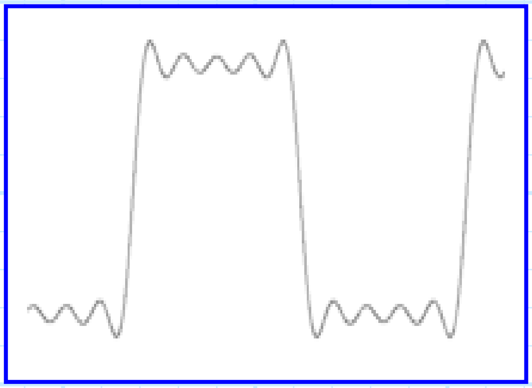
Of course, the **magnitude** of the sinusoidal oscillation will be **different** at different points within the circuit, as will the **relative phase**. But we know that **every** current and voltage in the circuit can be **precisely** expressed as a function of this form:



$$A \cos(\omega_o t + \varphi)$$

Q: *Isn't this pretty **obvious?***

A: **Why** should it be? Say our source function was instead a **square** wave, or **triangle** wave, or a **sawtooth** wave. We would find that (generally speaking) **nowhere** in the circuit would we find another current or voltage that was a **perfect** square wave (etc.)!



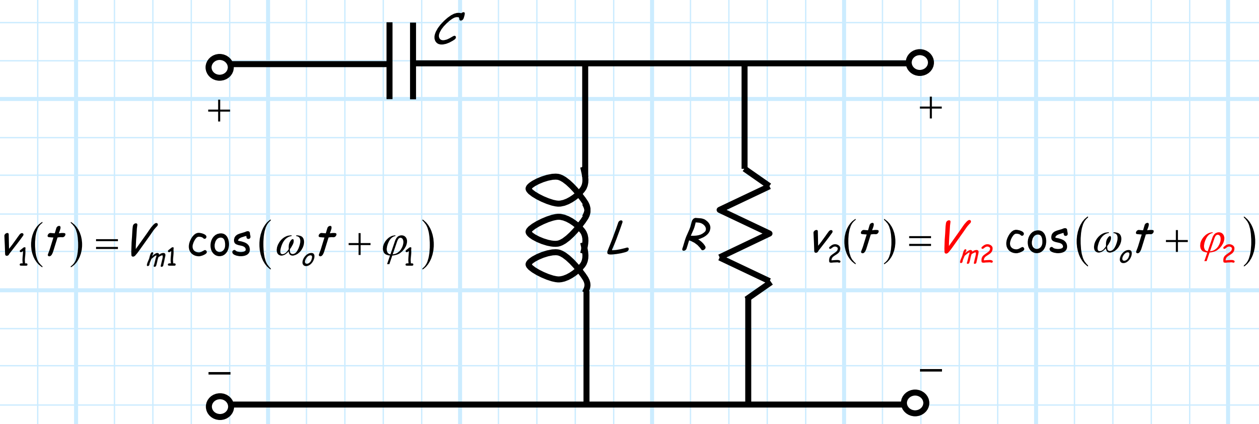
In fact, we would find that not only are the current and voltage functions within the circuit **different** than the source function (e.g. a sawtooth) they are (generally speaking) all different **from each other**.

We find then that a linear circuit will (generally speaking) **distort** any source function—**unless** that function is the **eigen function** (i.e., an sinusoidal function).

Thus, using an **eigen function** as circuit source greatly simplifies our linear circuit analysis problem. **All** we need to accomplish this is to **determine the magnitude** A and **relative phase** φ of the resulting (and otherwise **identical**) sinusoidal function!

A Complex Representation of Sinusoidal Functions

Q: *So, you say (for example) if a linear two-port circuit is driven by a sinusoidal source with arbitrary frequency ω_o, then the output will be identically sinusoidal, only with a different magnitude and relative phase.*



How do we determine the unknown magnitude V_{m2} and phase φ_2 of this output?

A: Say the input and output are related by the impulse response $g(t)$:

$$v_2(t) = \mathcal{L}\left[v_1(t)\right] = \int_{-\infty}^{t} g(t-t')\, v_1(t')\, dt'$$

We now know that **if** the input were **instead**:

$$v_1(t) = e^{j\omega_0 t}$$

then:

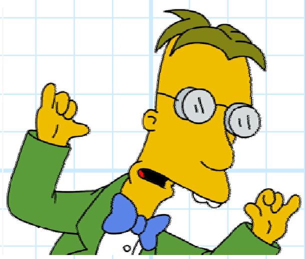
$$v_2(t) = \mathcal{L}\left[e^{j\omega_0 t}\right] = G(\omega_0)\, e^{j\omega_0 t}$$

where:

$$G(\omega_0) \doteq \int_0^\infty g(t)\, e^{-j\omega_0 t}\, dt$$

Thus, we simply multiply the input $v_1(t) = e^{j\omega_0 t}$ by the **complex** eigen value $G(\omega_0)$ to determine the **complex** output $v_2(t)$:

$$v_2(t) = G(\omega_0)\, e^{j\omega_0 t}$$



Q: *You professors drive me crazy with all this math involving* ***complex*** *(i.e., real and imaginary) voltage functions. In the lab I can only generate and measure* ***real-valued*** *voltages and* ***real-valued*** *voltage functions. Voltage is a* ***real-valued, physical*** *parameter!*

A: You are quite **correct**.

Voltage **is** a real-valued parameter, expressing electric potential (in Joules) per unit charge (in Coulombs).

Q: *So, all your* ***complex*** *formulations and* ***complex*** *eigen values and* ***complex*** *eigen functions may all be sound* ***mathematical abstractions****, but aren't they* ***worthless*** *to us* ***electrical engineers*** *who work in the* ***"real"*** *world (pun intended)?*

A: Absolutely not! Complex analysis actually **simplifies** our analysis of real-valued voltages and currents in **linear circuits** (but **only** for linear circuits!).

The key relationship comes from **Euler's Identity**:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Meaning:

$$\text{Re}\left\{e^{j\omega t}\right\} = \cos\omega t$$

Now, consider a **complex value** C. We of course **can** write this complex number in terms of it **real** and **imaginary** parts:

$$C = a + j\,b \qquad \therefore \quad a = \text{Re}\{C\} \quad \text{and} \quad b = \text{Im}\{C\}$$

But, we can **also** write it in terms of its **magnitude** $|C|$ and **phase** φ!

$$\boxed{C = |C|\,e^{j\varphi}}$$

where:

$$|C| = C\,C^* = a^2 + b^2$$

$$\varphi = \tan^{-1}\left[{}^{b}\!/\!_{a}\right]$$

Thus, the complex **function** $C\,e^{j\omega_0 t}$ is:

$$\begin{aligned} C\, e^{j\omega_0 t} &= |C|\, e^{j\varphi}\, e^{j\omega_0 t} \\ &= |C|\, e^{j\omega_0 t + \varphi} \\ &= |C| \cos(\omega_0 t + \varphi) + j\,|C| \sin(\omega_0 t + \varphi) \end{aligned}$$

Therefore we find:

$$|C| \cos(\omega_0 t + \varphi) = \text{Re}\left\{C\, e^{j\omega_0 t}\right\}$$

Now, consider again the **real-valued** voltage function:

$$v_1(t) = V_{m1} \cos(\omega t + \varphi_1)$$

This function is of course **sinusoidal** with a magnitude V_{m1} and phase φ_1. Using what we have learned above, we can **likewise** express this real function as:

$$\begin{aligned} v_1(t) &= V_{m1} \cos(\omega t + \varphi_1) \\ &= \text{Re}\left\{V_1\, e^{j\omega t}\right\} \end{aligned}$$

where V_1 is the **complex number**:

$$V_1 = V_{m1}\, e^{j\varphi_1}$$

Q: *I see! A **real-valued sinusoid** has a magnitude and phase, just like **complex number**. A **single** complex number (V) can be used to specify **both** of the fundamental (real-valued) parameters of our sinusoid (V_m, φ).*

What I don't see is how this helps us in our circuit analysis. After all:

$$v_2(t) = G(\omega_o)\ \left(V_1\, e^{j\omega_o t}\right)$$

which means:

$$v_2(t) \neq G(\omega_o)\ Re\left\{V_1\, e^{j\omega_o t}\right\}$$

*What then is the **real-valued** output $v_2(t)$ of our two-port network when the input $v_1(t)$ is the **real-valued** sinusoid:*

$$\begin{aligned} v_1(t) &= V_{m1}\cos(\omega_o t + \varphi_1) \\ &= \text{Re}\left\{V_1\, e^{j\omega_o t}\right\} \end{aligned} \quad ???$$

A: Let's go back to our **original** convolution integral:

$$v_2(t) = \int_{-\infty}^{t} g(t - t')\, v_1(t')\, dt'$$

If:

$$\begin{aligned} v_1(t) &= V_{m1}\cos(\omega_o t + \varphi_1) \\ &= \text{Re}\left\{V_1\, e^{j\omega_o t}\right\} \end{aligned}$$

then:

$$v_2(t) = \int_{-\infty}^{t} g(t - t')\, Re\left\{V_1\, e^{j\omega_o t'}\right\} dt'$$

Now, since the impulse function $g(t)$ is **real-valued** (this is really important!) it **can** be shown that:

$$v_2(t) = \int_{-\infty}^{t} g(t-t')\, Re\{V_1\, e^{j\omega_o t'}\}\, dt'$$

$$= Re\left\{\int_{-\infty}^{t} g(t-t')\, V_1\, e^{j\omega_o t'}\, dt'\right\}$$

Now, applying what we have **previously** learned;

$$v_2(t) = Re\left\{\int_{-\infty}^{t} g(t-t')\, V_1\, e^{j\omega_o t'}\, dt'\right\}$$

$$= Re\left\{V_1 \int_{-\infty}^{t} g(t-t')\ e^{j\omega_o t'}\, dt'\right\}$$

$$= Re\{V_1\, G(\omega_0)\, e^{j\omega_o t}\}$$

Thus, we **finally** can conclude the real-valued output $v_2(t)$ due to the **real-valued input**:

$$v_1(t) = V_{m1} \cos(\omega_o t + \varphi_1)$$

$$= \text{Re}\{V_1\, e^{j\omega_o t}\}$$

is:

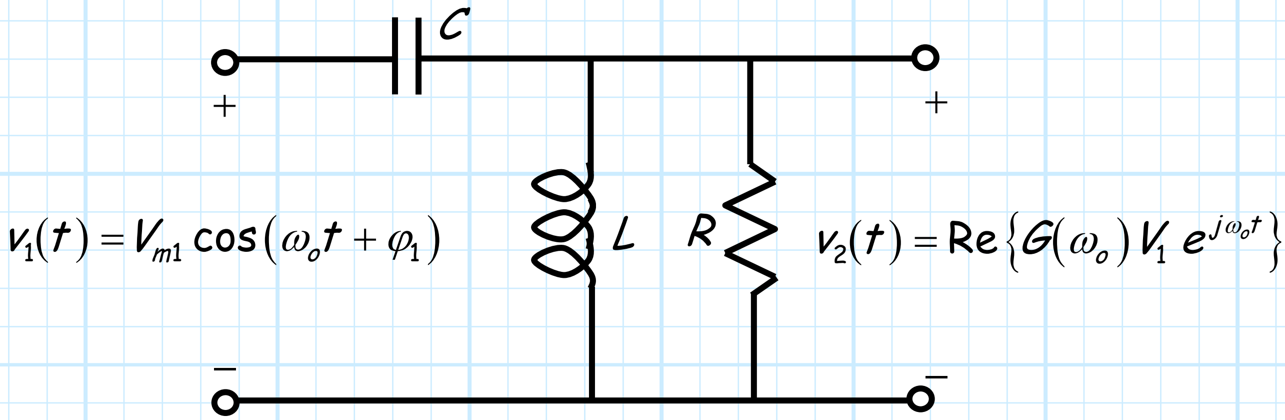
$$v_2(t) = Re\{V_2\ e^{j\omega_o t}\}$$

$$= V_{m2} \cos(\omega_o t + \varphi_2)$$

where:

$$V_2 = G(\omega_o)\, V_1$$

The **really important** result here is the last one!



The magnitude and phase of the **output** sinusoid (expressed as **complex** value V_2) is related to the magnitude and phase of the **input** sinusoid (expressed as **complex** value V_1) by the system **eigen value** $G(\omega_o)$:

$$\frac{V_2}{V_1} = G(\omega_o)$$

Therefore we find that **really** often in electrical engineering, we:

1. Use sinusoidal (i.e., eigen function) sources.

2. Express the voltages and currents created by these sources as complex **values** (i.e., **not** as real functions of time)!

For example, we might say " $V_3 = 2.0$", meaning:

$$V_3 = 2.0 = 2.0\, e^{j0} \quad \Rightarrow \quad v_3(t) = \operatorname{Re}\{2.0\, e^{j0} e^{j\omega_o t}\} = 2.0 \cos \omega_o t$$

Or "$I_L = -3.0$", meaning:

$$I_L = -2.0 = 3.0\, e^{j\pi} \quad \Rightarrow \quad i_L(t) = \text{Re}\left\{3.0\, e^{j\pi} e^{j\omega_o t}\right\} = 3.0\cos\left(\omega_o t + \pi\right)$$

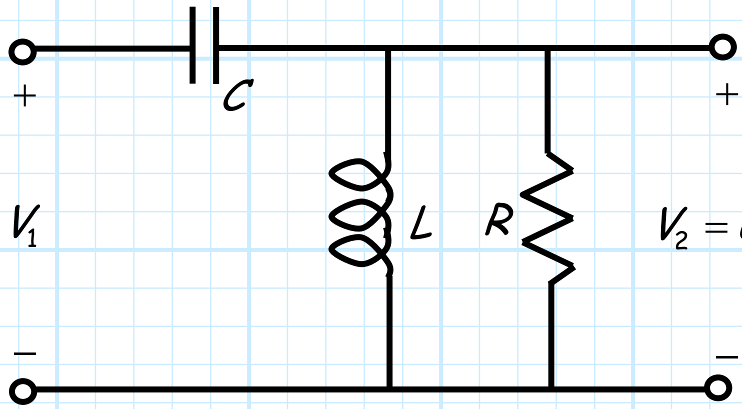
Or "$V_s = j$", meaning:

$$V_s = j = 1.0\, e^{j\left(\pi/2\right)} \quad \Rightarrow \quad v_s(t) = \text{Re}\left\{1.0\, e^{j\left(\pi/2\right)} e^{j\omega_o t}\right\} = 1.0\cos\left(\omega_o t + \pi/2\right)$$

* Remember, if a linear circuit is excited by a sinusoid (e.g., **eigen function** $\exp\left[j\omega_0 t\right]$), then the **only** unknowns are the magnitude and phase of the sinusoidal **currents** and **voltages** associated with **each element** of the circuit.

* These unknowns are **completely** described by complex values, as complex values **likewise** have a magnitude and phase.

* We can always **"recover"** the **real-valued** voltage or current function by multiplying the complex value by $\exp\left[j\omega_0 t\right]$ and then taking the real part, but typically we don't—after all, **no** new or unknown information is revealed by this operation!



Analysis of Circuits Driven by Arbitrary Functions

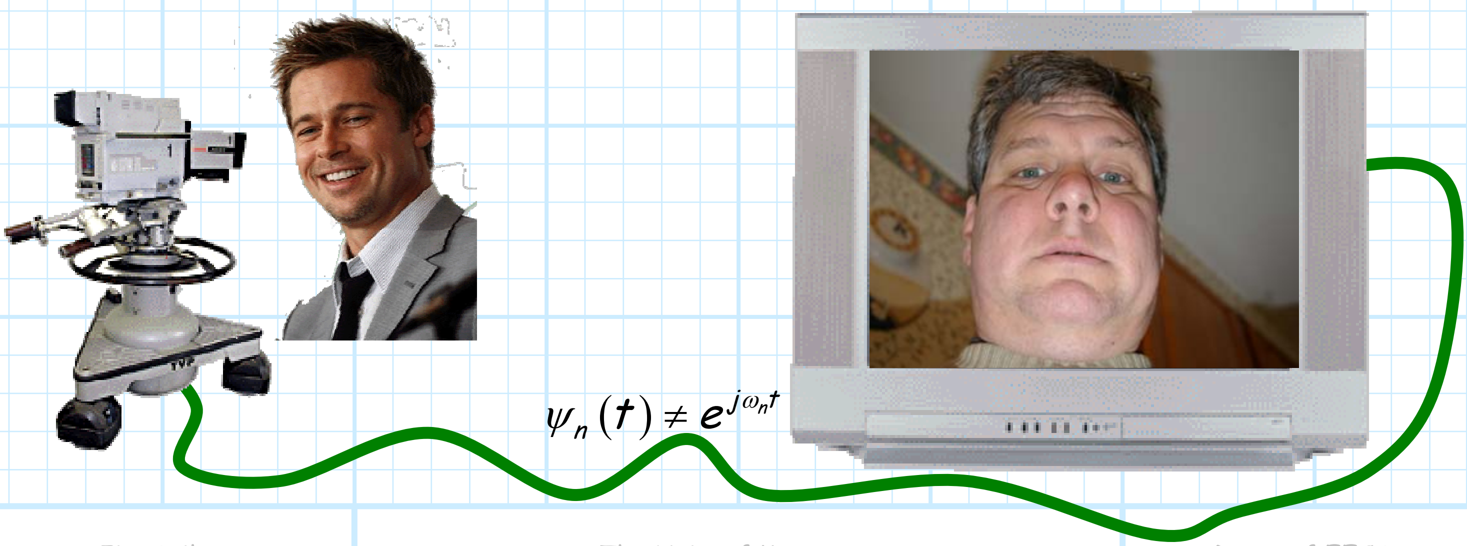
Q: *What happens if a linear circuit is excited by some function that is **not** an "eigen function"? Isn't limiting our analysis to sinusoids **too restrictive**?*

A: Not as restrictive as you might think.

Because sinusoidal functions are the eigen-functions of linear, time-invariant systems, they have become **fundamental** to much of our electrical engineering infrastructure—particularly with regard to **communications**.

For example, every radio and TV station is assigned its **very own eigen function** (i.e., its own frequency ω)!

It is **very** important that we use eigen functions for electromagnetic communication, otherwise the **received** signal might look **very** different from the one that was **transmitted**!



With sinusoidal functions (being eigen functions and all), we **know** that receive function will have **precisely** the same form as the one transmitted (albeit quite a bit **smaller**).

Thus, our assumption that a linear circuit is excited by a sinusoidal function is often a very **accurate** and **practical** one!

Q: *Still, we often find a circuit that is **not** driven by a sinusoidal source. How would we analyze **this** circuit?*

A: Recall the property of **linear operators**:

$$\mathcal{L}\left[a\, y_1 + b\, y_2\right] = a\,\mathcal{L}\left[y_1\right] + b\,\mathcal{L}\left[y_2\right]$$

We now know that we can **expand** the function:

$$v(t) = a_0\,\psi_0(t) + a_1\,\psi_1(t) + a_2\,\psi_2(t) + \cdots = \sum_{n=-\infty}^{\infty} a_n\,\psi_n(t)$$

and we found that:

$$\mathcal{L}\left[v(t)\right] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n\,\psi_n(t)\right] = \sum_{n=-\infty}^{\infty} a_n\,\mathcal{L}\left[\psi_n(t)\right]$$

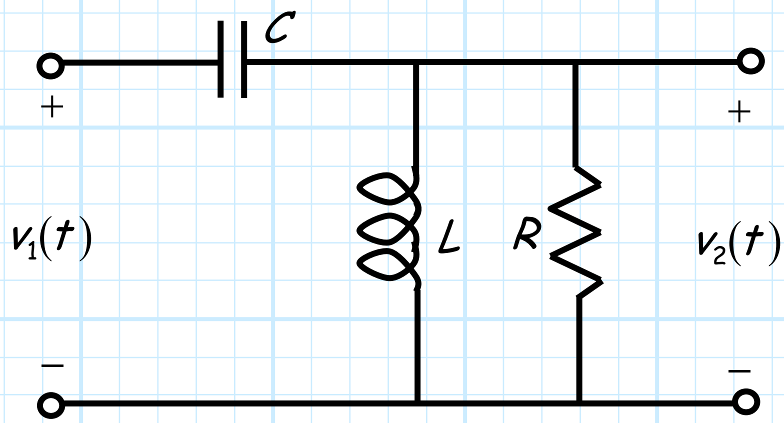
Finally, we found that any linear operation $\mathcal{L}[\psi_n(t)]$ is greatly simplified **if** we choose as our basis function the **eigen function** of linear systems:

$$\psi_n(t) = \begin{cases} e^{j\omega_n t} & for \quad 0 \le t \le T \\ 0 & for \quad t < 0, t > T \end{cases} \qquad \text{where} \qquad \omega_n = n\left(\frac{2\pi}{T}\right)$$

so that:

$$\mathcal{L}\left[\psi_n(t)\right] = G(\omega_n)\, e^{j\omega_n t}$$

Thus, for the example:



We follow these **analysis steps**:

1. **Expand** the input function $v_1(t)$ using the basis functions $\psi_n(t) = exp\left[j\omega_n t\right]$:

$$v_1(t) = V_{01}\, e^{j\omega_0 t} + V_{11}\, e^{j\omega_1 t} + V_{21}\, e^{j\omega_2 t} + \cdots = \sum_{n=-\infty}^{\infty} V_{n1}\, e^{j\omega_n t}$$

where:

$$V_{n1} = \frac{1}{T}\int_0^T v_1(t)\, e^{-j\omega_n t}\, dt$$

2. Evaluate the **eigen values** of the linear system:

$$G(\omega_n) = \int_0^\infty g(t)\, e^{-j\omega_n t}\, dt$$

3. Perform the **linear operaton** (the convolution integral) that relates $v_2(t)$ to $v_1(t)$:

$$\begin{aligned} v_2(t) &= \mathcal{L}\left[v_1(t)\right] \\ &= \mathcal{L}\left[\sum_{n=-\infty}^{\infty} V_{n1}\, e^{j\omega_n t}\right] \\ &= \sum_{n=-\infty}^{\infty} V_{n1}\, \mathcal{L}\left[e^{j\omega_n t}\right] \\ &= \sum_{n=-\infty}^{\infty} V_{n1}\, G(\omega_n)\, e^{j\omega_n t} \end{aligned}$$

Summarizing:

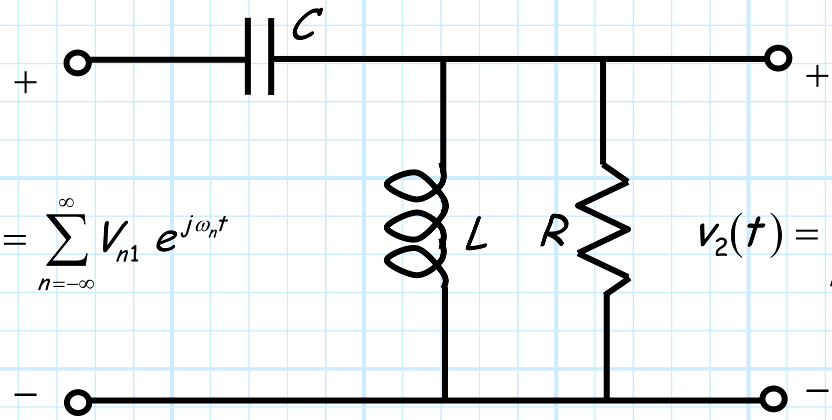
$$v_2(t) = \sum_{n=-\infty}^{\infty} V_{n2}\, e^{j\omega_n t}$$

where:

$$V_{n2} = G(\omega_n)\, V_{n1}$$

and:

$$V_{n1} = \frac{1}{T}\int_0^T v_1(t)\, e^{-j\omega_n t} dt \qquad G(\omega_n) = \int_0^\infty g(t)\, e^{-j\omega_n t}\, dt$$



$$v_1(t) = \sum_{n=-\infty}^{\infty} V_{n1}\, e^{j\omega_n t} \qquad v_2(t) = \sum_{n=-\infty}^{\infty} G_1(\omega_n)\, V_{n1}\, e^{j\omega_n t}$$

As stated earlier, the signal expansion used here is the **Fourier Series**.

Say that the **timewidth** T of the signal $v_1(t)$ becomes **infinite**. In this case we find our analysis becomes:

$$v_2(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} V_2(\omega)\, e^{j\omega t} d\omega$$

where:

$$V_2(\omega) = G(\omega)\, V_1(\omega)$$

and:

$$V_1(\omega) = \int_{-\infty}^{+\infty} v_1(t) e^{-j\omega t} dt \qquad G(\omega) = \int_{-\infty}^{+\infty} g(t)\, e^{-j\omega t}\, dt$$

The signal expansion in this case is the **Fourier Transform**.

We find that as $T \to \infty$ the number of **discrete** system eigen values $G(\omega_n)$ become so numerous that they form a **continuum**—$G(\omega)$ is a **continuous** function of frequency ω.

We thus call the function $G(\omega)$ the **eigen spectrum** or **frequency response** of the circuit.

Q: *You* ***claim*** *that all this fancy mathematics (e.g., eigen functions and eigen values) make analysis of linear systems and circuits much* ***easier****, yet to apply these techniques, we must* ***determine*** *the eigen values or eigen spectrum:*

$$G(\omega_n) = \int_0^{\infty} g(t)\, e^{-j\omega_n t}\, dt \qquad G(\omega) = \int_{-\infty}^{+\infty} g(t)\, e^{-j\omega t}\, dt$$

Neither *of these operations look* ***at all*** *easy. And in addition to performing the integration, we must* ***somehow*** *determine the* ***impulse function*** $g(t)$ *of the linear system as well !*

Just how are we supposed to do ***that****?*

A: An insightful question! Determining the impulse response $g(t)$ and then the frequency response $G(\omega)$ **does** appear to be **exceedingly** difficult—and for many linear systems it indeed **is**!

However, much to our great **relief**, we can determine the eigen spectrum $G(\omega)$ of linear circuits **without** having to perform a difficult integration. In fact, we **don't** even need to know the impulse response $g(t)$!

The Eigen Spectrum of Linear Circuits

Recall the linear operators that define a **capacitor**:

$$\mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] = i_C(t) = C\,\frac{d\,v_C(t)}{d\,t}$$

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[i_C(t)\right] = v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\,dt'$$

We now know that the **eigen function** of these linear, time-invariant operators—like **all** linear, time-invariant operartors—is $\exp[j\omega t]$.

The question now is, **what** is the **eigen spectrum** of each of these operators? It is this spectrum that **defines** the physical behavior of a given capacitor!

For $v_C(t) = \exp[j\omega t]$, we find:

$$\begin{aligned} i_C(t) &= \mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] \\ &= C\,\frac{d\,e^{j\omega t}}{d\,t} \\ &= (j\omega C)\,e^{j\omega t} \end{aligned}$$

Just as we expected, the eigen function $\exp[j\omega t]$ **"survives"** the linear operation **unscathed**—the current function $i(t)$ has **precisely** the same form as the voltage function $v(t) = \exp[j\omega t]$.

The **only** difference between the **current** and **voltage** is the multiplication of the **eigen spectrum**, denoted as $\mathcal{G}_{\mathcal{Y}}^{C}(\omega)$.

$$i(t) = \mathcal{L}_{\mathcal{Y}}^{C}\left[v(t) = e^{j\omega t}\right] = \mathcal{G}_{\mathcal{Y}}^{C}(\omega)\, e^{j\omega t}$$

Since we **just** determined that for this case:

$$i(t) = \left(j\omega C\right) e^{j\omega t}$$

it is **evident** that the eigen spectrum of the linear operation:

$$i(t) = \mathcal{L}_{\mathcal{Y}}^{C}\left[v(t)\right] = C\,\frac{d\,v(t)}{d\,t}$$

is:

$$\mathcal{G}_{\mathcal{Y}}^{C}(\omega) = j\omega C = \omega C\, e^{j\pi/2} \quad !!!$$

So for **example**, if:

$$\begin{aligned} v(t) &= V_m \cos\left(\omega_o t + \varphi\right) \\ &= \text{Re}\left\{\left(V_m\, e^{j\varphi}\right) e^{j\omega_o t}\right\} \end{aligned}$$

we will find that:

$$\mathcal{L}_{\mathcal{Y}}^{C}\left[\left(V_m\, e^{j\varphi}\right)e^{j\omega_o t}\right] = G_{\mathcal{Y}}^{C}(\omega_o)\left(V_m\, e^{j\varphi}\right)e^{j\omega_o t}$$
$$= \left(\omega C\, e^{j\pi/2}\right)\left(V_m\, e^{j\varphi}\right)e^{j\omega_o t}$$
$$= \left(\omega C\, V_m\, e^{j\left(\pi/2+\varphi\right)}\right)e^{j\omega_o t}$$

Therefore:

$$i_C(t) = \text{Re}\left\{\omega C\, V_m e^{j\left(\varphi+\pi/2\right)}e^{j\omega_o t}\right\}$$
$$= \omega C\ V_m \cos\left(\omega_o t + \varphi + \pi/2\right)$$
$$= -\omega C\ V_m \sin\left(\omega_o t + \varphi\right)$$

Hopefully, this example again emphasizes that these **real-valued** sinusoidal functions can be completely expressed in terms of **complex values**. For example, the complex value:

$$V_C = V_m e^{j\varphi}$$

means that the magnitude of the sinusoidal **voltage** is $|V_C| = V_m$, and its relative phase is $\angle V_C = \varphi$.

The complex value:

$$I_C = G_{\mathcal{Y}}^{C}(\omega)\, V_C$$
$$= \left(\omega C\, e^{j\pi/2}\right)V_C$$

likewise means that the **magnitude** of the sinusoidal **current** is:

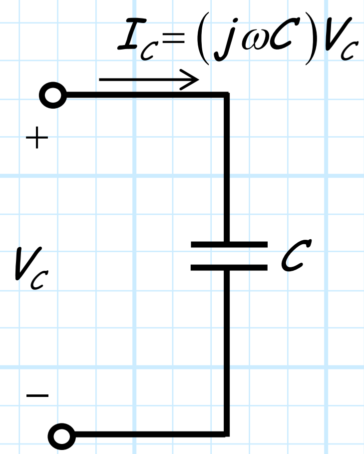
$$
\begin{aligned}
\left|I_C\right| &= \left|G_y^C(\omega)\,V_C\right| \\
&= \left|G_y^C(\omega)\right|\left|V_C\right| \\
&= \omega C\, V_m
\end{aligned}
$$

And the relative **phase** of the sinusoidal **current** is:

$$
\begin{aligned}
\angle I_C &= \angle G_y^C(\omega) + \angle V_C \\
&= \pi/2 + \varphi
\end{aligned}
$$

We can thus **summarize** the behavior of a capacitor with the simple **complex equation**:

$$
\begin{aligned}
I_C &= \left(j\omega C\right) V_C \\
&= \left(\omega C\, e^{j\pi/2}\right) V_C
\end{aligned}
$$



Now let's return to the **second** of the two linear operators that describe a capacitor:

$$
v_C(t) = \mathcal{L}_z^C\left[i_C(t)\right] = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\,dt'
$$

Now, if the capacitor **current** is the eigen function $i_C(t) = \exp[j\omega t]$, we find:

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[e^{j\omega t}\right]=\frac{1}{C}\int_{-\infty}^{t}e^{j\omega t'}\,dt'$$
$$=\left(\frac{1}{j\omega C}\right)e^{j\omega t}$$

where we assume $i(t=-\infty)=0$.

Thus, we can conclude that:

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[e^{j\omega t}\right]=G_{\mathcal{Z}}^{C}(\omega)\,e^{j\omega t}=\left(\frac{1}{j\omega C}\right)e^{j\omega t}$$

Hopefully, it is evident that the **eigen spectrum** of this linear operator is:

$$G_{\mathcal{Z}}^{C}(\omega)=\frac{1}{j\omega C}=\frac{-j}{\omega C}=\frac{1}{\omega C}e^{j\left(3\pi/2\right)}$$

And so:

$$V_C=\left(\frac{1}{j\omega C}\right)I_C$$

Q: *Wait a second! Isn't this essentially the* ***same*** *result as the one derived for operator* $\mathcal{L}_{\mathcal{Y}}^{C}$ *??*

A: It's **precisely** the same! For both operators we find:

$$\frac{V_C}{I_C}=\frac{1}{j\omega C}$$

This should **not** be surprising, as **both** operators $\mathcal{L}_y^C$ and $\mathcal{L}_z^C$ relate the current through and voltage across the **same** device (a capacitor).

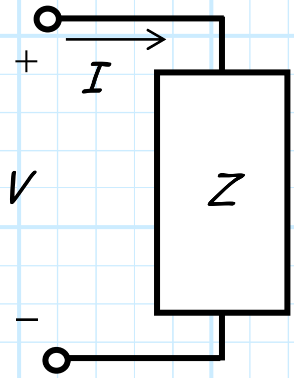
The **ratio** of complex voltage to complex current is of course referred to as the complex device **impedance** Z.

$$Z \doteq \frac{V}{I}$$

An **impedance** can be determined for **any** linear, time-invariant **one-port** network—but **only** for linear, time-invariant one-port networks!

Generally speaking, impedance is a **function of frequency**. In fact, the impedance of a one-port network is simply the **eigen spectrum** $G_z(\omega)$ of the linear operator $\mathcal{L}_z$:

$$V = Z\, I$$



$$\mathcal{L}_z\left[i(t)\right] = v(t)$$

$$Z = G_z(\omega)$$

Note that impedance is a **complex** value that provides us with **two** things:

1. The **ratio of the magnitudes** of the sinusoidal voltage and current:

$$|Z| = \frac{|V|}{|I|}$$

2. The **difference in phase** between the sinusoidal voltage and current:

$$\angle Z = \angle V - \angle I$$

Q: *What about the linear operator:*

$$\mathcal{L}_{\mathcal{Y}}\left[v(t)\right] = i(t) \quad ??$$

A: Hopefully it is now evident to **you** that:

$$G_{\mathcal{Y}}(\omega) = \frac{1}{G_{\mathcal{Z}}(\omega)} = \frac{1}{Z}$$

The inverse of impedance is **admittance** Y:

$$Y \doteq \frac{1}{Z} = \frac{I}{V}$$

Now, returning to the **other two** linear circuit elements, we find (and **you** can verify) that for resistors:

$$\mathcal{L}_{\mathcal{Y}}^{R}\left[v_R(t)\right] = i_R(t) \quad \Rightarrow G_{\mathcal{Y}}^{R}(\omega) = 1/R$$

$$\mathcal{L}_{\mathcal{Z}}^{R}\left[i_R(t)\right] = v_R(t) \quad \Rightarrow G_{\mathcal{Z}}^{R}(\omega) = R$$

and for inductors:

$$\mathcal{L}_{\mathcal{Y}}^{L}\left[v_L(t)\right] = i_L(t) \quad \Rightarrow \mathcal{G}_{\mathcal{Y}}^{L}(\omega) = \frac{1}{j\omega L}$$

$$\mathcal{L}_{\mathcal{Z}}^{L}\left[i_L(t)\right] = v_L(t) \quad \Rightarrow \mathcal{G}_{\mathcal{Z}}^{L}(\omega) = j\omega L$$

meaning:

$$Z_R = \frac{1}{Y_R} = R = R\, e^{j0} \qquad \text{and} \qquad Z_L = \frac{1}{Y_L} = j\omega L = \omega L\, e^{j(\pi/2)}$$

Now, note that the relationship

$$Z = \frac{V}{I}$$

forms a **complex "Ohm's Law"** with regard to complex currents and voltages.

Additionally, ICBST (It Can Be Shown That) **Kirchoff's Laws** are likewise valid for complex currents and voltages:

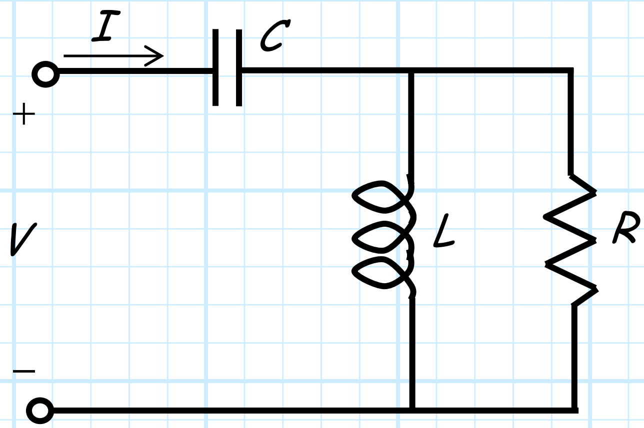
$$\sum_n I_n = 0 \qquad\qquad \sum_n V_n = 0$$

where of course the summation represents **complex addition**.

As a result, the impedance (i.e., the eigen spectrum) of **any** one-port device can be determined by simply applying a **basic** knowledge of **linear circuit analysis**!

Returning to the example:

$$Z = \frac{V}{I}$$



And thus using out **basic** circuits knowledge, we find:

$$Z = Z_C + Z_R \| Z_L = 1/_{j\omega C} + R \| j\omega L$$

Thus, the eigen spectrum of the linear operator:

$$\mathcal{L}_Z\left[i(t)\right] = v(t)$$

For **this** one-port network is:

$$G_Z(\omega) = 1/_{j\omega C} + R \| j\omega L$$

Look what we did! We were able to determine $G_Z(\omega)$ **without** explicitly determining impulse response $g_Z(t)$, or having to perform **any** integrations!

Now, if we actually **need** to determine the voltage function $v(t)$ created by some **arbitrary** current function $i(t)$, we integrate:

$$\begin{aligned} v(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_{\mathcal{Z}}(\omega)\, I(\omega)\, e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\tfrac{1}{j\omega C} + R \| j\omega L \right) I(\omega)\, e^{j\omega t} d\omega \end{aligned}$$

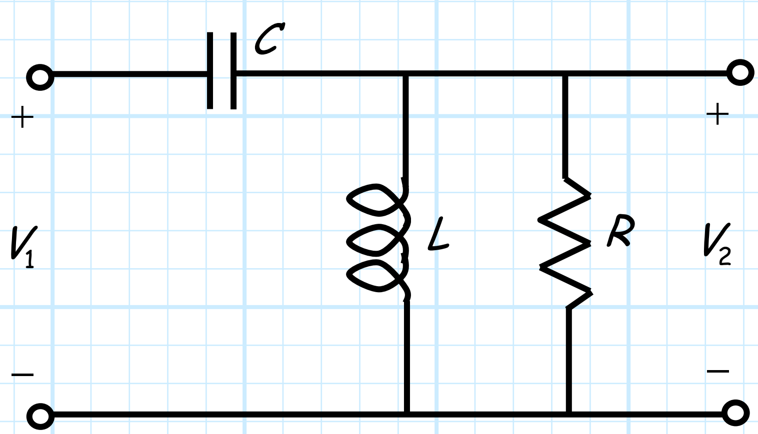
where:

$$I(\omega) = \int_{-\infty}^{+\infty} i(t)\, e^{-j\omega t} dt$$

Otherwise, if our current function is **time-harmonic** (i.e., sinusoidal with frequency ω), we can simply relate complex current I and complex voltage V with the equation:

$$\begin{aligned} V &= Z\, I \\ &= \left(\tfrac{1}{j\omega C} + R \| j\omega L \right) I \end{aligned}$$

Similarly, for our **two-port** example:



we can likewise determine from **basic** circuit theory the **eigen spectrum** of linear operator:

$$\mathcal{L}_{21}\left[v_1(t)\right] = v_2(t)$$

is:

$$G_{21}(\omega) = \frac{Z_L \| Z_R}{Z_C + Z_L \| Z_R} = \frac{j\omega L \| R}{\dfrac{1}{j\omega C} + j\omega L \| R}$$

so that:

$$V_2 = G_{21}(\omega)\, V_1$$

or more generally:

$$v_2(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G_{21}(\omega)\, V_1(\omega)\, e^{j\omega t}\, d\omega$$

where:

$$V_1(\omega) = \int_{-\infty}^{+\infty} v_1(t)\, e^{-j\omega t}\, dt$$

Finally, a few important **definitions** involving impedance and admittance:

$$\text{Re}\{Z\} \doteq \text{Resistance} \quad R$$

$$\text{Im}\{Z\} \doteq \text{Reactance} \quad X$$

$$\text{Re}\{Y\} \doteq \text{Admittance } G$$

$$\text{Im}\{Y\} \doteq \text{Susceptance } B$$

Therefore:

$$Z = R + jX \qquad\qquad Y = G + jB$$

But be **careful**!

Although:

$$Y = G + jB = \frac{1}{R + jX} = \frac{1}{Z}$$

keep in mind that:

$$G \neq \frac{1}{R} \qquad \text{and} \qquad B \neq \frac{1}{X}$$