4.3 – The Scattering Matrix

Reading Assignment: ***pp. 174-183***

Admittance and Impedance matrices use the quantities $I(z)$, $V(z)$, and $Z(z)$ (or $Y(z)$).

Q: *Is there an **equivalent** matrix for transmission line activity expressed in terms of $V^+(z)$, $V^-(z)$, and $\Gamma(z)$?*

A: Yes! Its called the **scattering matrix**.

HO: THE SCATTERING MATRIX

Q: *Can we likewise determine something **physical** about our device or network by simply **looking** at its scattering matrix?*

A: HO: MATCHED, RECIPROCAL, LOSSLESS

EXAMPLE: A LOSSLESS, RECIPROCAL DEVICE

Q: *Isn't all this linear algebra a bit **academic**? I mean, it can't help us design components, **can it?***

A: It sure can! An analysis of the scattering matrix can tell us if a certain device is **even possible** to construct, and if so, what the **form** of the device must be.

HO: THE MATCHED, LOSSLESS, RECIPROCAL 3-PORT NETWORK

HO: THE MATCHED, LOSSLESS, RECIPROCAL 4-PORT NETWORK

Q: *But how are scattering parameters **useful?** How do we use them to **solve or analyze** real microwave circuit problems?*

A: Study the **examples** provided below!

EXAMPLE: THE SCATTERING MATRIX

EXAMPLE: SCATTERING PARAMETERS

Q: *OK, but how can we **determine** the scattering matrix of a device?*

A: We must carefully apply our **transmission line theory**!

EXAMPLE: DETERMINING THE SCATTERING MATRIX

Q: *Determining the Scattering Matrix of a multi-port device would seem to be particularly laborious. Is there any way to simplify the process?*

A: Many (if not most) of the useful devices made by us humans exhibit a high degree of **symmetry**. This can greatly **simplify** circuit analysis—if we **know how** to exploit it!

HO: CIRCUIT SYMMETRY

EXAMPLE: USING SYMMETRY TO DETERMINING A SCATTERING MATRIX

Q: *Is there any **other** way to use circuit symmetry to our advantage?*

A: Absolutely! One of the most **powerful** tools in circuit analysis is **Odd-Even Mode** analysis.

HO: SYMMETRIC CIRCUIT ANALYSIS

HO: ODD-EVEN MODE ANALYSIS

EXAMPLE: ODD-EVEN MODE CIRCUIT ANALYSIS

Q: *Aren't you **finished** with this section yet?*

A: Just **one more** very important thing.

HO: GENERALIZED SCATTERING PARAMETERS

EXAMPLE: THE SCATTERING MATRIX OF A CONNECTOR

The Scattering Matrix

At **"low"** frequencies, we can completely characterize a **linear** device or network using an **impedance** matrix, which relates the currents and voltages at **each** device terminal to the currents and voltages at **all** other terminals.

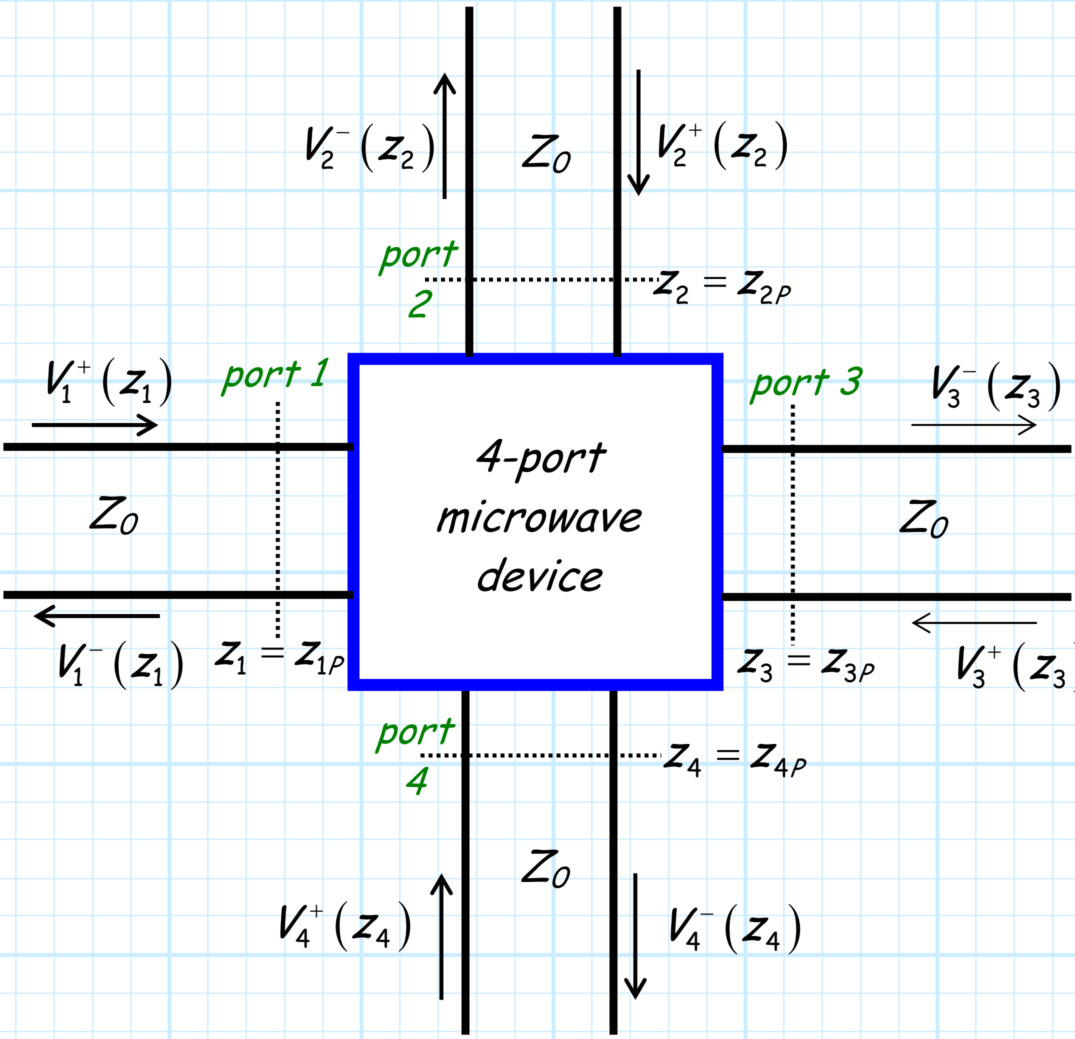
But, at microwave frequencies, it is **difficult** to measure total currents and voltages!

* Instead, we can measure the **magnitude** and **phase** of each of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.

* In other words, we can determine the relationship between the incident and reflected wave at **each** device terminal to the incident and reflected waves at **all** other terminals.

These relationships are completely represented by the **scattering matrix**. It **completely** describes the behavior of a linear, multi-port device at a **given frequency** ω, and a given line impedance Z_0.

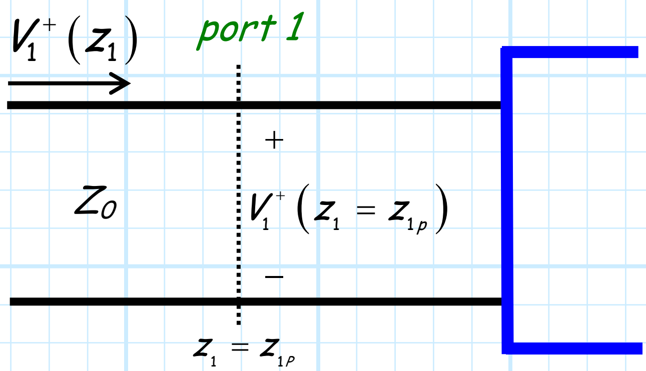
Consider now the **4-port** microwave device shown below:



Note that we have now characterized transmission line activity in terms of incident and "reflected" waves. Note the negative going "reflected" waves can be viewed as the waves **exiting** the multi-port network or device.

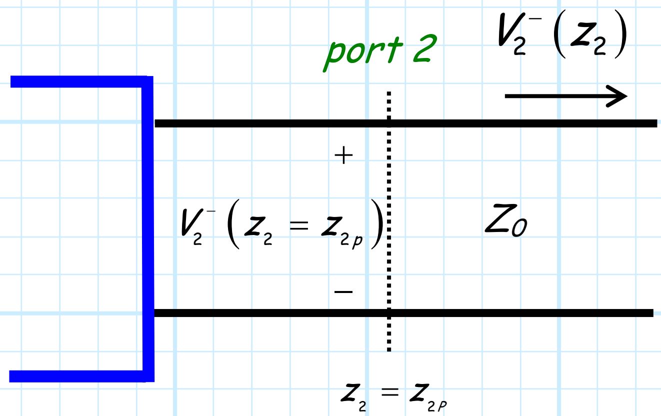
→ Viewing transmission line activity this way, we can fully characterize a multi-port device by its **scattering parameters**!

Say there exists an **incident** wave on **port 1** (i.e., $V_1^+(z_1) \neq 0$), while the incident waves on all other ports are known to be **zero** (i.e., $V_2^+(z_2) = V_3^+(z_3) = V_4^+(z_4) = 0$).



Say we measure/determine the voltage of the wave flowing **into port 1**, at the port 1 **plane** (i.e., determine $V_1^+(z_1 = z_{1P})$).

Say we then measure/determine the voltage of the wave flowing **out** of **port 2**, at the port 2 plane (i.e., determine $V_2^-(z_2 = z_{2P})$).



The complex ratio between $V_1^+(z_1 = z_{1P})$ and $V_2^-(z_2 = z_{2P})$ is know as the **scattering parameter** S_{21}:

$$S_{21} = \frac{V_2^-(z_2 = z_{2P})}{V_1^+(z_1 = z_{1P})} = \frac{V_{02}^- \, e^{+j\beta z_{2P}}}{V_{01}^+ \, e^{-j\beta z_{1P}}} = \frac{V_{02}^-}{V_{01}^+} \; e^{+j\beta(z_{2P}+z_{1P})}$$

Likewise, the scattering parameters S_{31} and S_{41} are:

$$S_{31} = \frac{V_3^-(z_3 = z_{3P})}{V_1^+(z_1 = z_{1P})} \qquad \text{and} \qquad S_{41} = \frac{V_4^-(z_4 = z_{4P})}{V_1^+(z_1 = z_{1P})}$$

We of course could **also** define, say, scattering parameter S_{34} as the ratio between the complex values $V_4^+(z_4 = z_{4P})$ (the wave **into** port 4) and $V_3^-(z_3 = z_{3P})$ (the wave **out of** port 3), given that the input to all other ports (1,2, and 3) are zero.

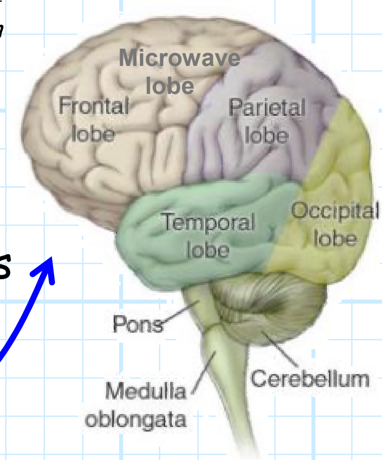
Thus, more **generally**, the ratio of the wave incident on port n to the wave emerging from port m is:

$$S_{mn} = \frac{V_m^-(z_m = z_{mP})}{V_n^+(z_n = z_{nP})} \qquad \text{(given that } V_k^+(z_k) = 0 \text{ for all } k \neq n\text{)}$$

Note that frequently the port positions are assigned a **zero** value (e.g., $z_{1P} = 0$, $z_{2P} = 0$). This of course **simplifies** the scattering parameter calculation:

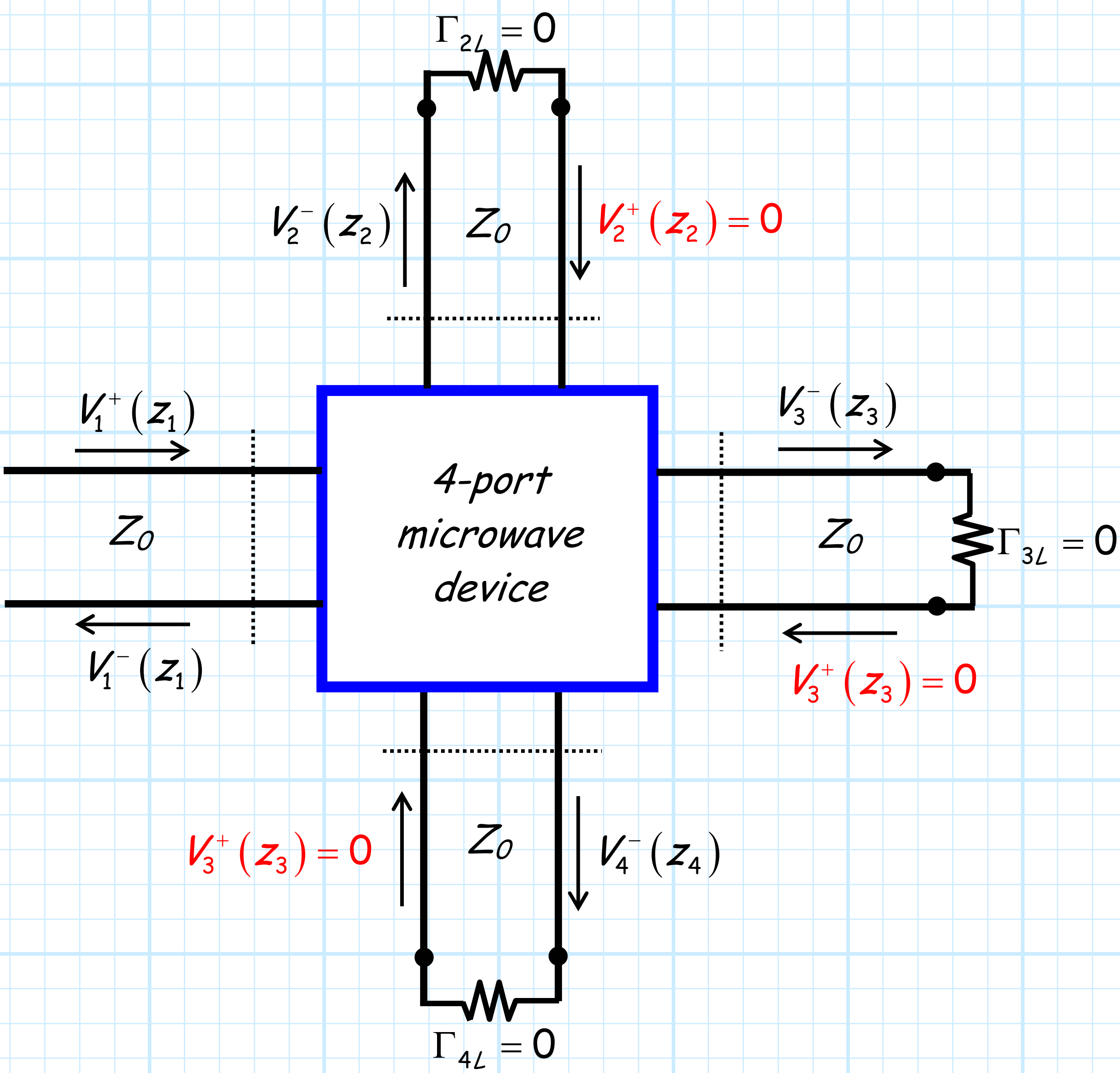
$$S_{mn} = \frac{V_m^-(z_m = 0)}{V_n^+(z_n = 0)} = \frac{V_{0m}^- e^{+j\beta 0}}{V_{0n}^+ e^{-j\beta 0}} = \frac{V_{0m}^-}{V_{0n}^+}$$

We will **generally assume** that the port locations are defined as $z_{nP} = 0$, and thus use the **above** notation. But **remember** where this expression came from!



Q: *But how do we ensure that **only one** incident wave is non-zero ?*

A: **Terminate** all other ports with a **matched load**!



Note that **if** the ports are terminated in a **matched load** (i.e., $Z_L = Z_0$), then $\Gamma_{nL} = 0$ and therefore:

$$V_n^+(z_n) = 0$$

In other words, terminating a port ensures that there will be **no signal** incident on that port!

Q: *Just between you and me, I think you've messed this up! **In all** previous handouts you said that if $\Gamma_L = 0$, the wave in the **minus** direction would be zero:*

$$V^-(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

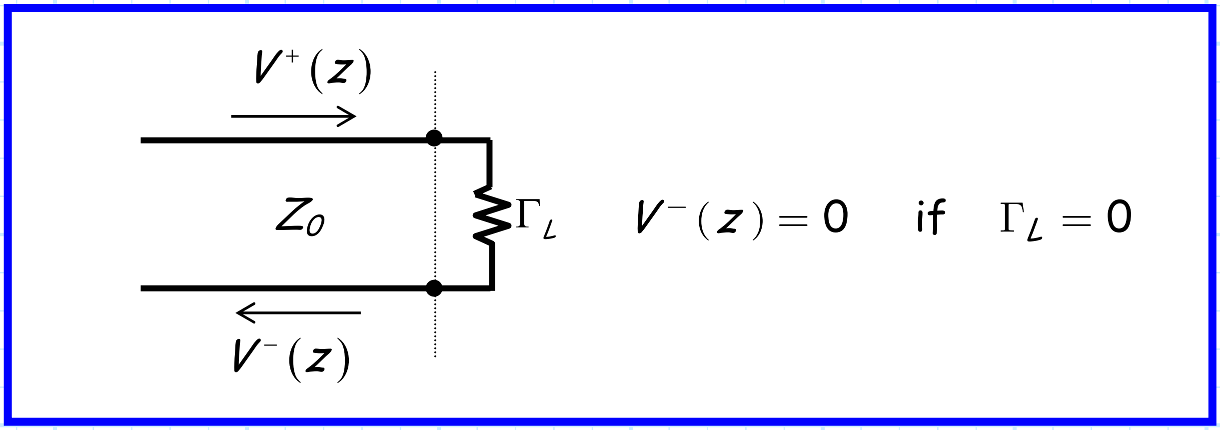
*but just **now** you said that the wave in the **positive** direction would be zero:*

$$V^+(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

*Of course, there is **no way** that **both** statements can be correct!*

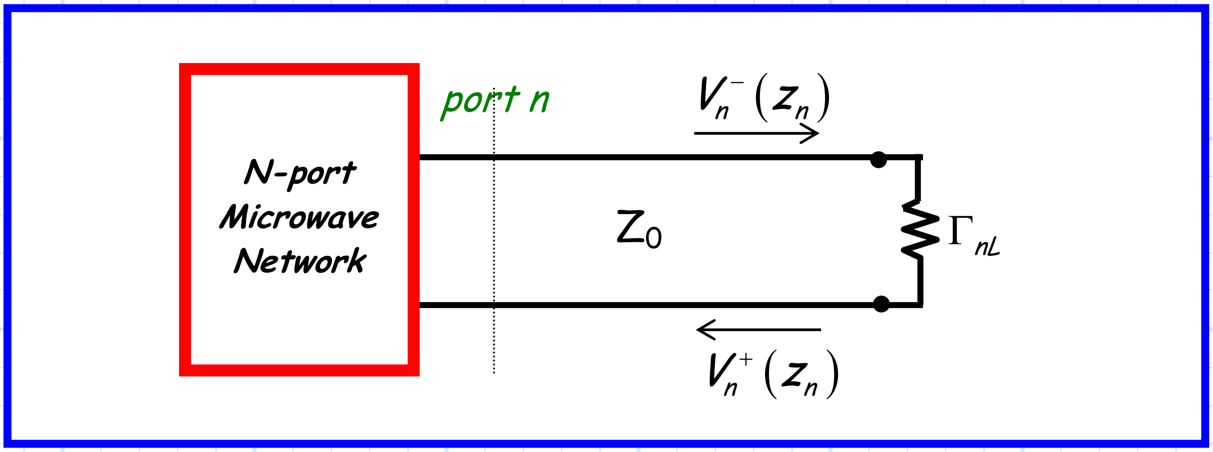
A: Actually, **both** statements are correct! You must be careful to understand the **physical definitions** of the plus and minus directions—in other words, the propagation directions of waves $V_n^+(z_n)$ and $V_n^-(z_n)$!

For example, we **originally** analyzed this case:



In this original case, the wave **incident** on the load is $V^+(z)$ (**plus** direction), while the **reflected** wave is $V^-(z)$ (**minus** direction).

Contrast this with the case we are **now** considering:

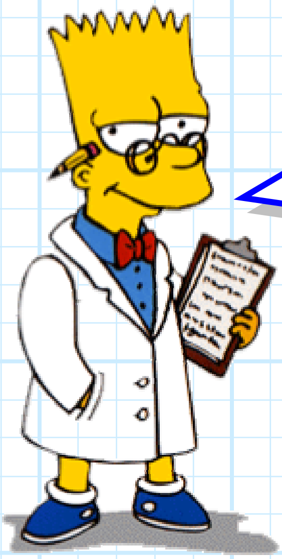


For this current case, the situation is **reversed**. The wave incident on the load is **now** denoted as $V_n^-(z_n)$ (coming **out** of port *n*), while the wave reflected off the load is **now** denoted as $V_n^+(z_n)$ (going **into** port *n*).

As a result, $V_n^+(z_n) = 0$ when $\Gamma_{nL} = 0$!

Perhaps we could more **generally** state that for some load Γ_L:

$$V^{reflected}(z = z_L) = \Gamma_L \, V^{incident}(z = z_L)$$



Now, **back** to our discussion of **S-parameters**. We found that **if** $z_{nP} = 0$ for all ports n, the scattering parameters could be directly written in terms of wave **amplitudes** V_{0n}^+ and V_{0m}^-.

$$S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \qquad \text{(when } V_k^+(z_k) = 0 \text{ for all } k \neq n\text{)}$$

Which we can now **equivalently** state as:

$$S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \qquad \text{(when all ports, except port } n\text{, are terminated in \textbf{matched loads})}$$

One more **important** note—notice that for the ports terminated in matched loads (i.e., those ports with **no** incident wave), the voltage of the exiting **wave** is also the **total** voltage!

$$\begin{aligned} V_m(z_m) &= V_{0m}^{+}\, e^{-j\beta z_n} + V_{0m}^{-}\, e^{+j\beta z_n} \\ &= 0 + V_{0m}^{-}\, e^{+j\beta z_m} \\ &= V_{0m}^{-}\, e^{+j\beta z_m} \qquad \text{(for all terminated ports)} \end{aligned}$$

Thus, the value of the exiting wave **at** each terminated **port** is likewise the value of the total voltage **at** those ports:

$$\begin{aligned} V_m(0) &= V_{0m}^{+} + V_{0m}^{-} \\ &= 0 + V_{0m}^{-} \\ &= V_{0m}^{-} \qquad \text{(for all terminated ports)} \end{aligned}$$

And so, we can express **some** of the scattering parameters equivalently as:

$$S_{mn} = \frac{V_m(0)}{V_{0n}^{+}} \qquad \text{(for \textbf{terminated} port } m\text{, i.e., for } m \neq n\text{)}$$

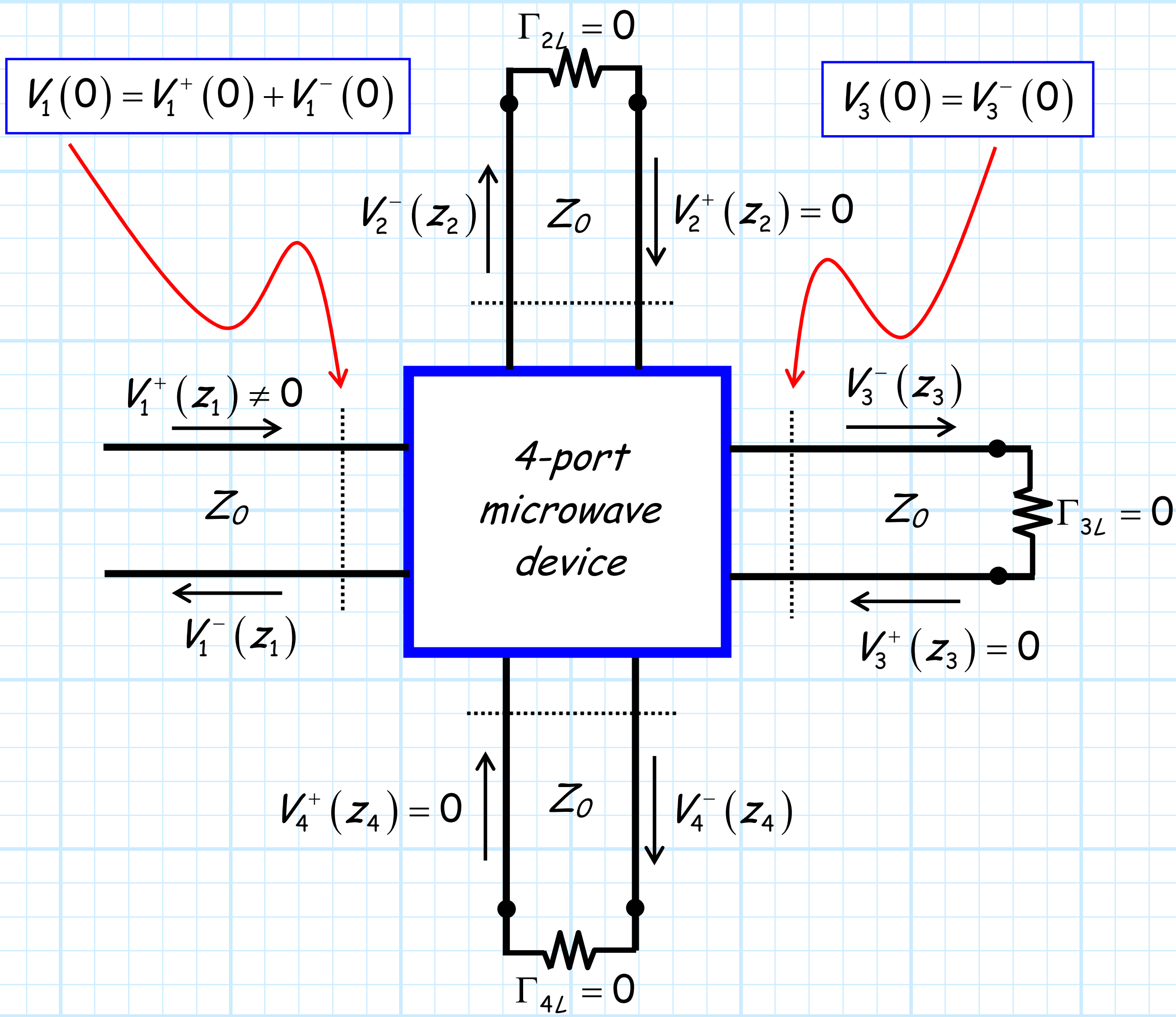
You might find this result **helpful** if attempting to determine scattering parameters where $m \neq n$ (e.g., S_{21}, S_{43}, S_{13}), as we can often use traditional **circuit theory** to easily determine the **total** port voltage $V_m(0)$.

However, we **cannot** use the expression above to determine the scattering parameters when $m = n$ (e.g., S_{11}, S_{22}, S_{33}).

Think about this! The scattering parameters for these cases are:

$$S_{nn} = \frac{V_{0n}^-}{V_{0n}^+}$$

Therefore, port n is a port where there actually **is** some incident wave V_{0n}^+ (port n is **not** terminated in a matched load!). And thus, the total voltage is **not** simply the value of the exiting wave, as **both** an incident wave and exiting wave exists at port n.



Typically, it is **much** more difficult to determine/measure the scattering parameters of the form S_{nn}, as opposed to scattering parameters of the form S_{mn} (where $m \neq n$) where there is **only** an **exiting** wave from port m!

We can use the scattering matrix to determine the solution for a more **general** circuit—one where the ports are **not** terminated in matched loads!

Q: *I'm not understanding the importance scattering parameters. How are they useful to us* ***microwave engineers?***

A: Since the device is **linear**, we can apply **superposition**. The output at any port due to **all** the incident waves is simply the coherent **sum** of the output at that port due to **each** wave!

For example, the **output** wave at port 3 can be determined by (assuming $z_{nP} = 0$):

$$V_{03}^{-} = S_{34} V_{04}^{+} + S_{33} V_{03}^{+} + S_{32} V_{02}^{+} + S_{31} V_{01}^{+}$$

More **generally**, the output at port m of an N-port device is:

$$V_{0m}^{-} = \sum_{n=1}^{N} S_{mn} V_{0n}^{+} \qquad (z_{nP} = 0)$$

This expression can be written in **matrix** form as:

$$\mathbf{V}^- = \mathcal{S}\ \mathbf{V}^+$$

Where $\mathbf{V}^-$ is the **vector**:

$$\mathbf{V}^- = \left[V_{01}^-, V_{02}^-, V_{03}^-, \ldots, V_{0N}^-\right]^T$$

and $\mathbf{V}^+$ is the vector:

$$\mathbf{V}^+ = \left[V_{01}^+, V_{02}^+, V_{03}^+, \ldots, V_{0N}^+\right]^T$$

Therefore $\mathcal{S}$ is the **scattering matrix**:

$$\mathcal{S} = \begin{bmatrix} \mathcal{S}_{11} & \cdots & \mathcal{S}_{1n} \\ \vdots & \ddots & \vdots \\ \mathcal{S}_{m1} & \cdots & \mathcal{S}_{mn} \end{bmatrix}$$

The scattering matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the scattering matrix describes a multi-port device the way that Γ_L describes a single-port device (e.g., a load)!

But **beware**! The values of the scattering matrix for a particular device or network, just like Γ_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\mathcal{S}(\omega)=\begin{bmatrix} S_{11}(\omega) & \cdots & S_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ S_{m1}(\omega) & \cdots & S_{mn}(\omega) \end{bmatrix}$$

Also realize that—also just like Γ_L—the scattering matrix is dependent on **both** the **device/network** and the Z_0 value of the **transmission lines connected** to it.

Thus, a device connected to transmission lines with $Z_0 = 50\Omega$ will have a **completely different scattering matrix** than that same device connected to transmission lines with $Z_0 = 100\Omega$!!!

Matched, Lossless, Reciprocal Devices

As we discussed earlier, a device can be **lossless** or **reciprocal**. In addition, we can likewise classify it as being **matched**.

Let's examine **each** of these three characteristics, and how they relate to the **scattering matrix**.

Matched

A matched device is another way of saying that the **input impedance** at each port is **equal to** Z_0 when **all other** ports are terminated in matched loads. As a result, the **reflection coefficient** of each port is **zero**—no signal will be come out of a port if a signal is incident on that port (but **only** that port!).

In other words, we want:

$$V_m^- = S_{mm} V_m^+ = 0 \quad \text{for all } m$$

a result that occurs when:

$$S_{mm} = 0 \quad \text{for all } m \text{ if matched}$$

We find therefore that a matched device will exhibit a scattering matrix where all **diagonal elements** are **zero**.

Therefore:

$$\mathcal{S} = \begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

is an example of a scattering matrix for a **matched,** three port device.

Lossless

For a lossless device, all of the power that delivered to each device port must eventually find its way **out**!

In other words, power is not **absorbed** by the network—no power to be **converted to heat**!

Recall the **power incident** on some port m is related to the amplitude of the **incident wave** (V_{0m}^+) as:

$$P_m^+ = \frac{|V_{0m}^+|^2}{2Z_0}$$

While power of the **wave exiting** the port is:

$$P_m^- = \frac{|V_{0m}^-|^2}{2Z_0}$$

Thus, the power **delivered** to (absorbed by) that port is the **difference** of the two:

$$\Delta P_m = P_m^+ - P_m^- = \frac{\left|V_{0m}^+\right|^2}{2Z_0} - \frac{\left|V_{0m}^-\right|^2}{2Z_0}$$

Thus, the **total power incident** on an N-port device is:

$$P^+ = \sum_{m=1}^{N} P_m^+ = \frac{1}{2Z_0} \sum_{m=1}^{N} \left|V_{0m}^+\right|^2$$

Note that:

$$\sum_{m=1}^{N} \left|V_{0m}^+\right|^2 = \left(\mathbf{V}^+\right)^H \mathbf{V}^+$$

where operator H indicates the **conjugate transpose** (i.e., Hermetian transpose) operation, so that $\left(\mathbf{V}^+\right)^H \mathbf{V}^+$ is the **inner product** (i.e., dot product, or scalar product) of complex vector $\mathbf{V}^+$ with itself.

Thus, we can write the **total power incident** on the device as:

$$P^+ = \frac{1}{2Z_0} \sum_{m=1}^{N} \left|V_{0m}^+\right|^2 = \frac{\left(\mathbf{V}^+\right)^H \mathbf{V}^+}{2Z_0}$$

Similarly, we can express the **total power** of the **waves exiting** our M-port network to be:

$$P^- = \frac{1}{2Z_0} \sum_{m=1}^{N} \left|V_{0m}^-\right|^2 = \frac{\left(\mathbf{V}^-\right)^H \mathbf{V}^-}{2Z_0}$$

Now, recalling that the incident and exiting wave amplitudes are **related** by the **scattering matrix** of the device:

$$\mathbf{V}^- = \mathcal{S}\ \mathbf{V}^+$$

Thus we find:

$$P^- = \frac{\left(\mathbf{V}^-\right)^H \mathbf{V}^-}{2Z_0} = \frac{\left(\mathbf{V}^+\right)^H \mathcal{S}^H \mathcal{S}\ \mathbf{V}^+}{2Z_0}$$

Now, the **total power delivered** to the network is:

$$\Delta P = \sum_{m=1}^{M} \Delta P = P^+ - P^-$$

Or explicitly:

$$\begin{aligned}\Delta P &= P^+ - P^- \\ &= \frac{\left(\mathbf{V}^+\right)^H \mathbf{V}^+}{2Z_0} - \frac{\left(\mathbf{V}^+\right)^H \mathcal{S}^H \mathcal{S}\ \mathbf{V}^+}{2Z_0} \\ &= \frac{1}{2Z_0}\ \left(\mathbf{V}^+\right)^H \left(\mathcal{I} - \mathcal{S}^H \mathcal{S}\right)\mathbf{V}^+\end{aligned}$$

where $\mathcal{I}$ is the **identity matrix**.

Q: *Is there actually some* ***point*** *to this long, rambling, complex presentation?*

A: Absolutely! If our M-port device is lossless then the total power exiting the device must **always** be equal to the total power incident on it.

If network is **lossless,** then $P^+ = P^-$.

Or stated another way, the total **power delivered** to the device (i.e., the power absorbed by the device) must always be **zero** if the device is lossless!

If network is **lossless,** then $\Delta P = 0$

Thus, we can conclude from our math that for a **lossless device**:

$$\Delta P = \frac{1}{2Z_0} \left(\mathbf{V}^+\right)^H \left(\mathcal{I} - \mathcal{S}^H \mathcal{S}\right)\mathbf{V}^+ = 0 \qquad \text{for all } \mathbf{V}^+$$

This is true **only** if:

$$\mathcal{I} - \mathcal{S}^H \mathcal{S} = 0 \qquad \Rightarrow \qquad \mathcal{S}^H \mathcal{S} = \mathcal{I}$$

Thus, we can conclude that the **scattering matrix** of a **lossless** device has the **characteristic**:

If a network is **lossless**, then $\mathcal{S}^H \mathcal{S} = \mathcal{I}$

Q: *Huh? What exactly is this supposed to* ***tell*** *us?*

A: A matrix that satisfies $\mathcal{S}^H \mathcal{S} = \mathcal{I}$ is a special kind of matrix known as a **unitary matrix**.

If a network is **lossless**, then its scattering matrix $\mathcal{S}$ is **unitary**.

Q: *How do I **recognize** a unitary matrix if I **see** one?*

A: The **columns** of a unitary matrix form an **orthonormal set**!

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{22} \\ S_{31} & S_{33} & S_{33} & S_{33} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

matrix columns

In other words, each **column** of the scattering matrix will have a **magnitude equal to one**:

$$\sum_{m=1}^{N} \left| S_{mn} \right|^2 = 1 \quad \text{for all } n$$

while the inner product (i.e., dot product) of **dissimilar columns** must be **zero**.

$$\sum_{n=1}^{N} S_{ni} S_{nj}^{*} = S_{1i} S_{1j}^{*} + S_{2i} S_{2j}^{*} + \cdots + S_{Ni} S_{Nj}^{*} = 0 \quad \text{for all } i \neq j$$

In other words, dissimilar columns are **orthogonal**.

Consider, for example, a lossless **three-port** device. Say a signal is incident on port 1, and that **all** other ports are **terminated**. The power **incident** on port 1 is therefore:

$$P_1^+ = \frac{\left|V_{01}^+\right|^2}{2Z_0}$$

while the power **exiting** the device at each port is:

$$P_m^- = \frac{\left|V_{0m}^-\right|^2}{2Z_0} = \frac{\left|S_{m1}V_{01}^-\right|^2}{2Z_0} = \left|S_{m1}\right|^2 P_1^+$$

The **total** power exiting the device is therefore:

$$\begin{aligned} P^- &= P_1^- + P_2^- + P_3^- \\ &= \left|S_{11}\right|^2 P_1^+ + \left|S_{21}\right|^2 P_1^+ + \left|S_{31}\right|^2 P_1^+ \\ &= \left(\left|S_{11}\right|^2 + \left|S_{21}\right|^2 + \left|S_{31}\right|^2\right) P_1^+ \end{aligned}$$

Since this device is **lossless**, then the incident power (**only** on port 1) is **equal** to exiting power (i.e, $P^- = P_1^+$). This is true **only** if:

$$\left|S_{11}\right|^2 + \left|S_{21}\right|^2 + \left|S_{31}\right|^2 = 1$$

Of course, this will likewise be true if the incident wave is placed on **any** of the **other** ports of this lossless device:

$$\left|S_{12}\right|^2 + \left|S_{22}\right|^2 + \left|S_{32}\right|^2 = 1$$
$$\left|S_{13}\right|^2 + \left|S_{23}\right|^2 + \left|S_{33}\right|^2 = 1$$

We can state in general then that:

$$\sum_{m=1}^{3} \left|S_{mn}\right|^2 = 1 \quad \text{for all } n$$

In other words, the columns of the scattering matrix must have **unit magnitude** (a requirement of all **unitary** matrices). It is apparent that this must be true for energy to be conserved.

An **example** of a (unitary) scattering matrix for a **lossless** device is:

$$\mathcal{S} = \begin{bmatrix} 0 & 1/2 & j\sqrt{3}/2 & 0 \\ 1/2 & 0 & 0 & j\sqrt{3}/2 \\ j\sqrt{3}/2 & 0 & 0 & 1/2 \\ 0 & j\sqrt{3}/2 & 1/2 & 0 \end{bmatrix}$$

Reciprocal

Recall **reciprocity** results when we build a **passive** (i.e., unpowered) device with **simple** materials.

For a reciprocal network, we find that the elements of the scattering matrix are **related** as:

$$S_{mn} = S_{nm}$$

For example, a **reciprocal** device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as:

$$\mathcal{S}^T = \mathcal{S} \qquad \text{if reciprocal}$$

where T indicates (non-conjugate) transpose.

An **example** of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10-j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$

Example: A Lossless, Reciprocal Network

A **lossless**, **reciprocal** 3-port device has $\mathcal{S}$-parameters of $S_{11} = 1/2$, $S_{31} = 1/\sqrt{2}$, and $S_{33} = 0$. It is likewise known that all scattering parameters are **real**.

→ Find the remaining **6** scattering parameters.

Q: *This problem is clearly* ***impossible*** *—you have not provided us with sufficient* ***information****!*

A: Yes I have! Note I said the device was **lossless** and **reciprocal**!

Start with what we **currently** know:

$$\mathcal{S} = \begin{bmatrix} \frac{1}{2} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ \frac{1}{\sqrt{2}} & S_{32} & 0 \end{bmatrix}$$

Because the device is **reciprocal**, we then also know:

$$S_{21} = S_{12} \qquad S_{13} = S_{31} = \frac{1}{\sqrt{2}} \qquad S_{32} = S_{23}$$

And therefore:

$$\mathcal{S} = \begin{bmatrix} \frac{1}{2} & S_{21} & \frac{1}{\sqrt{2}} \\ S_{21} & S_{22} & S_{32} \\ \frac{1}{\sqrt{2}} & S_{32} & 0 \end{bmatrix}$$

Now, since the device is **lossless**, we know that:

$$\begin{aligned} 1 &= |S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 \\ &= \left(\tfrac{1}{2}\right)^2 + |S_{21}|^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 \end{aligned}$$

Columns have **unit** magnitude.

$$\begin{aligned} 1 &= |S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 \\ &= |S_{21}|^2 + |S_{22}|^2 + |S_{32}|^2 \end{aligned}$$

$$\begin{aligned} 1 &= |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 \\ &= \left(\tfrac{1}{2}\right)^2 + |S_{32}|^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 \end{aligned}$$

and:

$$\begin{aligned} 0 &= S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^* \\ &= \tfrac{1}{2}\, S_{21}^* + S_{21}S_{22}^* + \tfrac{1}{\sqrt{2}}\, S_{32}^* \end{aligned}$$

Columns are **orthogonal**.

$$\begin{aligned} 0 &= S_{11}S_{13}^* + S_{21}S_{23}^* + S_{31}S_{33}^* \\ &= \tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}\right) + S_{21}S_{32}^* + \tfrac{1}{\sqrt{2}}\,(0) \end{aligned}$$

$$\begin{aligned} 0 &= S_{12}S_{13}^* + S_{22}S_{23}^* + S_{32}S_{33}^* \\ &= S_{21}\left(\tfrac{1}{\sqrt{2}}\right) + S_{22}S_{32}^* + S_{32}\,(0) \end{aligned}$$

These **six** expressions simplify to:

$$|S_{21}| = \tfrac{1}{2}$$

$$1 = |S_{21}|^2 + |S_{22}|^2 + |S_{32}|^2$$

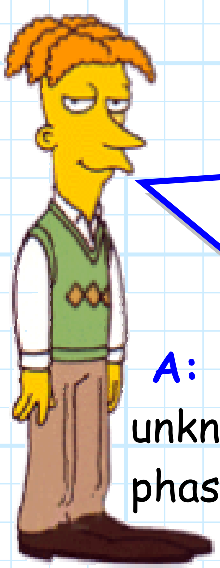
$$|S_{32}| = \tfrac{1}{\sqrt{2}}$$

$$0 = \tfrac{1}{2} S_{21} + S_{21} S_{22} + \tfrac{1}{\sqrt{2}} S_{32}$$

$$0 = \tfrac{1}{(2\sqrt{2})} + S_{21} S_{32}$$

$$0 = S_{21}\left(\tfrac{1}{\sqrt{2}}\right) + S_{22} S_{32}$$

where we have used the fact that since the elements are all **real**, then $S_{21}^* = S_{21}$ (etc.).



Q: *I count the expressions and find 6 equations yet only a paltry 3 unknowns. Your typical buffoonery appears to have led to an over-constrained condition for which there is **no** solution!*

A: Actually, we have **six** real equations and **six** real unknowns, since scattering element has a magnitude and phase. In this case we know the values are **real**, and thus the phase is either 0° or 180° (i.e., $e^{j0} = 1$ or $e^{j\pi} = -1$); however, we do not know which one!

From the first three equations, we can find the **magnitudes**:

$$|S_{21}| = 1/2 \qquad |S_{22}| = 1/2 \qquad |S_{32}| = 1/\sqrt{2}$$

and from the last three equations we find the **phase**:

$$S_{21} = 1/2 \qquad S_{22} = 1/2 \qquad S_{32} = -1/\sqrt{2}$$

Thus, the scattering matrix for this **lossless**, **reciprocal** device is:

$$\mathcal{S} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

A Matched, Lossless Reciprocal 3-Port Network

Consider a 3-port device. Such a device would have a scattering matrix :

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Assuming the device is passive and made of simple (isotropic) materials, the device will be **reciprocal,** so that:

$$S_{21} = S_{12} \qquad S_{31} = S_{13} \qquad S_{23} = S_{32}$$

Likewise, if it is **matched,** we know that:

$$S_{11} = S_{22} = S_{33} = 0$$

As a result, a **lossless**, **reciprocal** device would have a scattering matrix of the form:

$$\mathcal{S} = \begin{bmatrix} 0 & S_{21} & S_{31} \\ S_{21} & 0 & S_{32} \\ S_{31} & S_{32} & 0 \end{bmatrix}$$

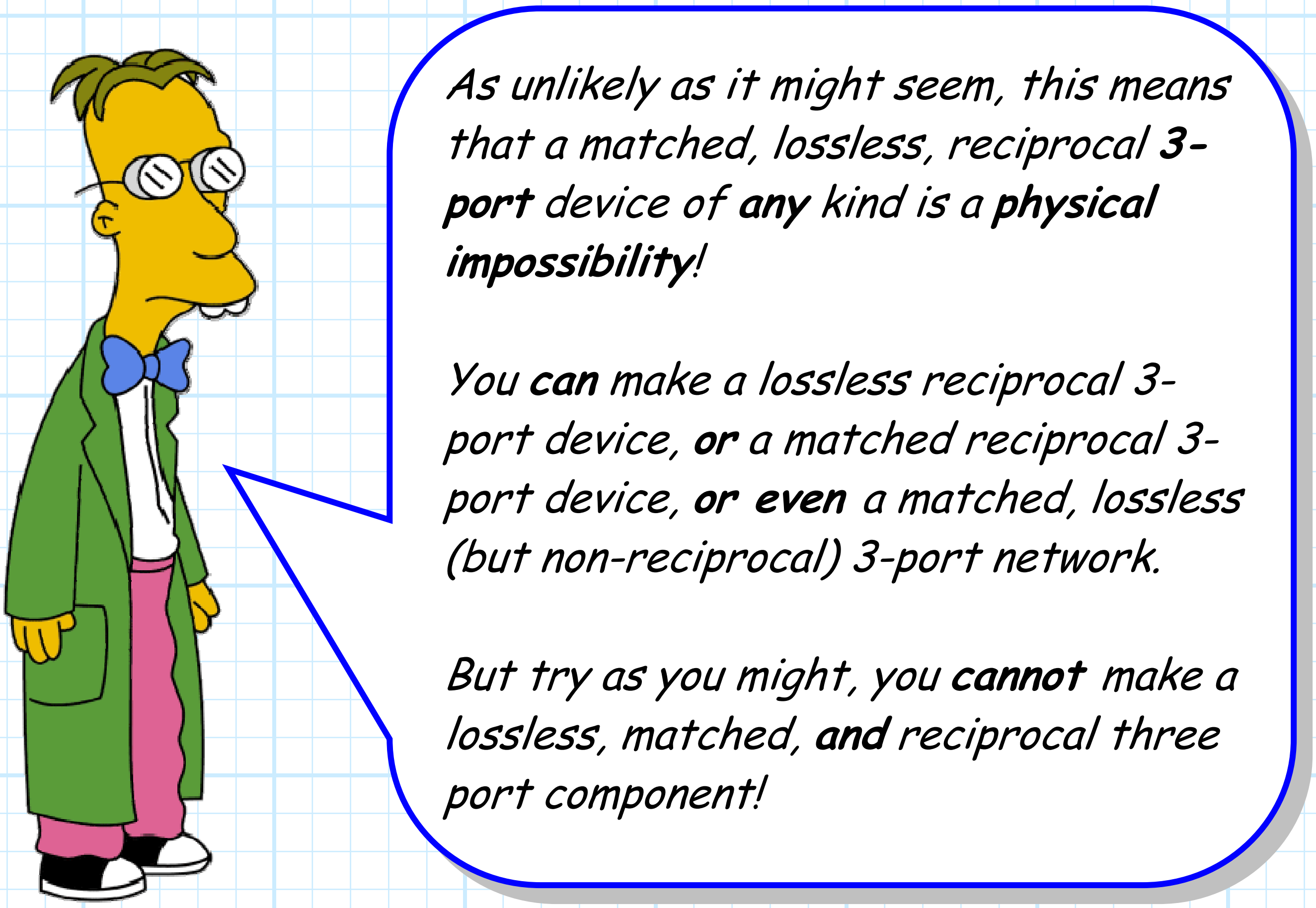
Just **3** non-zero scattering parameters define the **entire** matrix!

Likewise, if we wish for this network to be **lossless**, the scattering matrix must be **unitary**, and therefore:

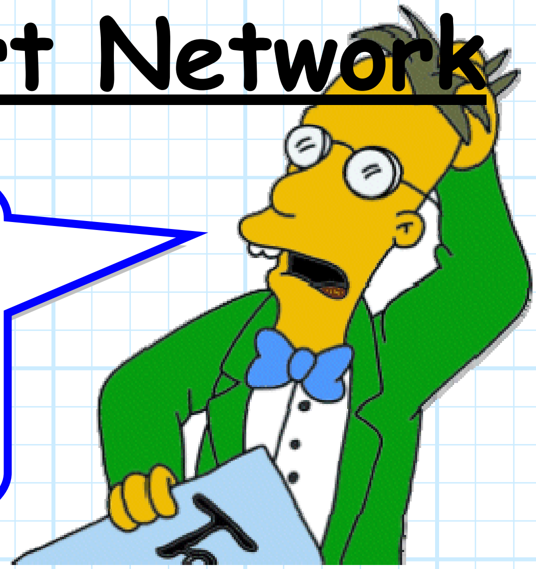
$$
\begin{aligned}
|S_{21}|^2 + |S_{31}|^2 &= 1 \qquad & S_{31}^* S_{32} &= 0 \\
|S_{21}|^2 + |S_{32}|^2 &= 1 \qquad & S_{21}^* S_{32} &= 0 \\
|S_{31}|^2 + |S_{32}|^2 &= 1 \qquad & S_{21}^* S_{31} &= 0
\end{aligned}
$$

Since each complex value S is represented by **two real numbers** (i.e., real and imaginary parts), the equations above result in **9** real equations. The problem is, the 3 complex values S_{21}, S_{31} and S_{32} are represented by only **6** real unknowns.

We have **over constrained** our problem ! There are **no solutions** to these equations !



The Matched, Lossless, Reciprocal 4-Port Network



The first solution is referred to as the **symmetric** solution:

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

Note for this symmetric solution, every row and every column of the scattering matrix has the **same** four values (i.e., α, $j\beta$, and two zeros)!

The second solution is referred to as the **anti-symmetric** solution:

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

Note that for this anti-symmetric solution, **two** rows and **two** columns have the same four values (i.e., α, β, and two zeros), while the **other** two row and columns have (slightly) **different** values (α, $-\beta$, and two zeros)

It is **quite** evident that each of these solutions are **matched** and **reciprocal**. However, to ensure that the solutions are indeed **lossless**, we must place an **additional** constraint on the values of α, β. Recall that a **necessary** condition for a lossless device is:

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \quad \text{for all } n$$

Applying this to the **symmetric** case, we find:

$$|\alpha|^2 + |\beta|^2 = 1$$

Likewise, for the **anti-symmetric** case, we also get

$$|\alpha|^2 + |\beta|^2 = 1$$

It is evident that if the scattering matrix is **unitary** (i.e., lossless), the values α and β **cannot** be independent, but must **related** as:

$$\boxed{|\alpha|^2 + |\beta|^2 = 1}$$

Generally speaking, we will find that $|\alpha| \geq |\beta|$. Given the constraint on these two values, we can thus conclude that:

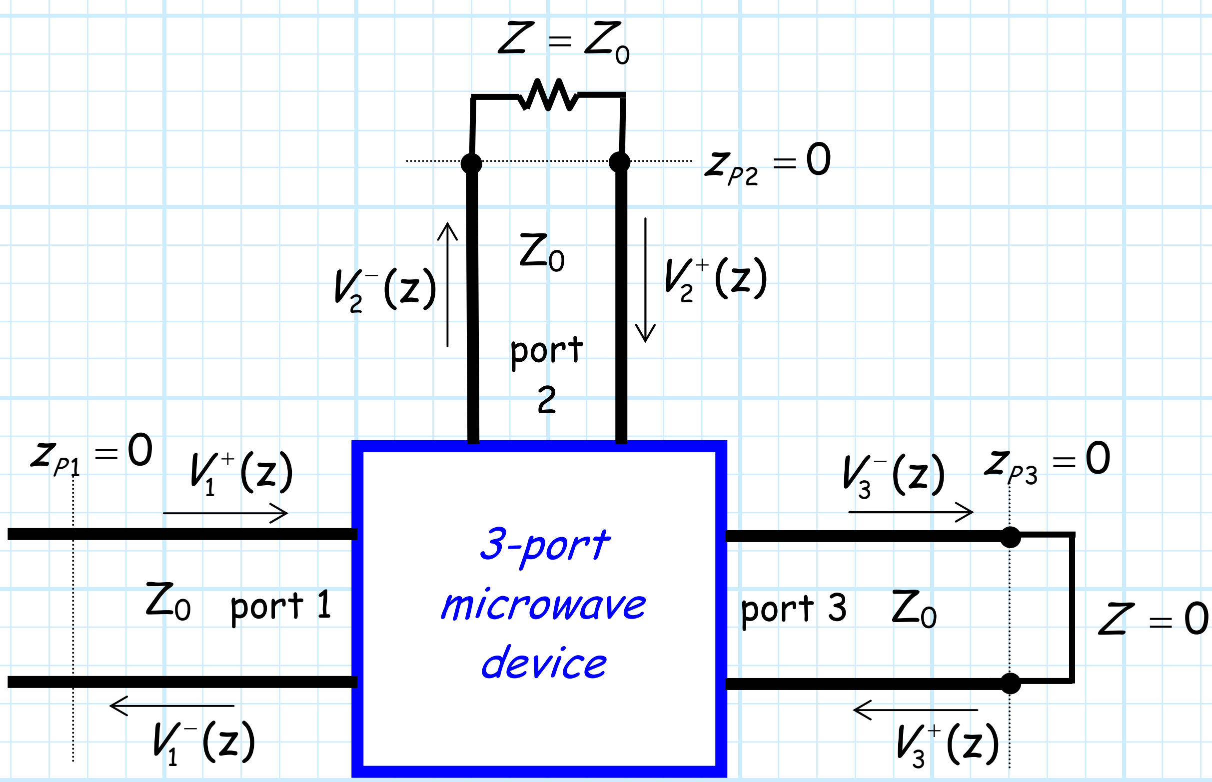
$$0 \leq |\beta| \leq 1/\sqrt{2} \quad \text{and} \quad 1/\sqrt{2} \leq |\alpha| \leq 1$$

Example: The Scattering Matrix

Say we have a 3-port network that is completely characterized at some frequency ω by the **scattering matrix**:

$$\mathcal{S} = \begin{bmatrix} 0.0 & 0.2 & 0.5 \\ 0.5 & 0.0 & 0.2 \\ 0.5 & 0.5 & 0.0 \end{bmatrix}$$

A **matched load** is attached to port 2, while a **short circuit** has been placed at port 3:



Because of the **matched** load at port 2 (i.e., $\Gamma_L = 0$), we know that:

$$\frac{V_2^+(z_2=0)}{V_2^-(z_2=0)} = \frac{V_{02}^+}{V_{02}^-} = 0$$

and therefore:

$$V_{02}^+ = 0$$

NO!! Remember, the signal $V_2^-(z)$ is **incident** on the matched load, and $V_2^+(z)$ is the **reflected** wave from the load (i.e., $V_2^+(z)$ is incident on **port 2**). Therefore, $V_{02}^+ = 0$ **is correct**!

Likewise, because of the **short** circuit at port 3 ($\Gamma_L = -1$):

$$\frac{V_3^+(z_3=0)}{V_3^-(z_3=0)} = \frac{V_{03}^+}{V_{03}^-} = -1$$

and therefore:

$$V_{03}^+ = -V_{03}^-$$

Problem:

a) Find the **reflection** coefficient at port 1, i.e.:

$$\Gamma_1 \doteq \frac{V_{01}^-}{V_{01}^+}$$

b) Find the **transmission** coefficient from port 1 to port 2, i.e.,

$$T_{21} \doteq \frac{V_{02}^-}{V_{01}^+}$$

*I am amused by the trivial problems that **you** apparently find so difficult. **I** know that:*

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = S_{11} = 0.0$$

and

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = S_{21} = 0.5$$

NO!!! The above statement is **not correct**!

Remember, $V_{01}^- / V_{01}^+ = S_{11}$ **only** if ports 2 and 3 are terminated in **matched** loads! In this problem port 3 is terminated with a **short circuit**.

Therefore:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} \neq S_{11}$$

and similarly:

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} \neq S_{21}$$

To determine the values T_{21} and Γ_1, we must start with the **three** equations provided by the **scattering matrix**:

$$V_{01}^- = 0.2\,V_{02}^+ + 0.5\,V_{03}^+$$

$$V_{02}^- = 0.5\,V_{01}^+ + 0.2\,V_{03}^+$$

$$V_{03}^- = 0.5\,V_{01}^+ + 0.5\,V_{02}^+$$

and the **two** equations provided by the **attached loads**:

$$V_{02}^+ = 0$$

$$V_{03}^+ = -V_{03}^-$$

We can divide all of these equations by V_{01}^{+}, resulting in:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = \qquad 0.2\,\frac{V_{02}^+}{V_{01}^+} + 0.5\,\frac{V_{03}^+}{V_{01}^+}$$

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = 0.5 \qquad + 0.2\,\frac{V_{03}^+}{V_{01}^+}$$

$$\frac{V_{03}^-}{V_{01}^+} = 0.5 + 0.5\,\frac{V_{02}^+}{V_{01}^+}$$

$$\frac{V_{02}^+}{V_{01}^+} = 0$$

$$\frac{V_{03}^+}{V_{01}^+} = -\frac{V_{03}^-}{V_{01}^+}$$

Look what we have—**5** equations and **5** unknowns! Inserting equations 4 and 5 into equations 1 through 3, we get:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = -0.5\,\frac{V_{03}^+}{V_{01}^+}$$

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = 0.5 - 0.2\,\frac{V_{03}^+}{V_{01}^+}$$

$$\frac{V_{03}^-}{V_{01}^+} = 0.5$$

Solving, we find:

$$\Gamma_1 = -0.5(0.5) = -0.25$$

$$\mathcal{T}_{21} = 0.5 - 0.2(0.5) = 0.4$$

Example: Scattering Parameters

Consider a **two-port device** with a scattering matrix (at some specific frequency ω_0):

$$\mathcal{S}(\omega=\omega_0)=\begin{bmatrix} 0.1 & j0.7 \\ j0.7 & -0.2 \end{bmatrix}$$

and $Z_0 = 50\Omega$.

Say that the transmission line connected to **port 2** of this device is terminated in a **matched** load, and that the wave **incident** on **port 1** is:

$$V_1^+(z_1) = -j2\, e^{-j\beta z_1}$$

where $z_{1P} = z_{2P} = 0$.

Determine:

1. the port voltages $V_1(z_1 = z_{1P})$ and $V_2(z_2 = z_{2P})$.

2. the port currents $I_1(z_1 = z_{1P})$ and $I_2(z_2 = z_{2P})$.

3. the net power flowing into port 1

1. Since the **incident** wave on port 1 is:

$$V_1^+(z_1) = -j2\, e^{-j\beta z_1}$$

we can conclude (since $z_{1P} = 0$):

$$\begin{aligned} V_1^+(z_1 = z_{1P}) &= -j2\, e^{-j\beta z_{1P}} \\ &= -j2\, e^{-j\beta(0)} \\ &= -j2 \end{aligned}$$

and since port 2 is **matched** (and **only** because its matched!), we find:

$$\begin{aligned} V_1^-(z_1 = z_{1P}) &= S_{11}\, V_1^+(z_1 = z_{1P}) \\ &= 0.1(-j2) \\ &= -j0.2 \end{aligned}$$

The voltage at port 1 is thus:

$$\begin{aligned} V_1(z_1 = z_{1P}) &= V_1^+(z_1 = z_{1P}) + V_1^-(z_1 = z_{1P}) \\ &= -j2.0 - j0.2 \\ &= -j2.2 \\ &= 2.2\, e^{-j\pi/2} \end{aligned}$$

Likewise, since port 2 is **matched**:

$$V_2^+(z_2 = z_{2P}) = 0$$

And also:

$$\begin{aligned} V_2^-(z_2 = z_{2P}) &= S_{21}\, V_1^+(z_1 = z_{1P}) \\ &= j0.7\,(-j2) \\ &= 1.4 \end{aligned}$$

Therefore:

$$\begin{aligned} V_2(z_2 = z_{2P}) &= V_2^+(z_2 = z_{2P}) + V_2^-(z_2 = z_{2P}) \\ &= 0 + 1.4 \\ &= 1.4 \\ &= 1.4\, e^{-j0} \end{aligned}$$

2. The port **currents** can be easily determined from the results of the previous section.

$$\begin{aligned} I_1(z_1 = z_{1P}) &= I_1^+(z_1 = z_{1P}) - I_1^-(z_1 = z_{1P}) \\ &= \frac{V_1^+(z_1 = z_{1P})}{Z_0} - \frac{V_1^-(z_1 = z_{1P})}{Z_0} \\ &= -j\,\frac{2.0}{50} + j\,\frac{0.2}{50} \\ &= -j\,\frac{1.8}{50} \\ &= -j0.036 \\ &= 0.036\, e^{-j\pi/2} \end{aligned}$$

and:

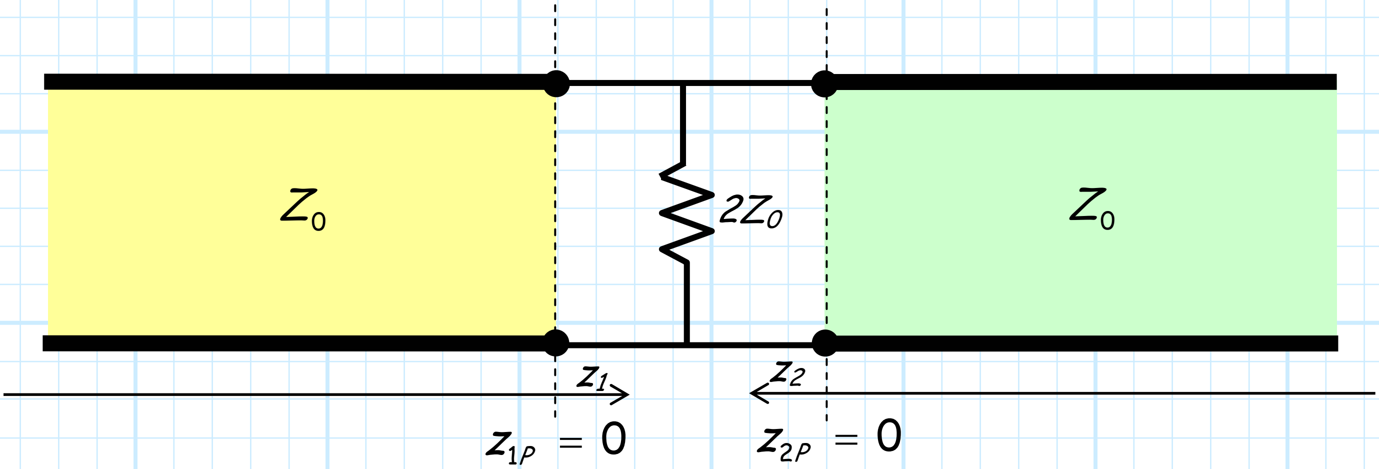
$$
\begin{aligned}
I_2(z_2 = z_{2P}) &= I_2^+(z_2 = z_{2P}) - I_2^-(z_2 = z_{2P}) \\
&= \frac{V_2^+(z_2 = z_{2P})}{Z_0} - \frac{V_2^-(z_2 = z_{2P})}{Z_0} \\
&= \frac{0}{50} - \frac{1.4}{50} \\
&= -0.028 \\
&= 0.028\, e^{+j\pi}
\end{aligned}
$$

3. The **net power** flowing into port 1 is:

$$
\begin{aligned}
\Delta P_1 &= P_1^+ - P_1^- \\
&= \frac{\left|V_{01}^+\right|^2}{2Z_0} - \frac{\left|V_{01}^-\right|^2}{2Z_0} \\
&= \frac{(2)^2 - (0.2)^2}{2(50)} \\
&= 0.0396 \quad \text{Watts}
\end{aligned}
$$

Example: Determining the Scattering Matrix

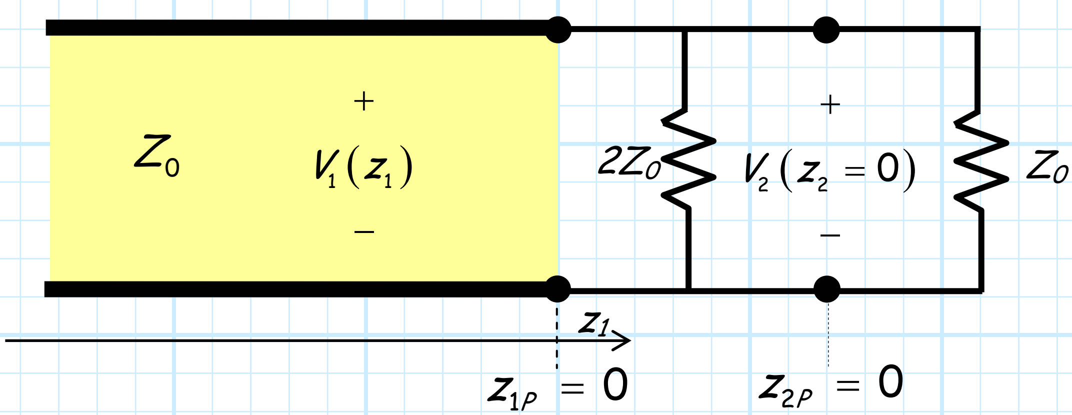
Let's determine the **scattering matrix** of this two-port device:



The **first** step is to terminate port **2** with a **matched** load, and then determine the values:

$$V_1^-(z_1 = z_{P1}) \quad \text{and} \quad V_2^-(z_2 = z_{P2})$$

in terms of $V_1^+(z_1 = z_{P1})$.



Recall that since port 2 is matched, we know that:

$$V_2^+(z_2 = z_{2P}) = 0$$

And thus:

$$\begin{aligned} V_2(z_2 = 0) &= V_2^+(z_2 = 0) + V_2^-(z_2 = 0) \\ &= 0 + V_2^-(z_2 = 0) \\ &= V_2^-(z_2 = 0) \end{aligned}$$

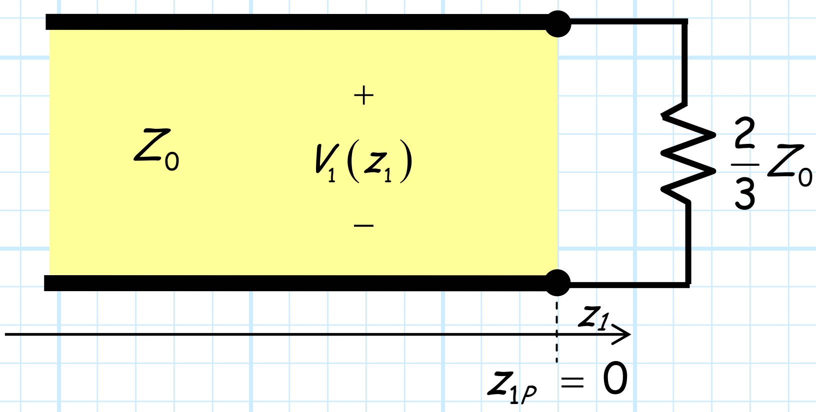
In other words, we **simply** need to determine $V_2(z_2 = 0)$ in order to find $V_2^-(z_2 = 0)$!

However, determining $V_1^-(z_1 = 0)$ is a bit **trickier**. Recall that:

$$V_1(z_1) = V_1^+(z_1) + V_1^-(z_1)$$

Therefore we find $V_1(z_1 = 0) \neq V_1^-(z_1 = 0)$!

Now, we can **simplify** this circuit:



And we know from the **telegraphers equations**:

$$V_1(z_1) = V_1^+(z_1) + V_1^-(z_1)$$
$$= V_{01}^+ e^{-j\beta z_1} + V_{01}^- e^{+j\beta z_1}$$

Since the load $2Z_0/3$ is located at $z_1 = 0$, we know that the **boundary condition** leads to:

$$V_1(z_1) = V_{01}^+ \left(e^{-j\beta z_1} + \Gamma_L e^{+j\beta z_1}\right)$$

where:

$$\begin{aligned}\Gamma_L &= \frac{\left(\frac{2}{3}\right) Z_0 - Z_0}{\left(\frac{2}{3}\right) Z_0 + Z_0} \\ &= \frac{\left(\frac{2}{3}\right) - 1}{\left(\frac{2}{3}\right) + 1} \\ &= \frac{-\frac{1}{3}}{\frac{5}{3}} \\ &= -0.2\end{aligned}$$

Therefore:

$$V_1^+(z_1) = V_{01}^+ e^{-j\beta z_1} \quad \text{and} \quad V_1^-(z_1) = V_{01}^+(-0.2) e^{+j\beta z_1}$$

and thus:

$$V_1^+(z_1 = 0) = V_{01}^+ e^{-j\beta(0)} = V_{01}^+$$

$$V_1^-(z_1 = 0) = V_{01}^+(-0.2) e^{+j\beta(0)} = -0.2 V_{01}^+$$

We can now determine S_{11} !

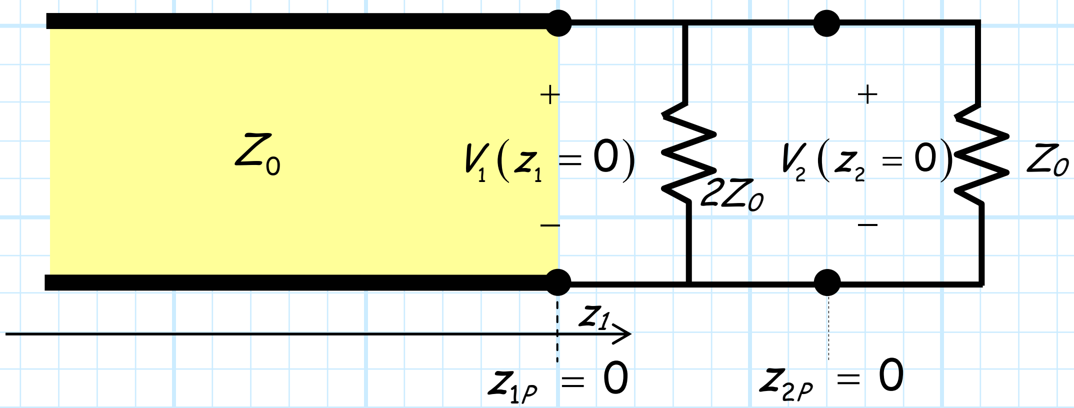
$$S_{11} = \frac{V_1^-(z_1 = 0)}{V_1^+(z_1 = 0)} = \frac{-0.2\, V_{01}^+}{V_{01}^+} = -0.2$$

Now its time to find $V_2^-(z_2=0)$!

Again, since port 2 is terminated, the **incident** wave on port 2 must be **zero**, and thus the value of the **exiting** wave at port 2 is equal to the **total** voltage at port 2:

$$V_2^-(z_2=0) = V_2\ (z_2=0)$$

This **total** voltage is relatively **easy** to determine. Examining the circuit, it is evident that $V_1(z_1=0) = V_2(z_2=0)$.



Therefore:

$$\begin{aligned} V_2(z_2=0) = V_1(z_1=0) &= V_{01}^+\left(e^{-j\beta(0)} - 0.2\, e^{+j\beta(0)}\right) \\ &= V_{01}^+(1-0.2) \\ &= V_{01}^+(0.8) \end{aligned}$$

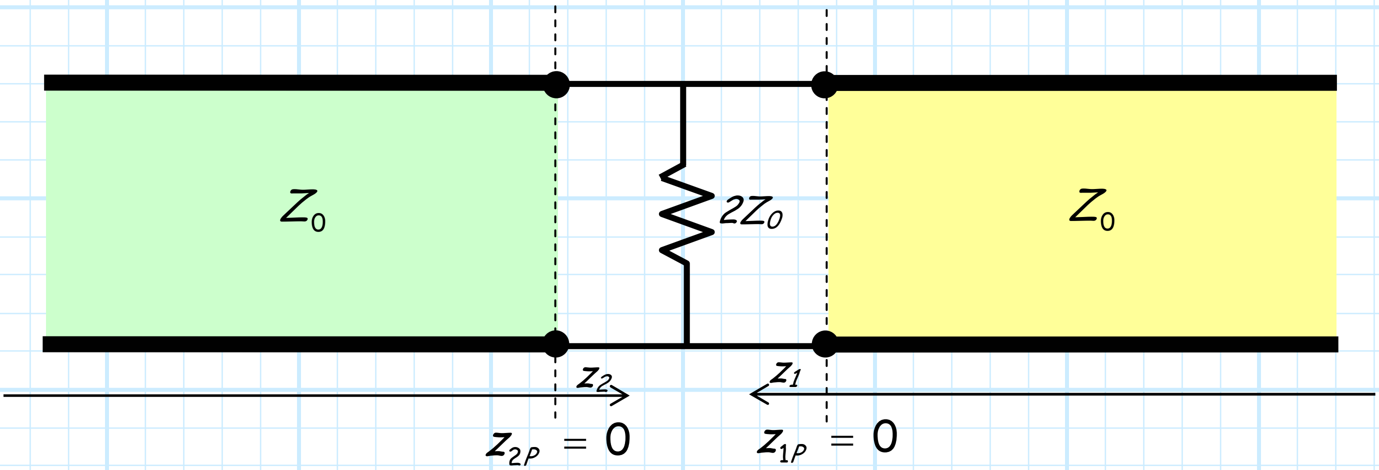
And thus the scattering parameter S_{21} is:

$$S_{21} = \frac{V_2^-(z_2=0)}{V_1^+(z_1=0)} = \frac{0.8\, V_{01}^+}{V_{01}^+} = 0.8$$

Now we **just** need to find S_{12} and S_{22}.

Q: *Yikes! This has been an awful lot of work, and you mean that we are only **half-way** done!?*

A: Actually, we are nearly finished! Note that this circuit is **symmetric**—there is really **no** difference between port 1 and port 2. If we "flip" the circuit, it remains **unchanged**!



Thus, we can conclude due to this **symmetry** that:

$$S_{11} = S_{22} = -0.2$$

and:

$$S_{21} = S_{12} = 0.8$$

Note this last equation is **likewise** a result of **reciprocity**.

Thus, the **scattering matrix** for this two port network is:

$$\mathcal{S} = \begin{bmatrix} -0.2 & 0.8 \\ 0.8 & -0.2 \end{bmatrix}$$

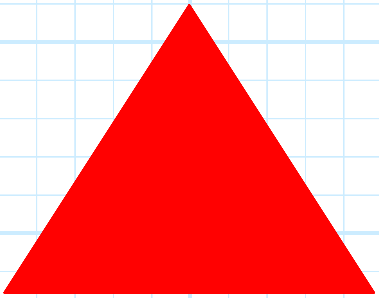
Circuit Symmetry

One of the most powerful concepts in for evaluating circuits is that of symmetry. **Normal** humans have a **conceptual** understanding of symmetry, based on an **esthetic** perception of structures and figures.

On the other hand, **mathematicians** (as they are wont to do) have defined symmetry in a very precise and unambiguous way. Using a branch of mathematics called **Group Theory**, first developed by the young genius **Évariste Galois** (1811-1832), **symmetry** is defined by a set of operations (a group) that leaves an object **unchanged**.

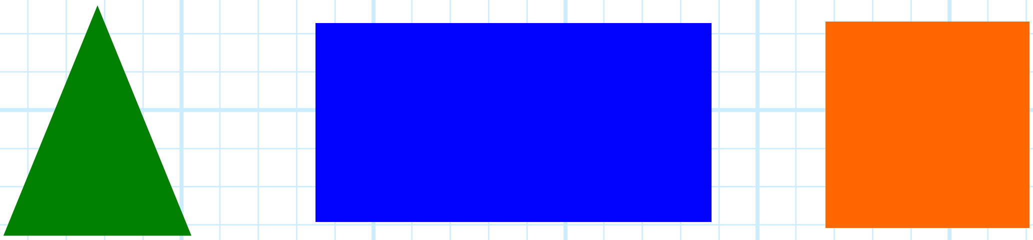
Initially, the symmetric "objects" under consideration by Galois were **polynomial functions**, but group theory can likewise be applied to evaluate the symmetry of **structures**.

For example, consider an ordinary **equilateral triangle**; we find that it is a highly **symmetric** object!

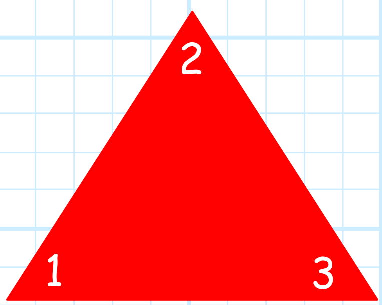


Q: ***Obviously*** *this is true. We don't need a mathematician to tell us that!*

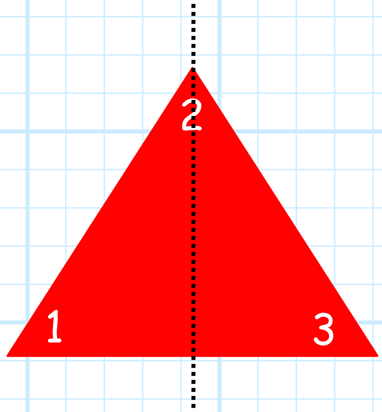
A: Yes, but **how** symmetric is it? How does the symmetry of an equilateral triangle **compare** to that of an isosceles triangle, a rectangle, or a square?



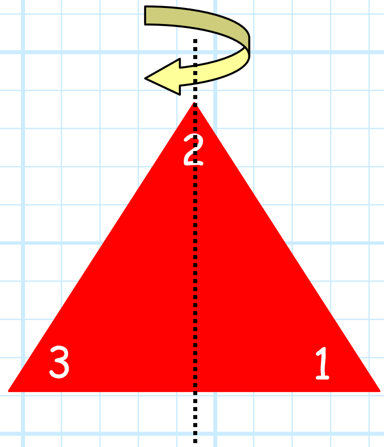
To determine its level of symmetry, let's first label each corner as corner **1**, corner **2**, and corner **3**.



First, we note that the triangle exhibits a plane of **reflection symmetry**:



Thus, if we "**reflect**" the triangle across this plane we get:



Note that although corners 1 and 3 have changed places, the triangle itself remains **unchanged**—that is, it has the same **shape**, same **size**, and same **orientation** after reflecting across the symmetric plane!

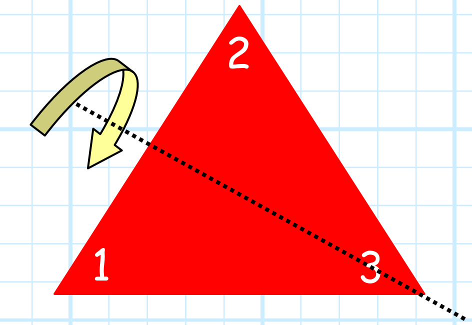
Mathematicians say that these two triangles are **congruent**.

Note that we can write this reflection operation as a **permutation** (an exchange of position) of the corners, defined as:

$$1 \rightarrow 3$$
$$2 \rightarrow 2$$
$$3 \rightarrow 1$$

Q: *But wait! Isn't there is **more** than just **one** plane of reflection symmetry?*

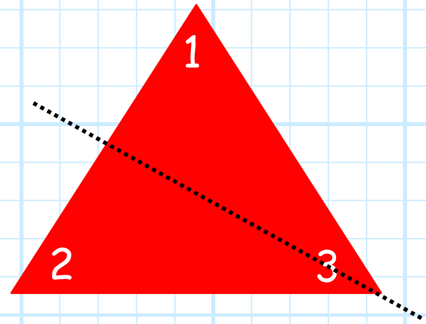
A: Definitely! There are **two more**:

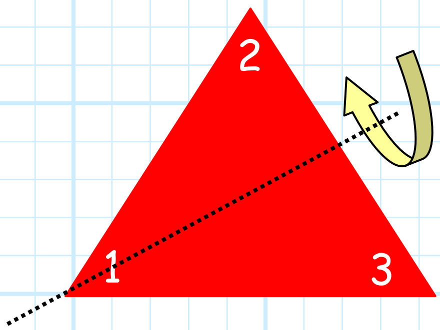


$1 \rightarrow 2$

$2 \rightarrow 1$

$3 \rightarrow 3$

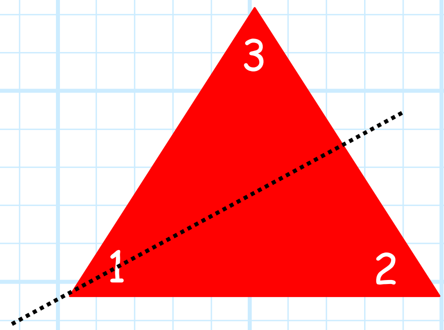




$1 \rightarrow 1$

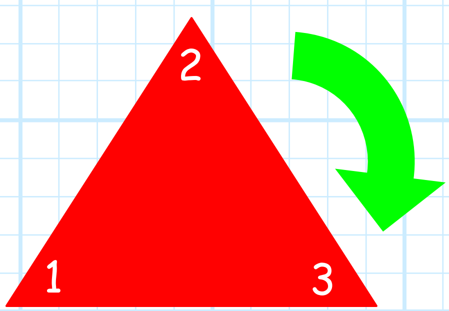
$2 \rightarrow 3$

$3 \rightarrow 2$



In addition, an equilateral triangle exhibits **rotation symmetry!**

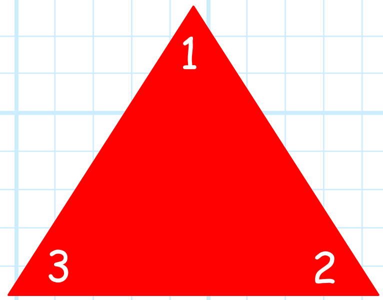
Rotating the triangle 120° clockwise also results in a **congruent** triangle:



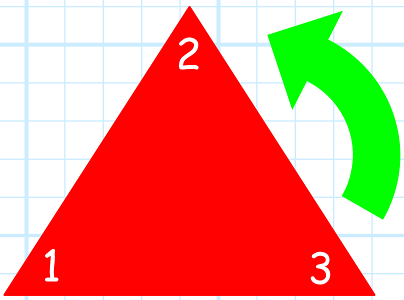
$1 \rightarrow 2$

$2 \rightarrow 3$

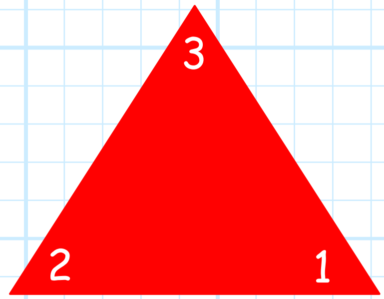
$3 \rightarrow 1$



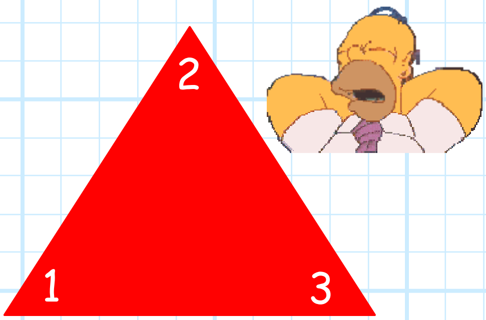
Likewise, rotating the triangle 120° **counter-clockwise** results in a congruent triangle:



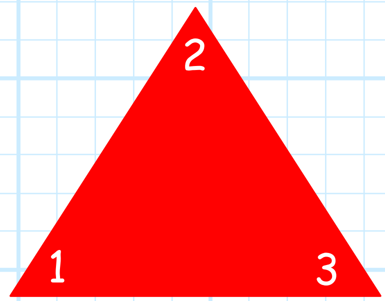
$$1 \rightarrow 3$$
$$2 \rightarrow 1$$
$$3 \rightarrow 2$$



Additionally, there is **one more** operation that will result in a congruent triangle—do **nothing**!



$$1 \rightarrow 1$$
$$2 \rightarrow 2$$
$$3 \rightarrow 3$$

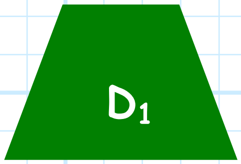


This seemingly **trivial** operation is known as the **identity operation**, and is an element of **every** symmetry group.

These 6 operations form the **dihedral symmetry group D_3** which has **order six** (i.e., it consists of six operations). An object that remains **congruent** when operated on by any and all of these six operations is said to have **D_3** symmetry.

An equilateral triangle has **D_3** symmetry!

By applying a similar analysis to a isosceles triangle, rectangle, and square, we find that:



An isosceles trapezoid has D_1 symmetry, a dihedral group of **order 2**.

A rectangle has D_2 symmetry, a dihedral group of **order 4**.

A square has D_4 symmetry, a dihedral group of **order 8**.

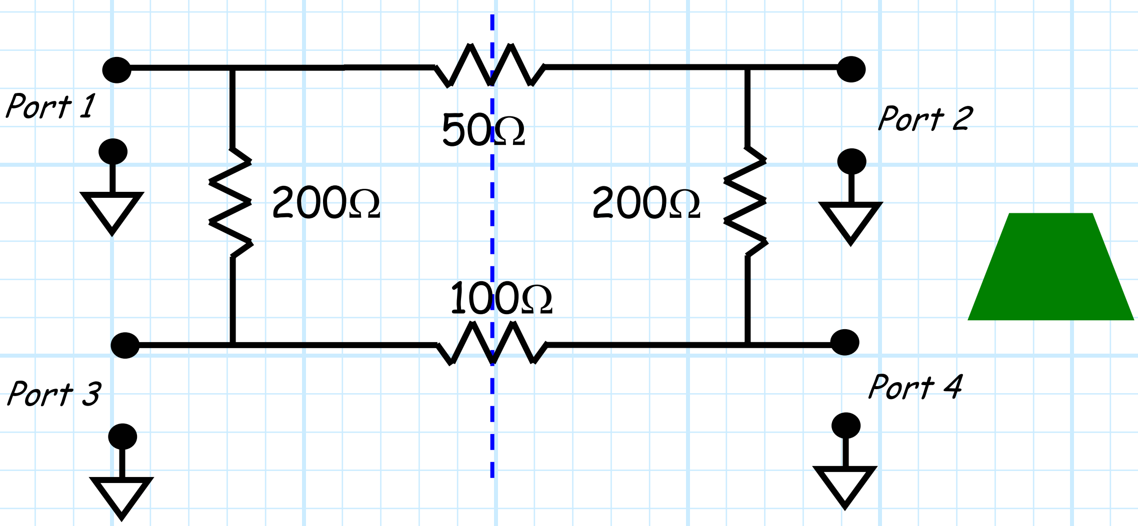
Thus, a square is the **most** symmetric object of the four we have discussed; the isosceles trapezoid is the **least**.

Q: *Well that's all just fascinating—but just what the heck does this have to do with **microwave circuits!?!***

A: Plenty! **Useful circuits** often display high levels of symmetry.

For example consider these D_1 symmetric multi-port circuits:

$1 \rightarrow 2$
$2 \rightarrow 1$
$3 \rightarrow 4$
$4 \rightarrow 3$

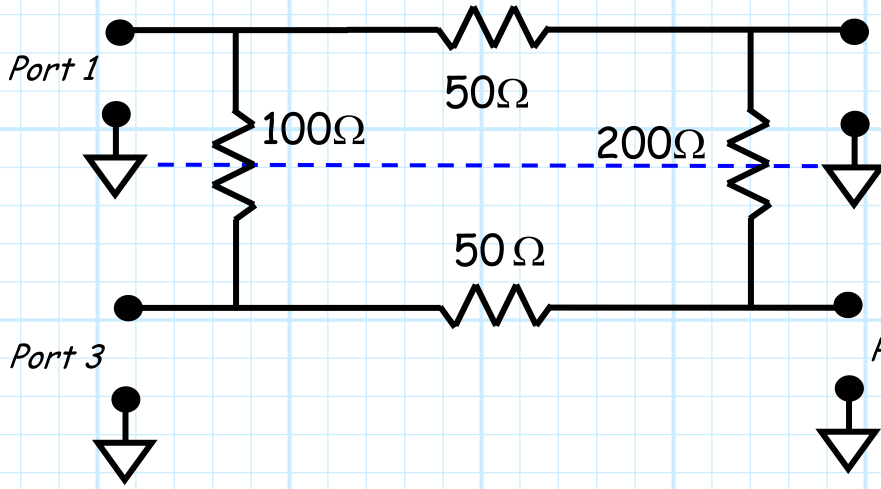


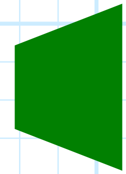
$1 \rightarrow 3$

$2 \rightarrow 4$

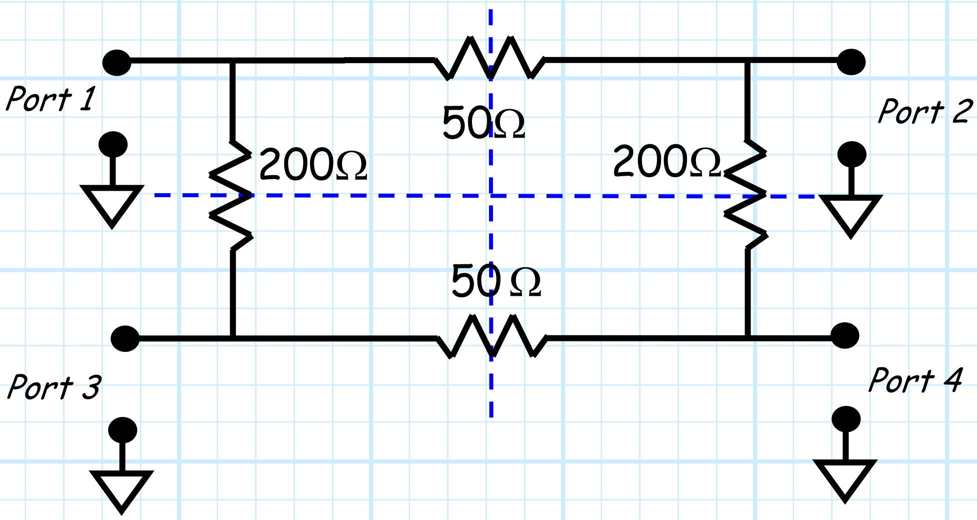
$3 \rightarrow 1$

$4 \rightarrow 2$





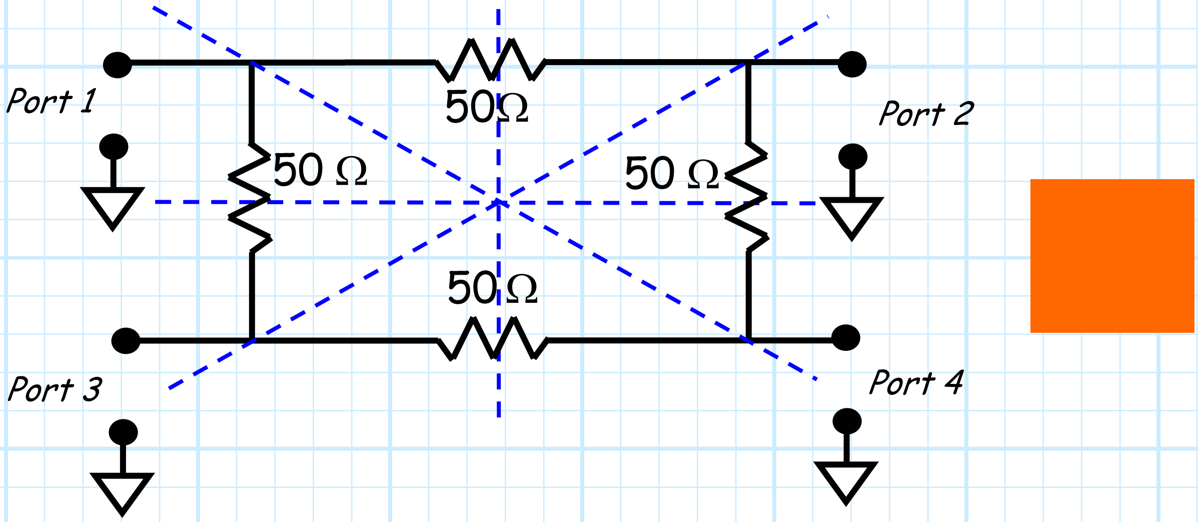
Or this circuit with **D_2** symmetry:



which is **congruent** under these permutations:

$1 \rightarrow 3$	$1 \rightarrow 2$	$1 \rightarrow 4$
$2 \rightarrow 4$	$2 \rightarrow 1$	$2 \rightarrow 3$
$3 \rightarrow 1$	$3 \rightarrow 4$	$3 \rightarrow 2$
$4 \rightarrow 2$	$4 \rightarrow 3$	$4 \rightarrow 1$

Or this circuit with **D4** symmetry:

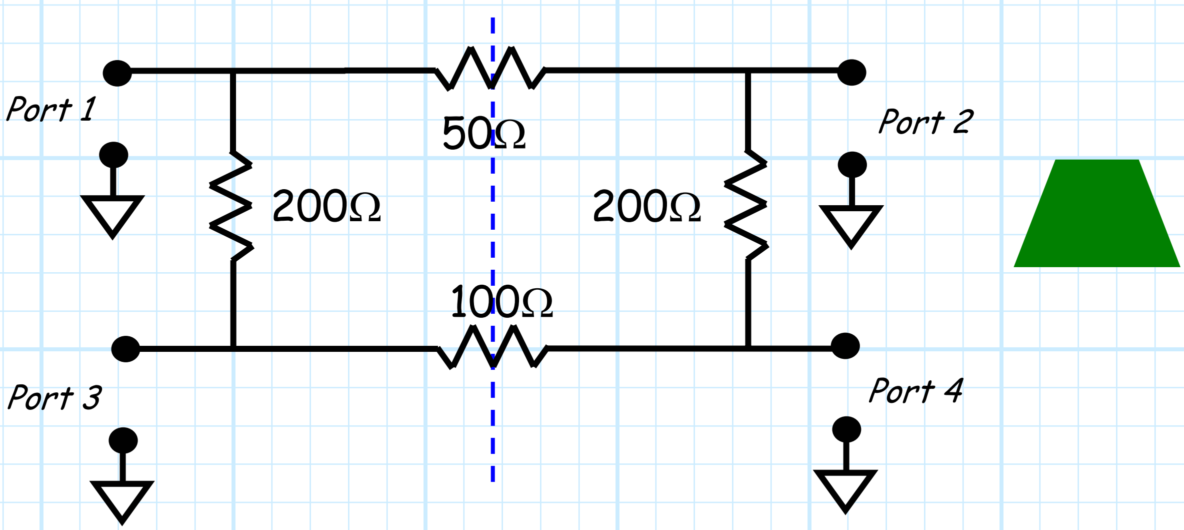


which is **congruent** under these permutations:

$1 \rightarrow 3$	$1 \rightarrow 2$	$1 \rightarrow 4$	$1 \rightarrow 4$	$1 \rightarrow 1$
$2 \rightarrow 4$	$2 \rightarrow 1$	$2 \rightarrow 3$	$2 \rightarrow 2$	$2 \rightarrow 3$
$3 \rightarrow 1$	$3 \rightarrow 4$	$3 \rightarrow 2$	$3 \rightarrow 3$	$3 \rightarrow 2$
$4 \rightarrow 2$	$4 \rightarrow 3$	$4 \rightarrow 1$	$4 \rightarrow 1$	$4 \rightarrow 4$

The **importance** of this can be seen when considering the scattering matrix, impedance matrix, or admittance matrix of these networks.

For **example**, consider again this **symmetric circuit**:



This four-port network has a single plane of **reflection symmetry** (i.e., D_1 symmetry), and thus is congruent under the permutation:

$$1 \rightarrow 2$$
$$2 \rightarrow 1$$
$$3 \rightarrow 4$$
$$4 \rightarrow 3$$

So, since (for example) $1 \rightarrow 2$, we find that for this circuit:

$$S_{11} = S_{22} \quad Z_{11} = Z_{22} \quad Y_{11} = Y_{22}$$

must be true!

Or, since $1 \rightarrow 2$ and $3 \rightarrow 4$ we find:

$$S_{13} = S_{24} \quad Z_{13} = Z_{24} \quad Y_{13} = Y_{24}$$

$$S_{31} = S_{42} \quad Z_{31} = Z_{42} \quad Y_{31} = Y_{42}$$

Continuing for **all** elements of the permutation, we find that for this symmetric circuit, the scattering matrix **must** have **this** form:

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{21} & S_{13} & S_{14} \\ S_{21} & S_{11} & S_{14} & S_{13} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

and the **impedance** and **admittance** matrices would likewise have this same form.

Note there are just **8** independent elements in this matrix. If we also consider **reciprocity** (a constraint independent of symmetry) we find that $S_{31} = S_{13}$ and $S_{41} = S_{14}$, and the matrix reduces further to one with just **6** independent elements:

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

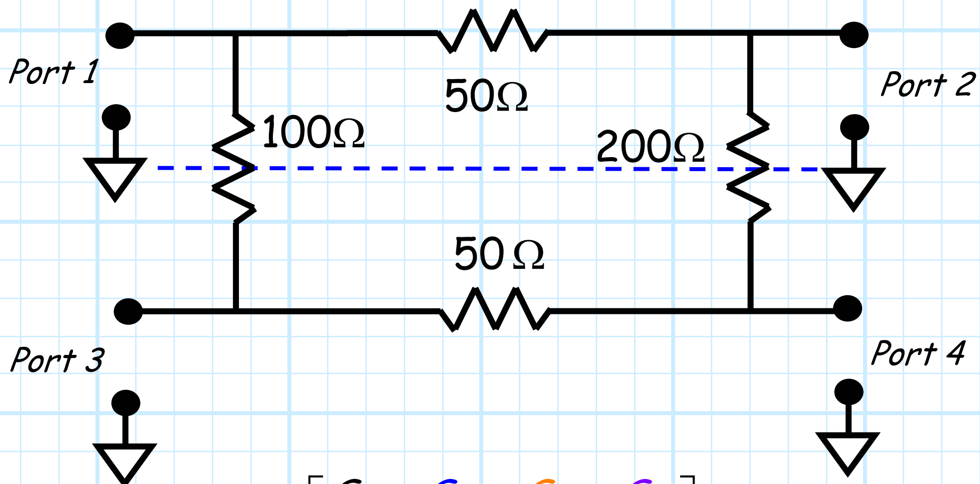
Or, for circuits with **this D_1** symmetry:

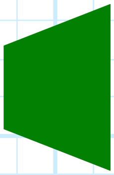
$1 \rightarrow 3$

$2 \rightarrow 4$

$3 \rightarrow 1$

$4 \rightarrow 2$



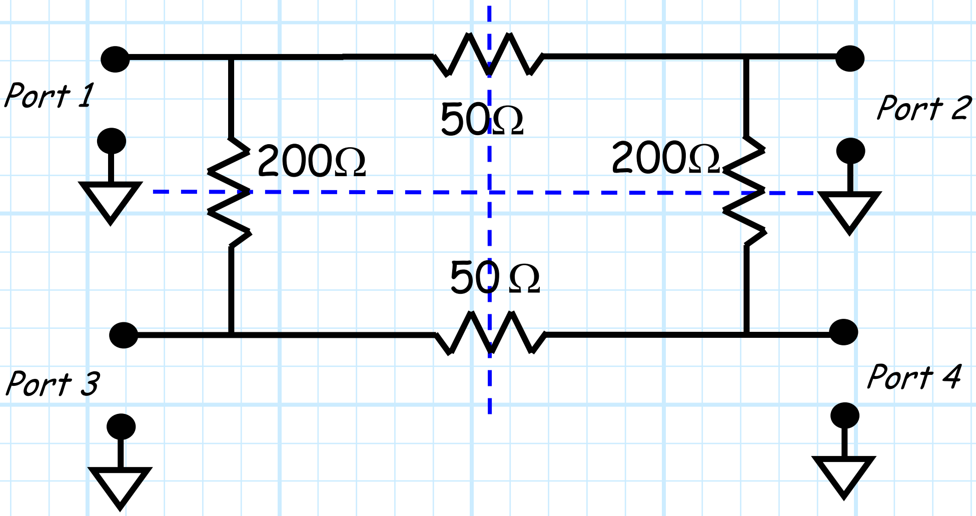


$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{22} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{11} & S_{21} \\ S_{41} & S_{31} & S_{21} & S_{22} \end{bmatrix}$$

Q: *Interesting. But **why do we care?***

A: This will greatly **simplify** the analysis of this symmetric circuit, as we need to determine **only** six matrix elements!

For a circuit with D_2 symmetry:

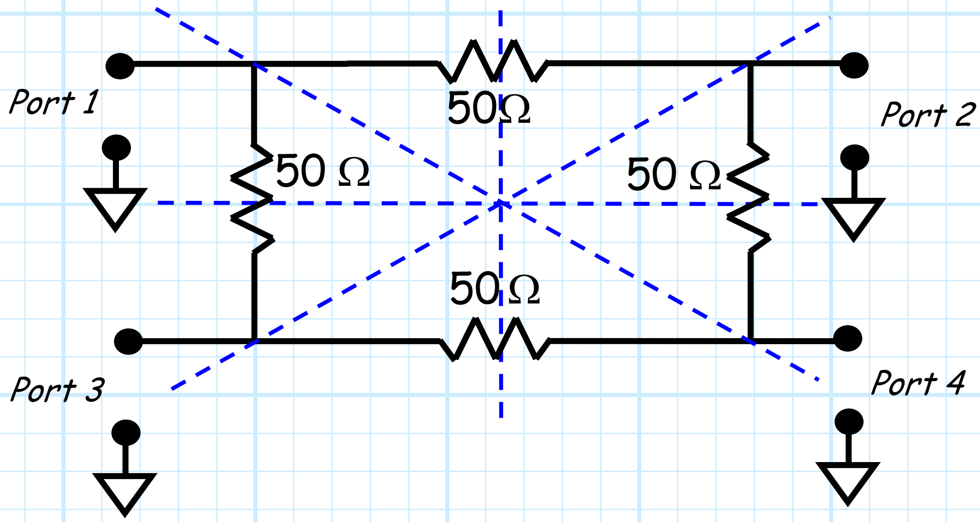


we find that the impedance (or scattering, or admittance) matrix has the form:

$$\mathcal{Z} = \begin{bmatrix} Z_{11} & Z_{21} & Z_{31} & Z_{41} \\ Z_{21} & Z_{11} & Z_{41} & Z_{31} \\ Z_{31} & Z_{41} & Z_{11} & Z_{21} \\ Z_{41} & Z_{31} & Z_{21} & Z_{11} \end{bmatrix}$$

Note here that there are just **four** independent values!

For a circuit with D_4 symmetry:



we find that the admittance (or scattering, or impedance) matrix has the form:

$$\mathcal{Y} = \begin{bmatrix} y_{11} & y_{21} & y_{21} & y_{41} \\ y_{21} & y_{11} & y_{41} & y_{21} \\ y_{21} & y_{41} & y_{11} & y_{21} \\ y_{41} & y_{21} & y_{21} & y_{11} \end{bmatrix}$$

Note here that there are just **three** independent values!

One more interesting thing (yet **another** one!); recall that we earlier found that a matched, lossless, reciprocal **4-port** device must have a scattering matrix with one of **two forms**:

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

The "symmetric" solution

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

The "anti-symmetric" solution

Compare these to the matrix forms above. The "symmetric solution" has the **same form** as the scattering matrix of a circuit with **D_2** symmetry!

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

Q: *Does this mean that a matched, lossless, reciprocal four-port device with the "symmetric" scattering matrix* ***must*** *exhibit* ***D_2*** *symmetry?*

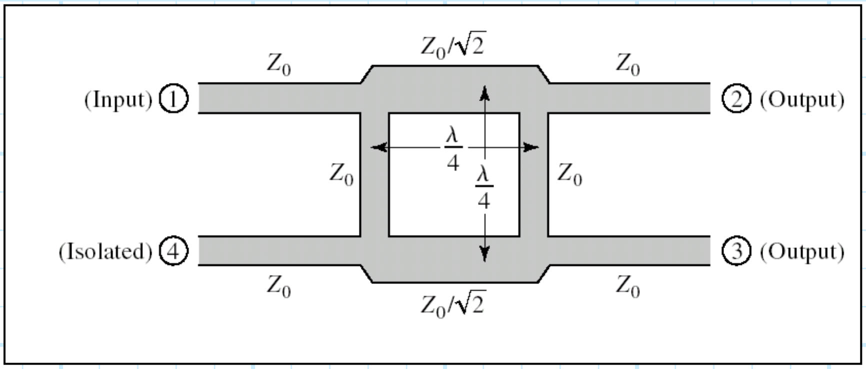
A: That's **exactly** what it means!

Not only can we determine from the **form** of the scattering matrix **whether** a particular design is possible (e.g., a matched, lossless, reciprocal 3-port device is impossible), we can also determine the **general structure** of a possible solutions (e.g. the circuit must have D_2 symmetry).

Likewise, the "anti-symmetric" matched, lossless, reciprocal four-port network **must** exhibit D_1 symmetry!

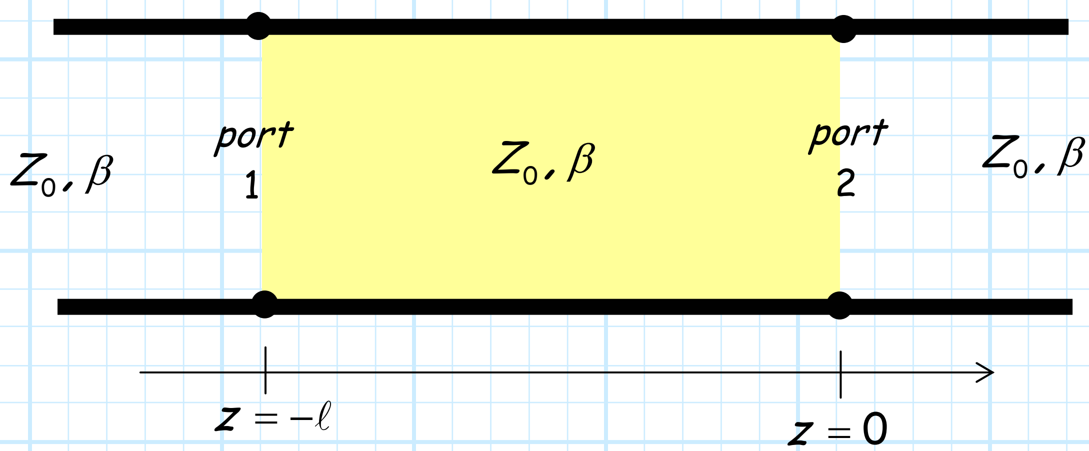
$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

We'll see just what these symmetric, matched, lossless, reciprocal four-port circuits actually are later in the course!



Example: Using Symmetry to Determine a Scattering Matrix

Say we wish to determine the scattering matrix of the simple two-port device shown below:



We note that that attaching transmission lines of characteristic impedance Z_0 to each port of our "circuit" forms a **continuous** transmission line of characteristic impedance Z_0.

Thus, the voltage **all along** this transmission line thus has the form:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

We begin by defining the location of port 1 as $z_{1P} = -\ell$, and the port location of port 2 as $z_{2P} = 0$:

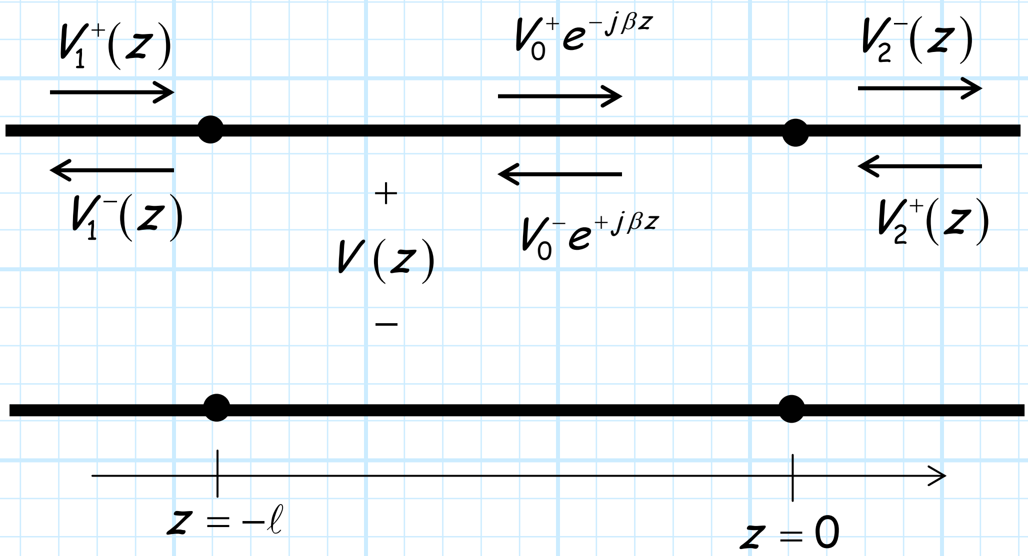
We can thus conclude:

$$V_1^+(z) = V_0^+ e^{-j\beta z} \qquad (z \leq -\ell)$$

$$V_1^-(z) = V_0^- e^{+j\beta z} \qquad (z \leq -\ell)$$

$$V_2^+(z) = V_0^- e^{+j\beta z} \qquad (z \geq 0)$$

$$V_2^-(z) = V_0^+ e^{-j\beta z} \qquad (z \geq 0)$$



Say the transmission line on port 2 is terminated in a **matched load**. We know that the **–z** wave must be **zero** $(V_0^- = 0)$, and so the voltage along the transmission line becomes simply the **+z** wave voltage:

$$V(z) = V_0^+ e^{-j\beta z}$$

and so:

$$V_1^+(z) = V_0^+ e^{-j\beta z} \qquad V_1^-(z) = 0 \qquad (z \le -\ell)$$

$$V_2^+(z) = 0 \qquad V_2^-(z) = V_0^+ e^{-j\beta z} \qquad (z \ge 0)$$

Now, **because** port 2 is terminated in a matched load, we can determine the scattering parameters S_{11} and S_{21}:

$$S_{11} = \left.\frac{V_1^-(z = z_{1P})}{V_1^+(z = z_{1P})}\right|_{V_2^+=0} = \left.\frac{V^-(z=-\ell)}{V^+(z=-\ell)}\right|_{V_2^+=0} = \frac{0}{V_0^+ \, e^{-j\beta(-\ell)}} = 0$$

$$S_{21} = \left.\frac{V_2^-(z = z_{2P})}{V_1^+(z = z_{1P})}\right|_{V_2^+=0} = \left.\frac{V_2^-(z=0)}{V_1^+(z=-\ell)}\right|_{V_2^+=0} = \frac{V_0^+ \, e^{-j\beta(0)}}{V_0^+ \, e^{-j\beta(-\ell)}} = \frac{1}{e^{+j\beta\ell}} = e^{-j\beta\ell}$$

From the **symmetry** of the structure, we can conclude:

$$S_{22} = S_{11} = 0$$

And from both reciprocity **and** symmetry:

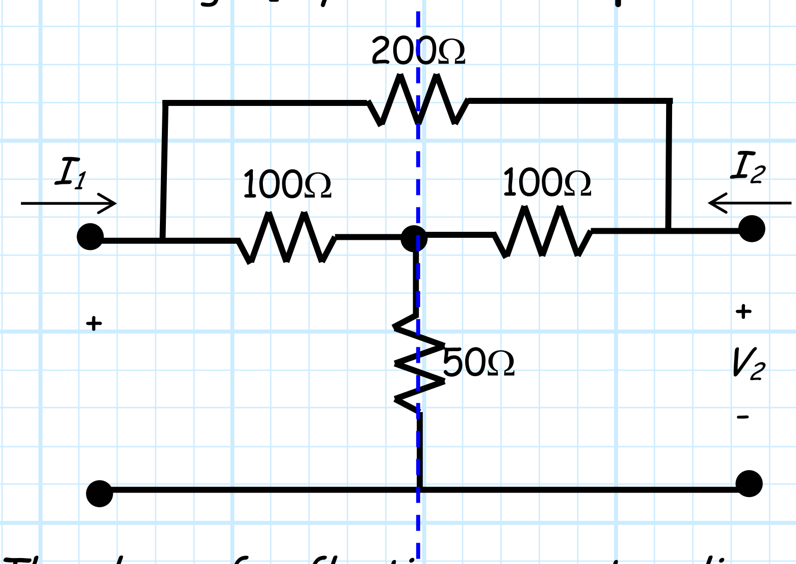
$$S_{12} = S_{21} = e^{-j\beta\ell}$$

Thus:

$$\mathcal{S} = \begin{bmatrix} 0 & e^{-j\beta\ell} \\ e^{-j\beta\ell} & 0 \end{bmatrix}$$

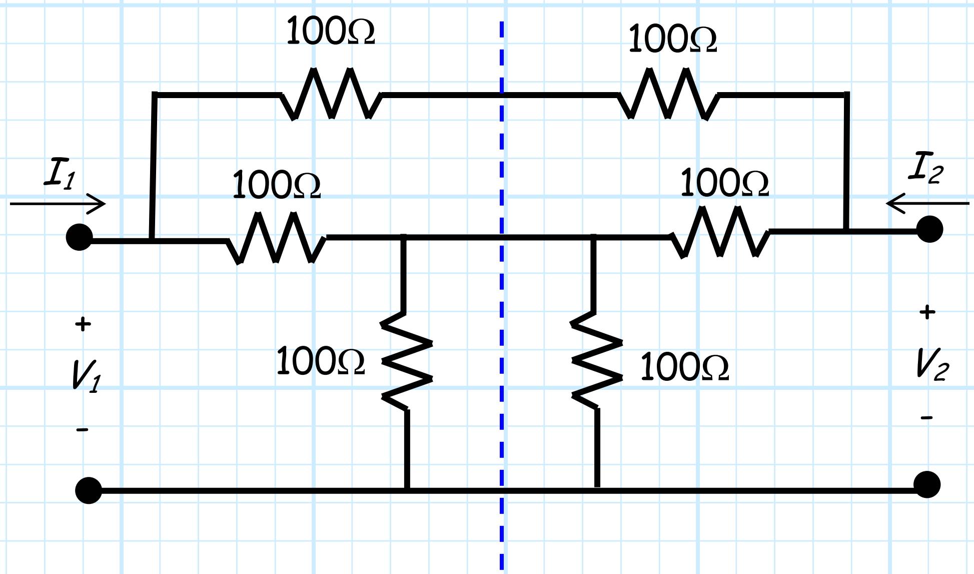
Symmetric Circuit Analysis

Consider the following D_1 symmetric **two-port** device:



Q: *Yikes! The plane of reflection symmetry slices through two resistors. What can we do about that?*

A: Resistors are easily split into two equal pieces: the 200Ω resistor into two 100Ω resistors in **series**, and the 50Ω resistor as two 100 Ω resistors in **parallel**.



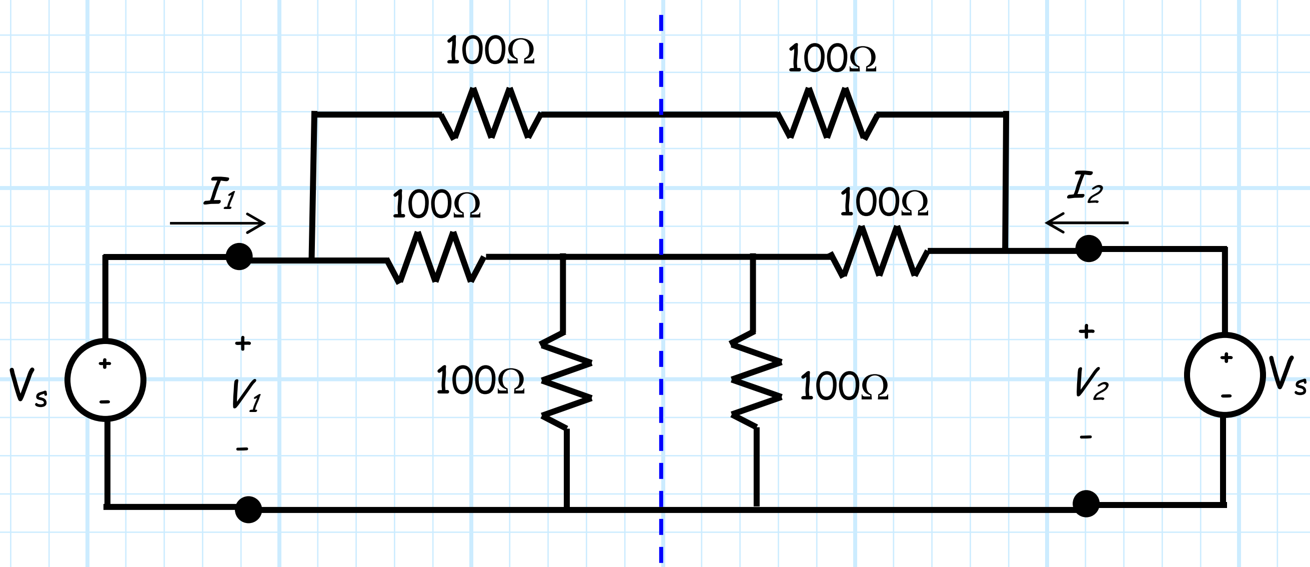
Recall that the **symmetry** of this 2-port device leads to **simplified** network matrices:

$$\mathcal{S}=\begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{11} \end{bmatrix} \qquad \mathcal{Z}=\begin{bmatrix} Z_{11} & Z_{21} \\ Z_{21} & Z_{11} \end{bmatrix} \qquad \mathcal{Y}=\begin{bmatrix} Y_{11} & Y_{21} \\ Y_{21} & Y_{11} \end{bmatrix}$$

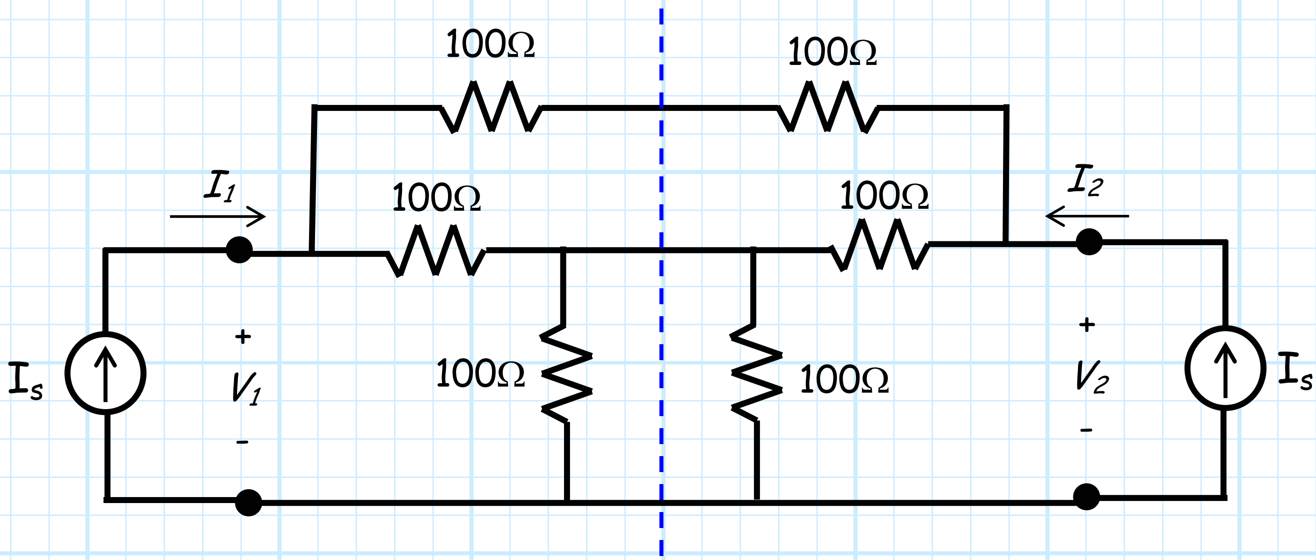
Q: *Yes, but can circuit symmetry likewise simplify the procedure of **determining** these elements? In other words, can symmetry be used to **simplify circuit analysis**?*

A: You bet!

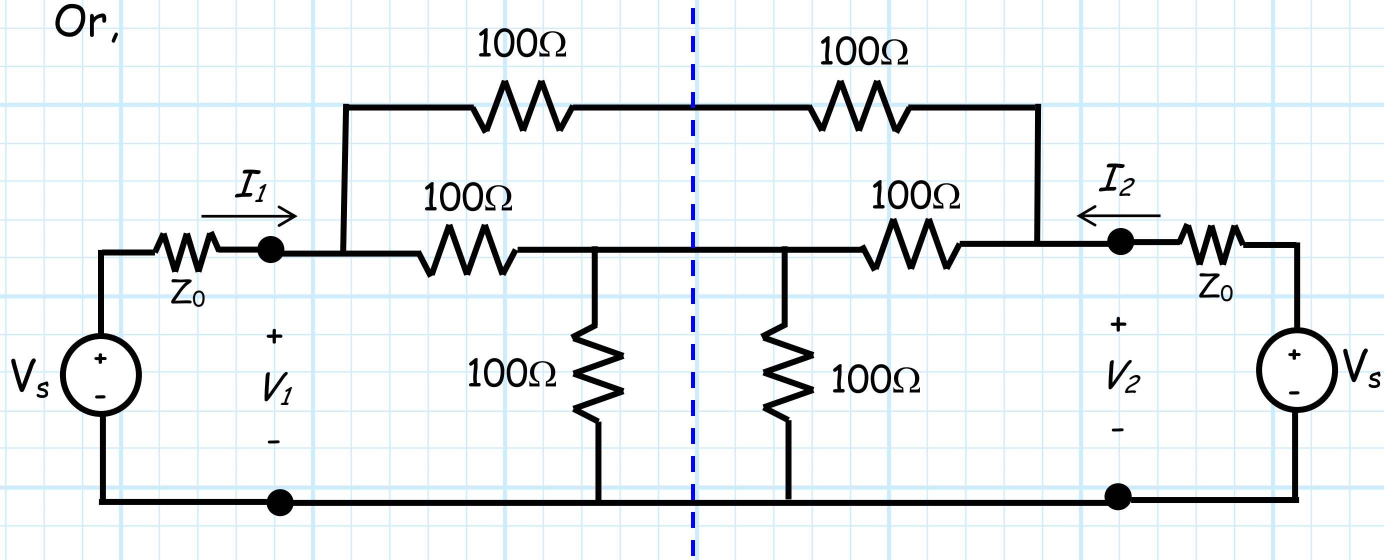
First, consider the case where we **attach sources** to circuit in a way that **preserves** the circuit **symmetry**:



Or,

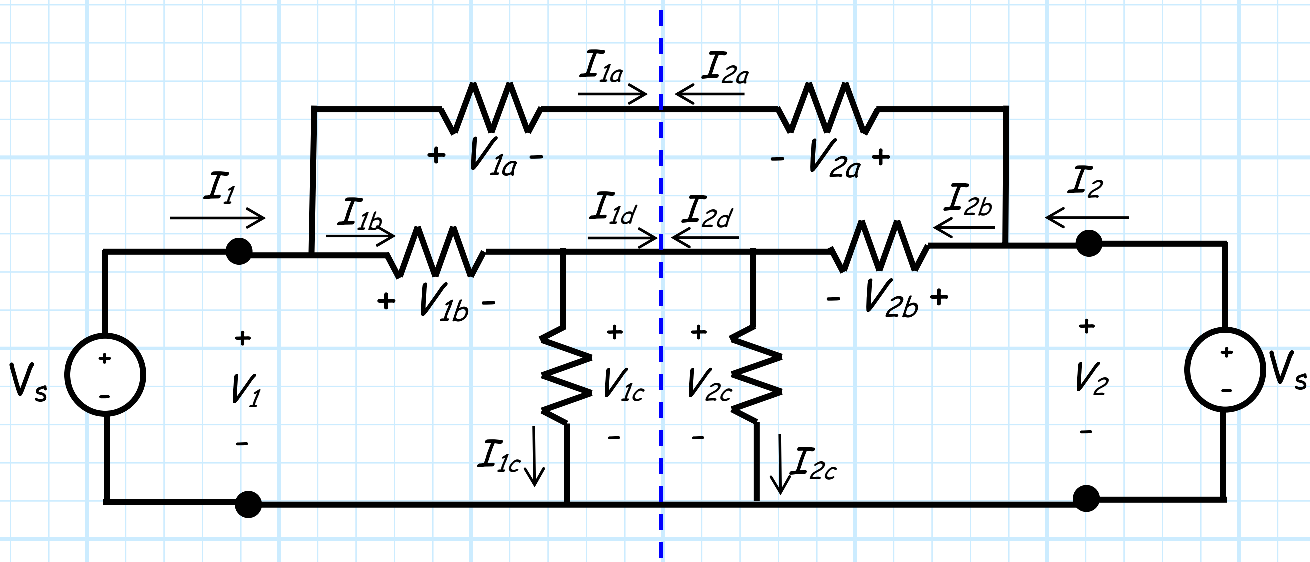


Or,



But remember! In order for **symmetry to be preserved**, the source values on both sides (i.e, I_s, V_s, Z_0) must be **identical**!

Now, consider the **voltages** and **currents** within this circuit under this symmetric configuration:



Since this circuit possesses **bilateral** (reflection) symmetry ($1 \rightarrow 2, 2 \rightarrow 1$), symmetric currents and voltages must be equal:

$$
\begin{aligned}
V_1 &= V_2 \\
V_{1a} &= V_{2a} \\
V_{1b} &= V_{2b} \\
V_{1c} &= V_{2c}
\end{aligned}
\qquad
\begin{aligned}
I_1 &= I_2 \\
I_{1a} &= I_{2a} \\
I_{1b} &= I_{2b} \\
I_{1c} &= I_{2c} \\
I_{1d} &= I_{2d}
\end{aligned}
$$

Q: ***Wait!*** *This* ***can't*** *possibly be correct! Look at currents I_{1a} and I_{2a}, as well as currents I_{1d} and I_{2d}. From KCL,* ***this*** *must be true:*

$$I_{1a} = -I_{2a} \qquad I_{1d} = -I_{2d}$$

Yet ***you*** *say that* ***this*** *must be true:*

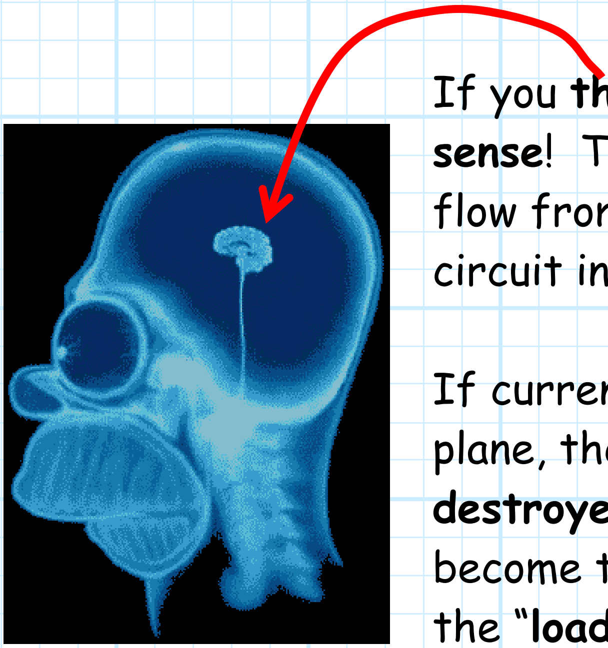
$$I_{1a} = I_{2a} \qquad I_{1d} = I_{2d}$$

*There is an **obvious contradiction** here! There is **no way** that both sets of equations can simultaneously be correct, **is there?***

A: Actually there **is**! There is **one** solution that will satisfy **both** sets of equations:

$$I_{1a} = I_{2a} = 0 \qquad\qquad I_{1d} = I_{2d} = 0$$

The currents are **zero**!

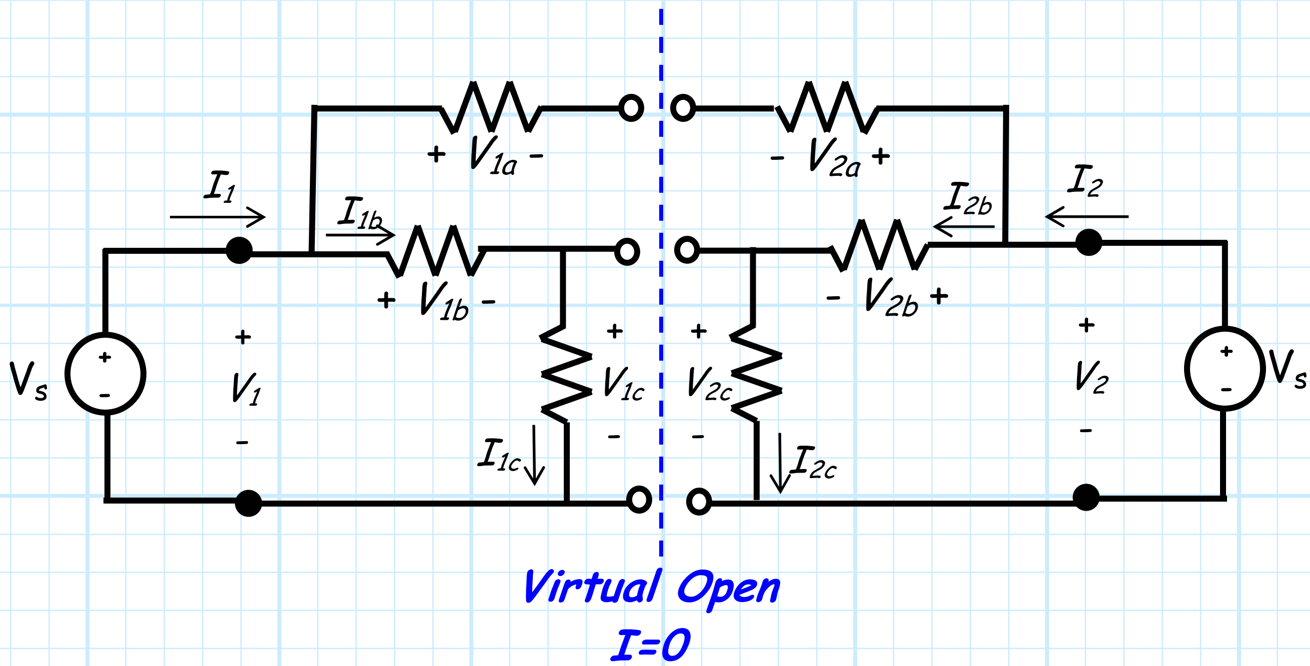


If you **think** about it, this makes **perfect sense**! The result says that **no current** will flow from one side of the symmetric circuit into the other.

If current **did** flow across the symmetry plane, then the circuit symmetry would be **destroyed**—one side would effectively become the "**source side**", and the other the "**load side**" (i.e., the source side delivers current to the load side).

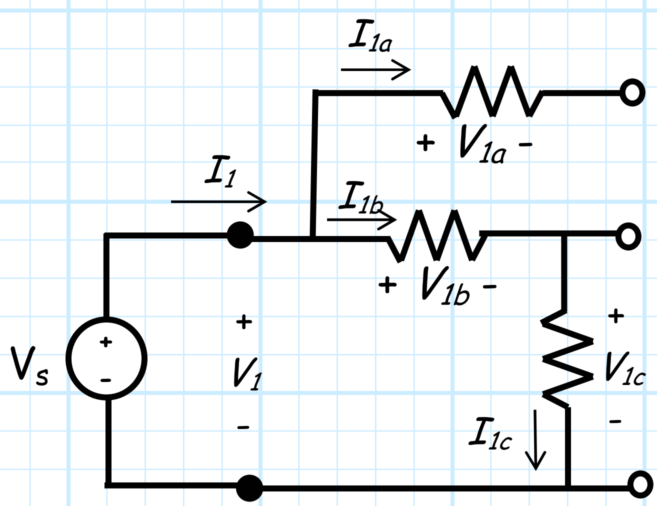
Thus, **no current** will flow **across** the reflection symmetry plane of a **symmetric circuit**—the symmetry plane thus acts as a **open circuit**!

The plane of symmetry thus becomes a **virtual open**!



Q: *So what?*

A: So what! This means that our circuit can be **split apart** into **two separate** but **identical** circuits. Solve **one** half-circuit, and you have **solved** the other!



$$V_1 = V_2 = V_s$$

$$V_{1a} = V_{2a} = 0$$

$$V_{1b} = V_{2b} = V_s/2$$

$$V_{1c} = V_{2c} = V_s/2$$

$$I_1 = I_2 = V_s/200$$

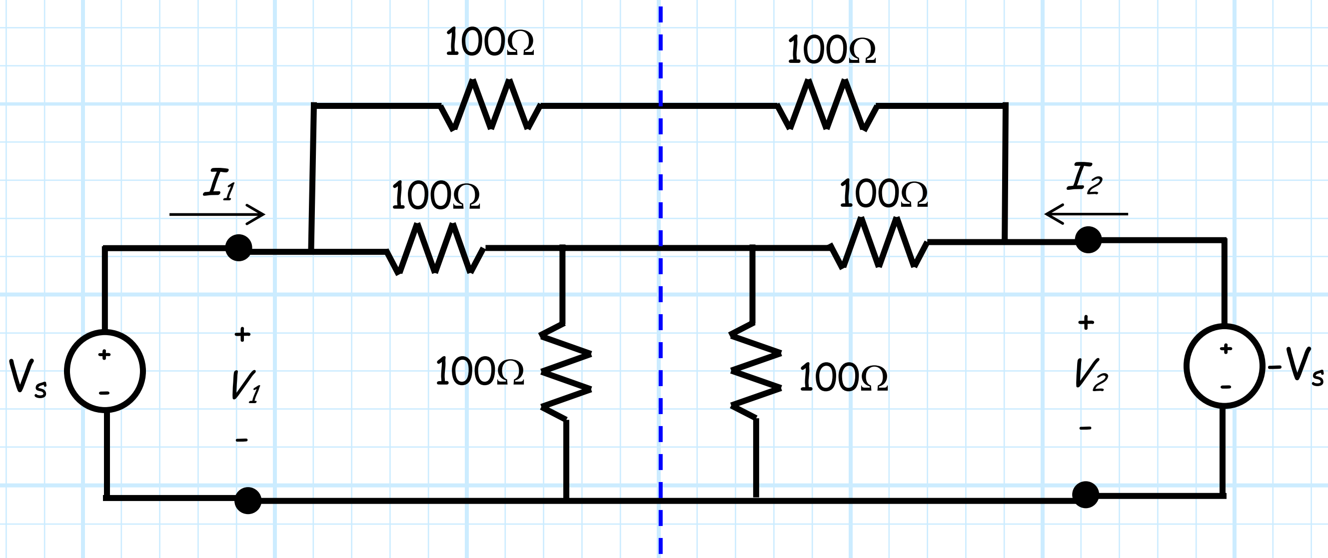
$$I_{1a} = I_{2a} = 0$$

$$I_{1b} = I_{2b} = V_s/200$$

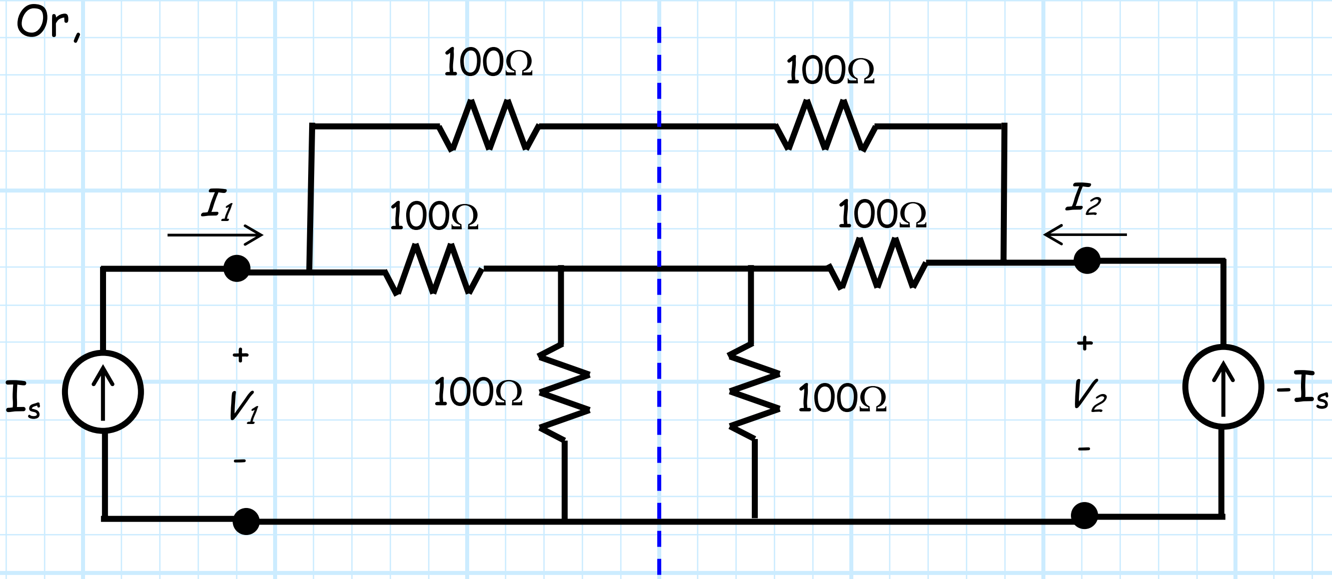
$$I_{1c} = I_{2c} = V_s/200$$

$$I_{1d} = I_{2d} = 0$$

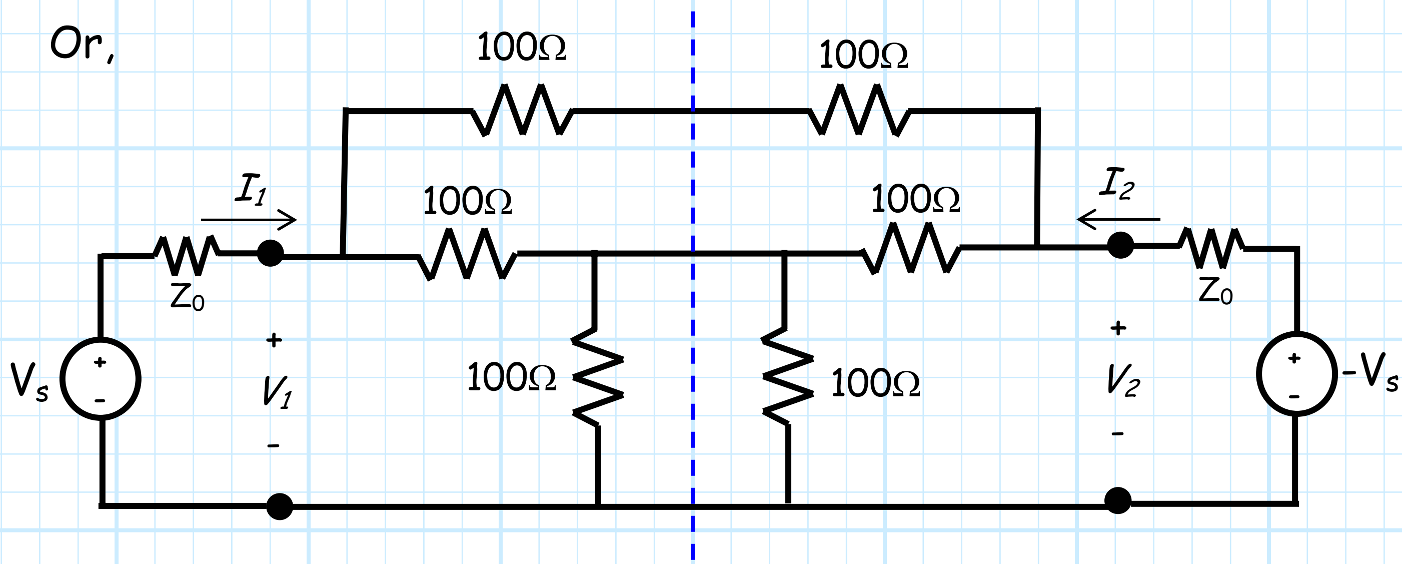
Now, consider **another** type of symmetry, where the sources are **equal** but **opposite** (i.e., **180 degrees** out of phase).



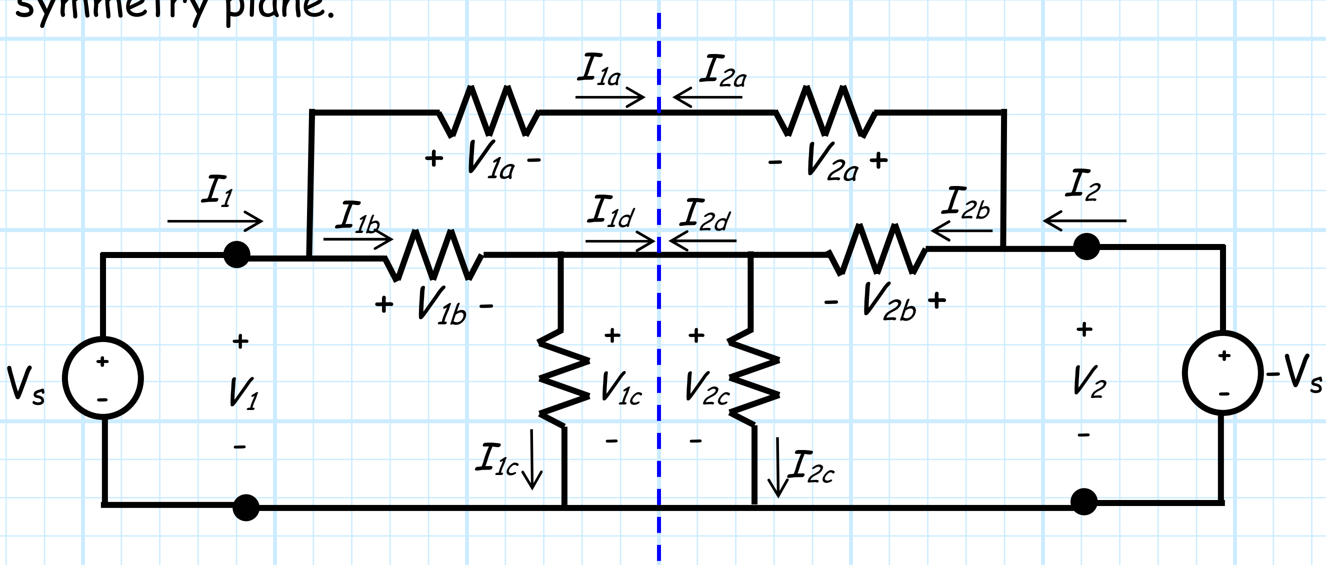
Or,



Or,



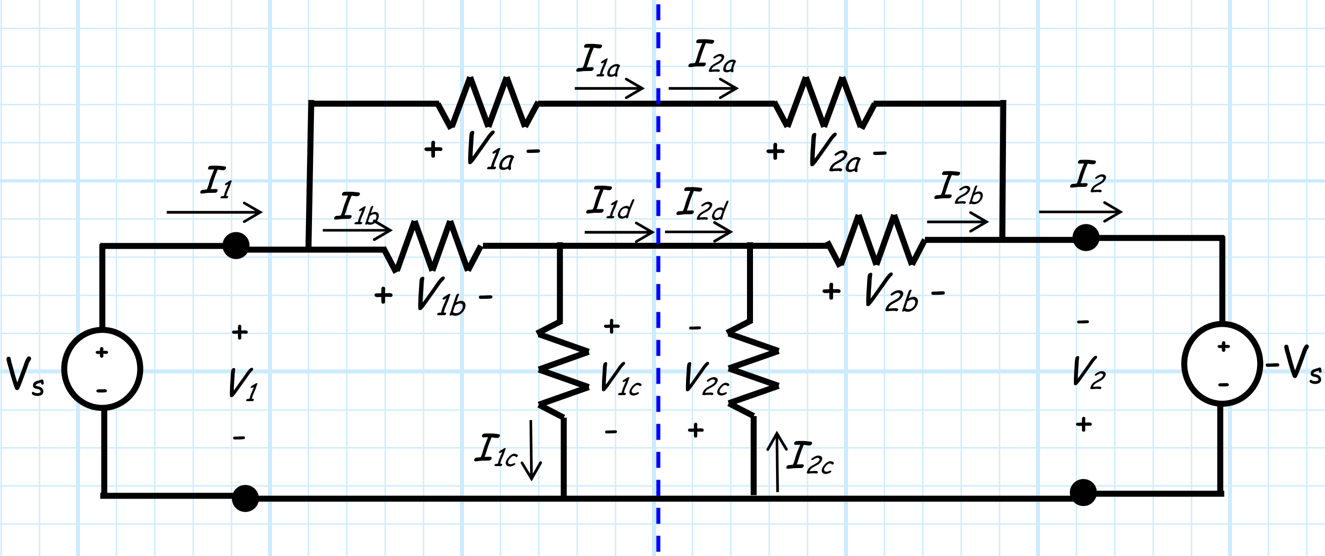
This situation still preserves the **symmetry** of the circuit—**somewhat.** The **voltages** and **currents** in the circuit will now posses **odd symmetry**—they will be **equal but opposite** (180 degrees out of phase) at symmetric points across the symmetry plane.



$$V_1 = -V_2$$
$$V_{1a} = -V_{2a}$$
$$V_{1b} = -V_{2b}$$
$$V_{1c} = -V_{2c}$$

$$I_1 = -I_2$$
$$I_{1a} = -I_{2a}$$
$$I_{1b} = -I_{2b}$$
$$I_{1c} = -I_{2c}$$
$$I_{1d} = -I_{2d}$$

Perhaps it would be easier to **redefine** the circuit variables as:



$$V_1 = V_2$$
$$V_{1a} = V_{2a}$$
$$V_{1b} = V_{2b}$$
$$V_{1c} = V_{2c}$$

$$I_1 = I_2$$
$$I_{1a} = I_{2a}$$
$$I_{1b} = I_{2b}$$
$$I_{1c} = I_{2c}$$
$$I_{1d} = I_{2d}$$

Q: *But wait! **Again** I see a problem. By **KVL** it is evident that:*

$$V_{1c} = -V_{2c}$$

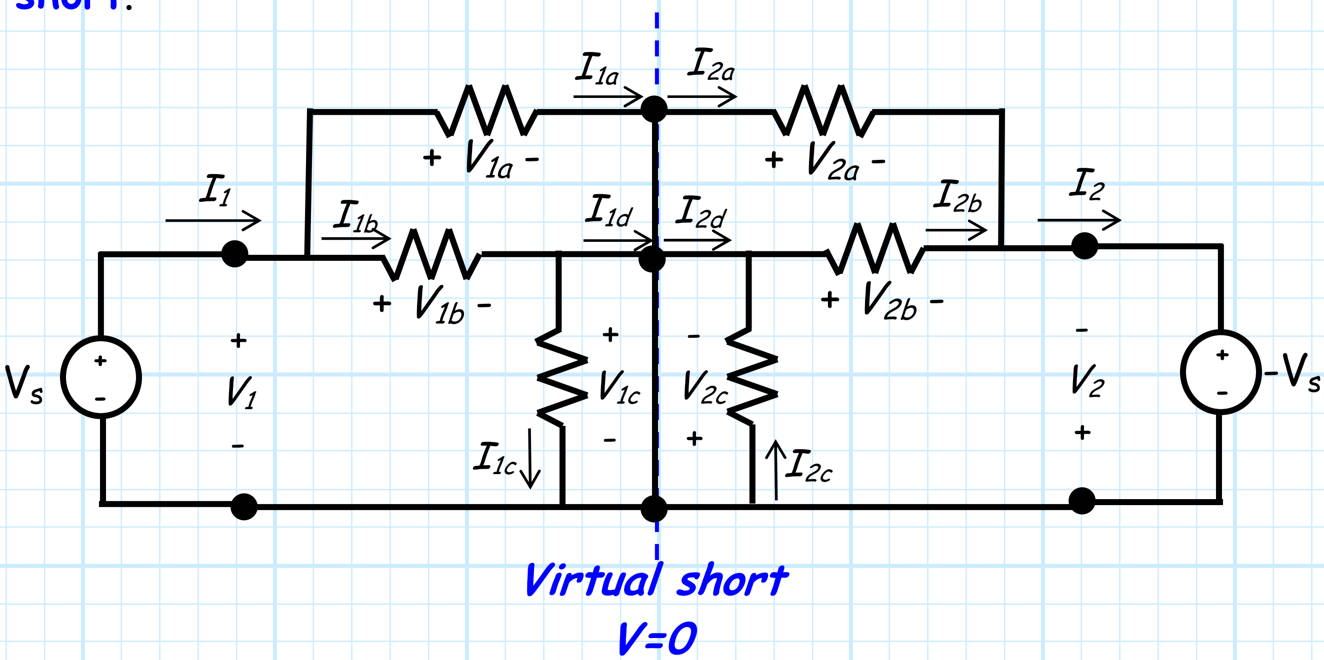
*Yet **you** say that $V_{1c} = V_{2c}$ must be true!*

A: Again, the solution to **both** equations is **zero**!

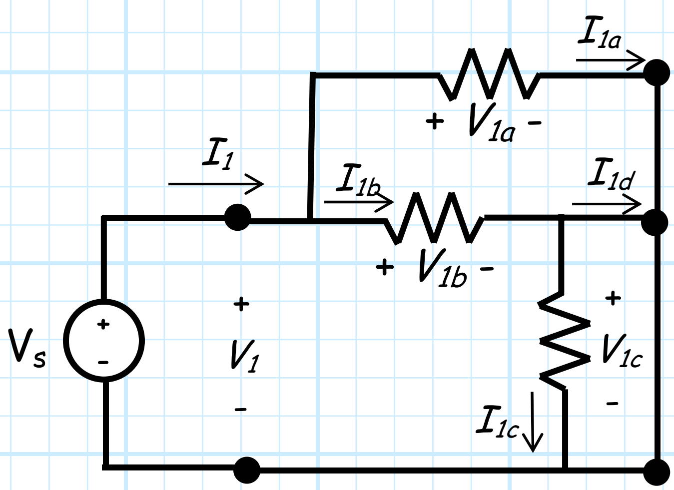
$$V_{1c} = V_{2c} = 0$$

For the case of **odd symmetry**, the symmetric plane must be a plane of **constant potential** (i.e., constant voltage)—just like a **short circuit**!

Thus, for odd symmetry, the symmetric plane forms a **virtual short**.



This **greatly** simplifies things, as we can again **break** the circuit into **two** independent and (effectively) identical circuits!



$V_1 = V_s$

$V_{1a} = V_s$

$V_{1b} = V_s$

$V_{1c} = 0$

$I_1 = V_s/50$

$I_{1a} = V_s/100$

$I_{1b} = V_s/100$

$I_{1c} = 0$

$I_{1d} = V_s/100$

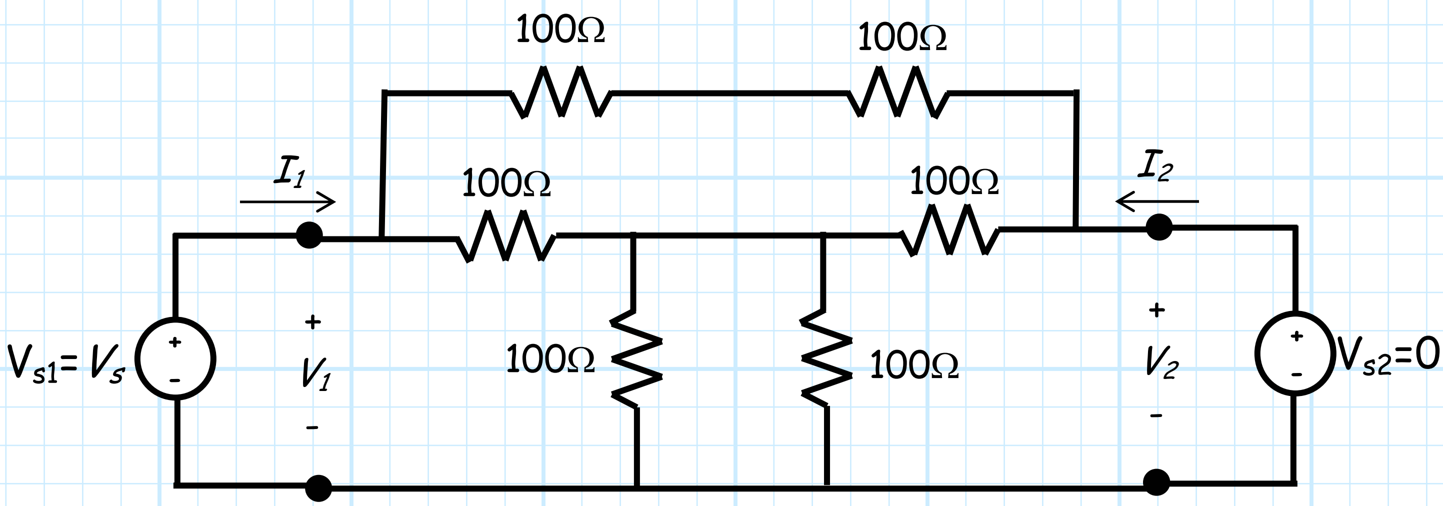
Odd/Even Mode Analysis

Q: *Although symmetric* ***circuits*** *appear to be plentiful in microwave engineering, it seems* ***unlikely*** *that we would often encounter symmetric* ***sources*** *. Do virtual shorts and opens typically ever occur?*

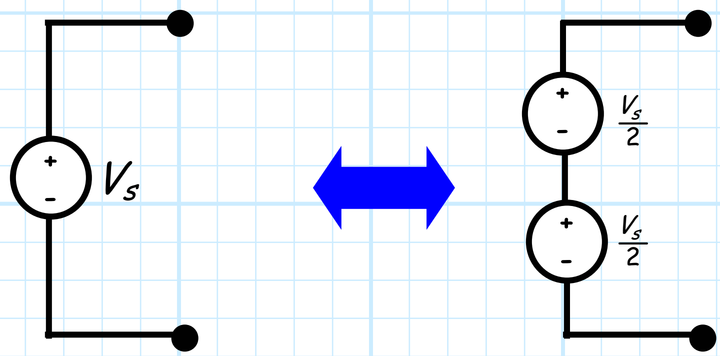
A: One word—**superposition**!

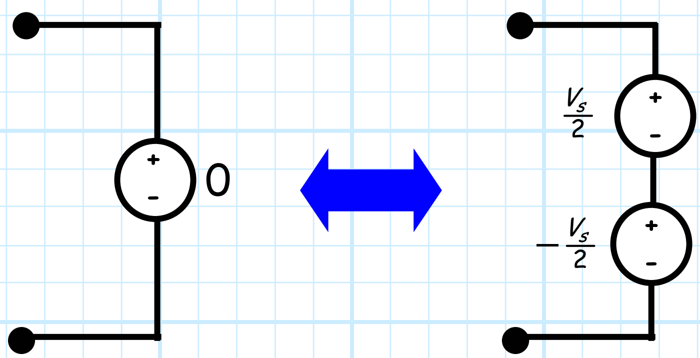
If the elements of our circuit are **independent** and **linear**, we can apply superposition to analyze **symmetric circuits** when **non-symmetric** sources are attached.

For example, say we wish to determine the **admittance matrix** of this circuit. We would place a **voltage source at port 1**, and a **short circuit at port 2**—a set of **asymmetric** sources if there ever was one!

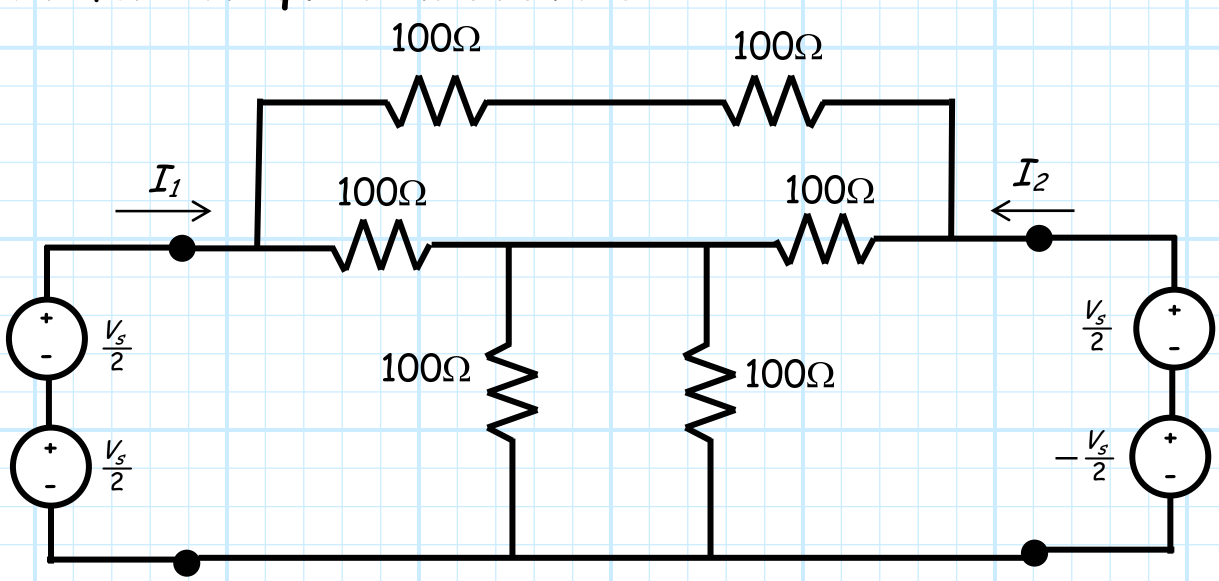


Here's the really **neat** part. We find that the source on port 1 can be model as **two equal** voltage sources in series, whereas the source at port 2 can be modeled as **two equal but opposite** sources in series.

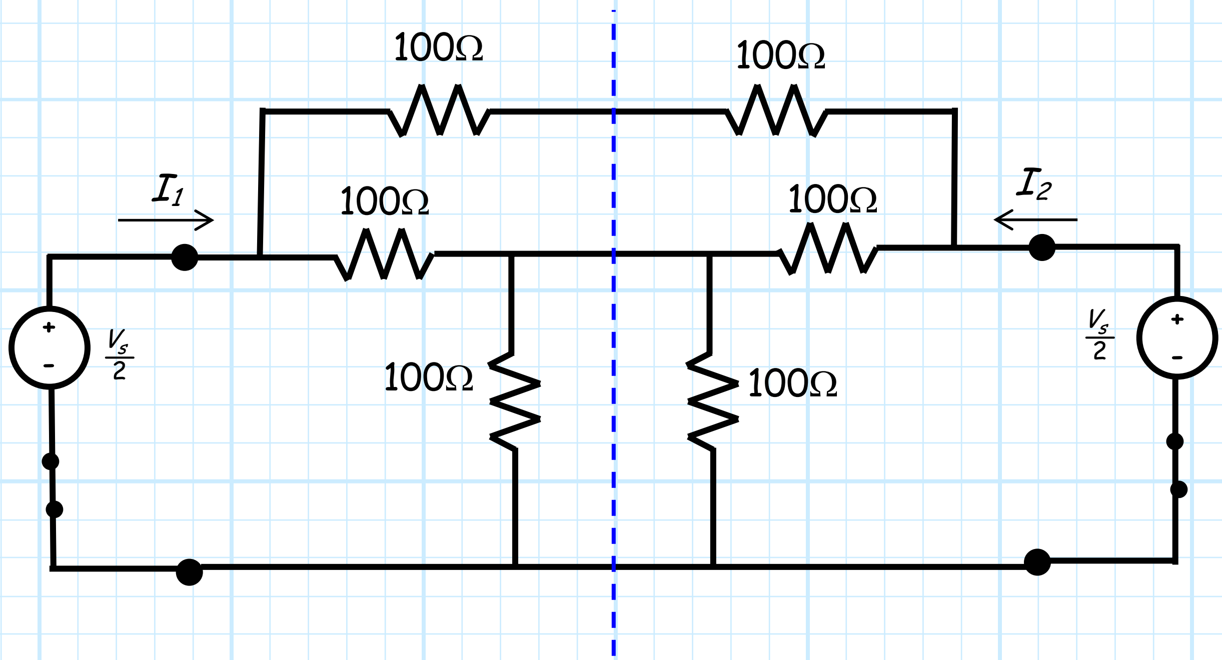




Therefore an **equivalent** circuit is:



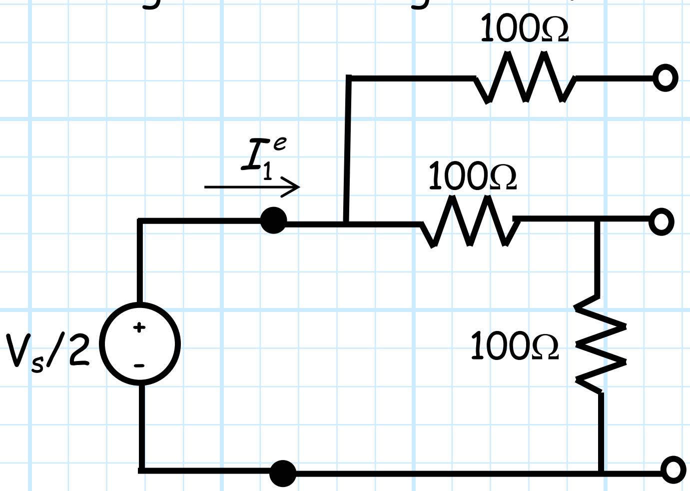
Now, the **above** circuit (due to the sources) is obviously **asymmetric**—**no** virtual ground, **nor** virtual short is present. But, let's say we **turn off** (i.e., set to V=0) the **bottom** source on **each side** of the circuit:



Our **symmetry** has been **restored**! The symmetry plane is a **virtual open**.

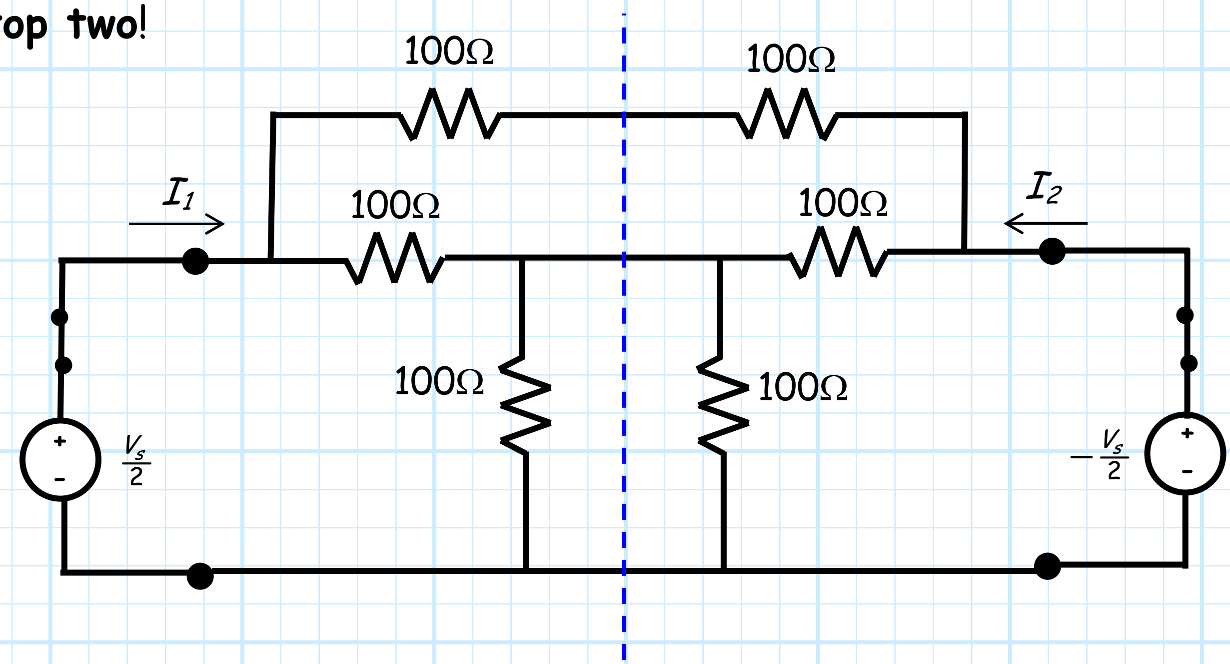
This circuit is referred to as its **even mode**, and analysis of it is known as the **even mode analysis**. The solutions are known as the even mode **currents** and **voltages**!

Evaluating the resulting **even mode** half circuit we find:



$$I_1^e = \frac{V_s}{2}\frac{1}{200} = \frac{V_s}{400} = I_2^e$$

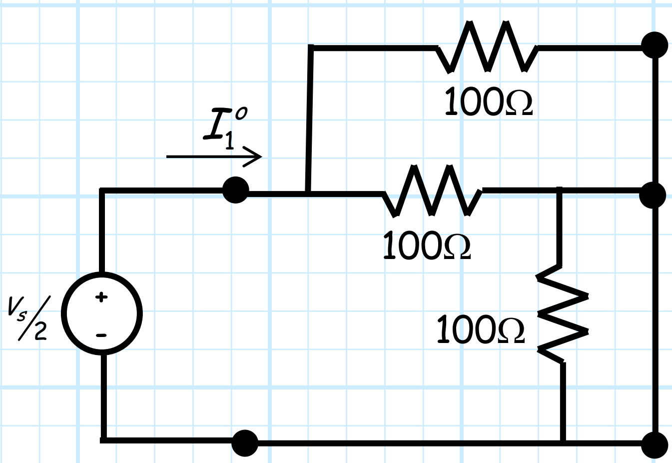
Now, let's turn the bottom sources **back on**—but turn **off** the **top two**!



We now have a circuit with **odd symmetry**—the symmetry plane is a **virtual short**!

This circuit is referred to as its **odd mode**, and analysis of it is known as the **odd mode analysis**. The solutions are known as the odd mode **currents** and **voltages**!

Evaluating the resulting **odd mode** half circuit we find:



$$I_1^o = \frac{V_s}{2}\frac{1}{50} = \frac{V_s}{100} = -I_2^o$$

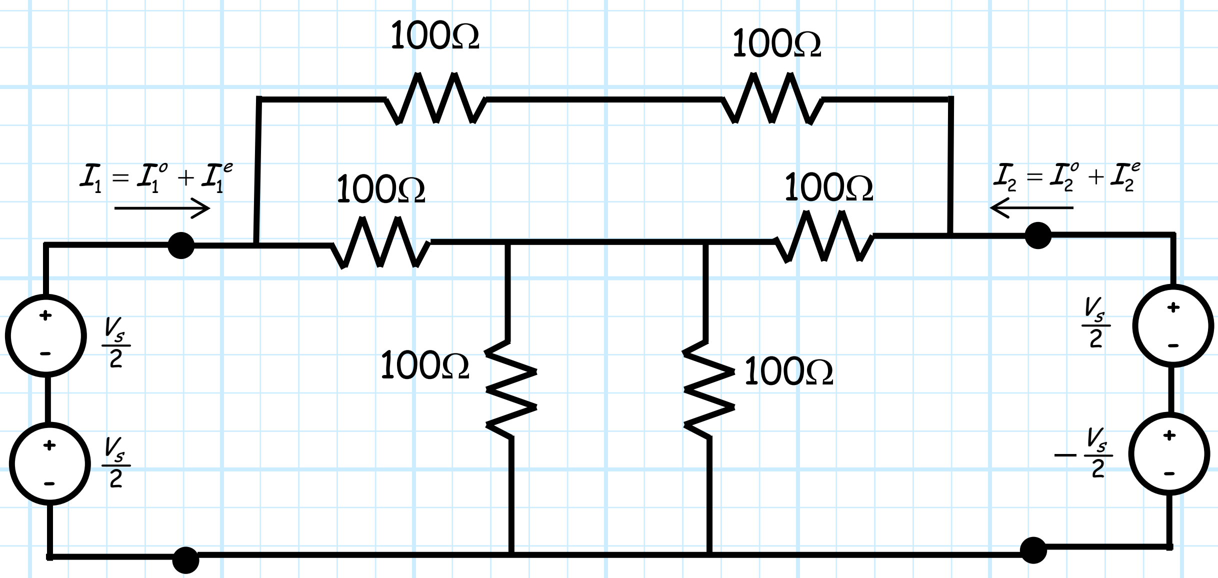
Q: *But what good is this "even mode" and "odd mode" analysis? After all, the source on port 1 is $V_{s1} = V_s$, and the source on port 2 is $V_{s2} = 0$. What are the currents I_1 and I_2 for **these** sources?*

A: Recall that these sources are the **sum** of the even and odd mode sources:

$$V_{s1} = V_s = \frac{V_s}{2} + \frac{V_s}{2} \qquad\qquad V_{s2} = 0 = \frac{V_s}{2} - \frac{V_s}{2}$$

and thus—since all the devices in the circuit are **linear**—we know from superposition that the currents I_1 and I_2 are simply the **sum** of the **odd** and **even** mode currents !

$$I_1 = I_1^e + I_1^o \qquad\qquad I_2 = I_2^e + I_2^o$$



Thus, **adding** the odd and even mode analysis results together:

$$I_1 = I_1^e + I_1^o = \frac{V_s}{400} + \frac{V_s}{100} = \frac{V_s}{80}$$

$$I_2 = I_2^e + I_2^o = \frac{V_s}{400} - \frac{V_s}{100} = -\frac{3\,V_s}{400}$$

And then the **admittance parameters** for this two port network is:

$$Y_{11} = \left.\frac{I_1}{V_{s1}}\right|_{V_{s2}=0} = \frac{V_s}{80}\frac{1}{V_s} = \frac{1}{80}$$

$$Y_{21} = \left.\frac{I_2}{V_{s1}}\right|_{V_{s2}=0} = -\frac{3\,V_s}{400}\frac{1}{V_s} = \frac{-3}{400}$$

And from the **symmetry** of the device we know:

$$Y_{22} = Y_{11} = \frac{1}{80}$$

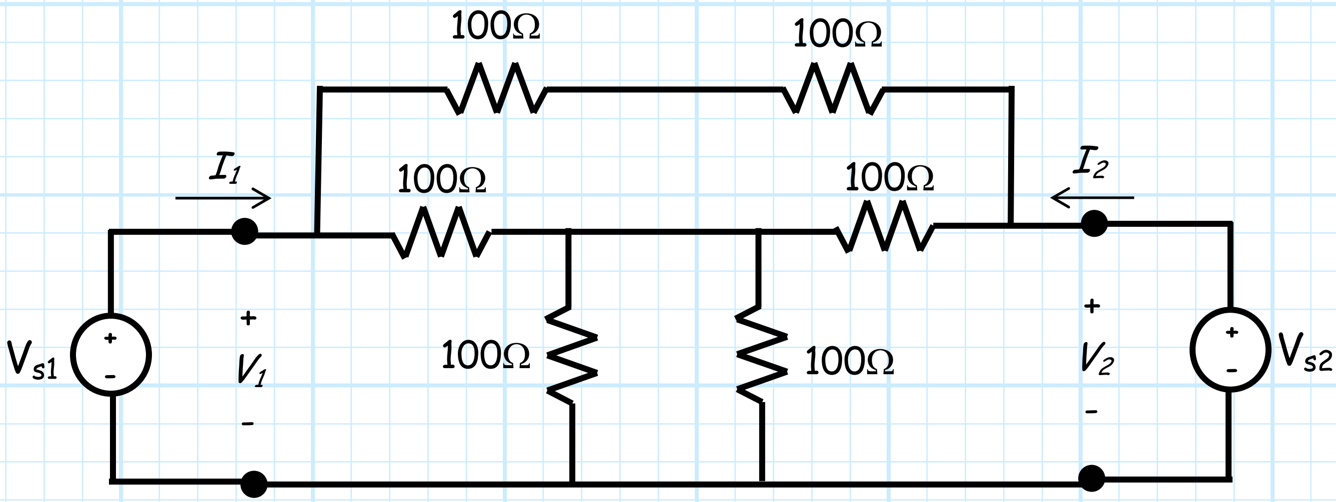
$$Y_{12} = Y_{21} = \frac{-3}{400}$$

Thus, the full **admittance matrix** is:

$$\mathcal{Y} = \begin{bmatrix} 1/80 & -3/400 \\ -3/400 & 1/80 \end{bmatrix}$$

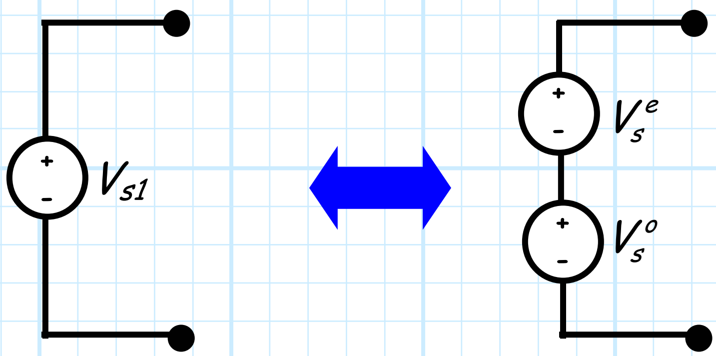
Q: *What happens if **both** sources are **non-zero**? Can we use symmetry then?*

A: Absolutely! Consider the problem below, where **neither** source is equal to zero:

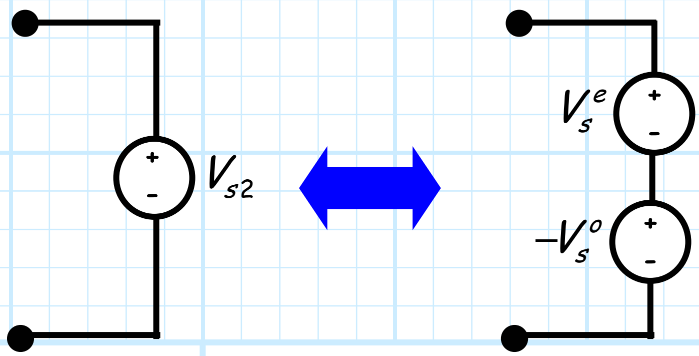


In this case we can define an even mode and an odd mode source as:

$$V_s^e = \frac{V_{s1} + V_{s2}}{2} \qquad\qquad V_s^o = \frac{V_{s1} - V_{s2}}{2}$$

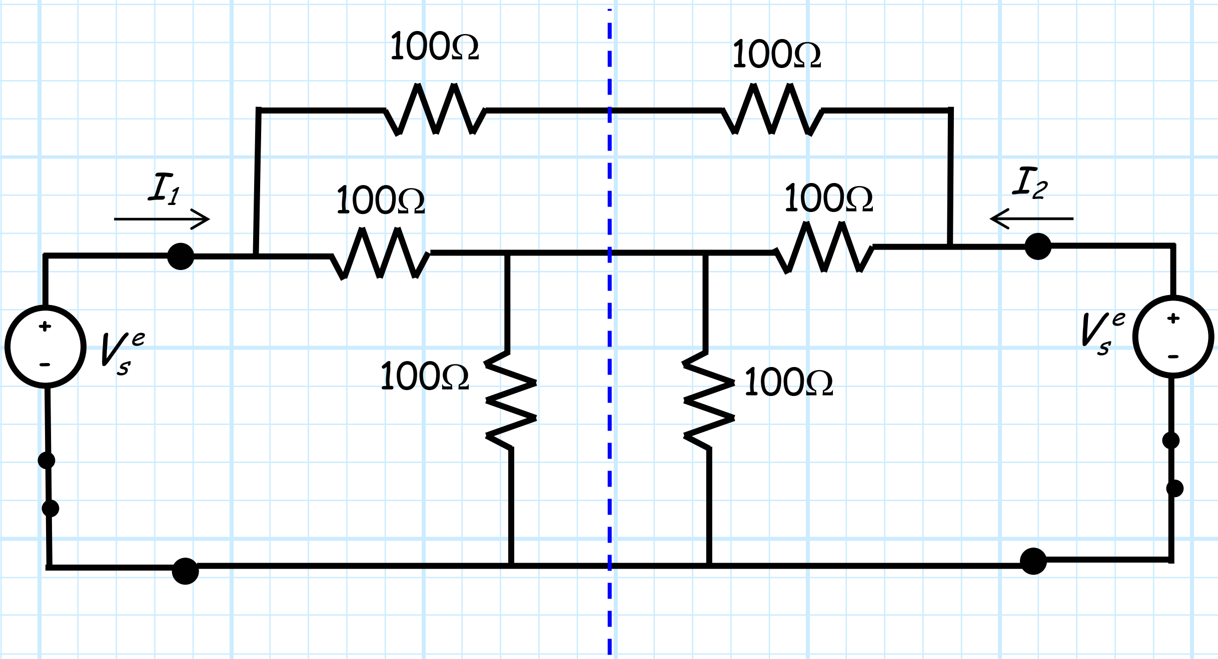


$$V_{s1} = V_s^e + V_s^o$$

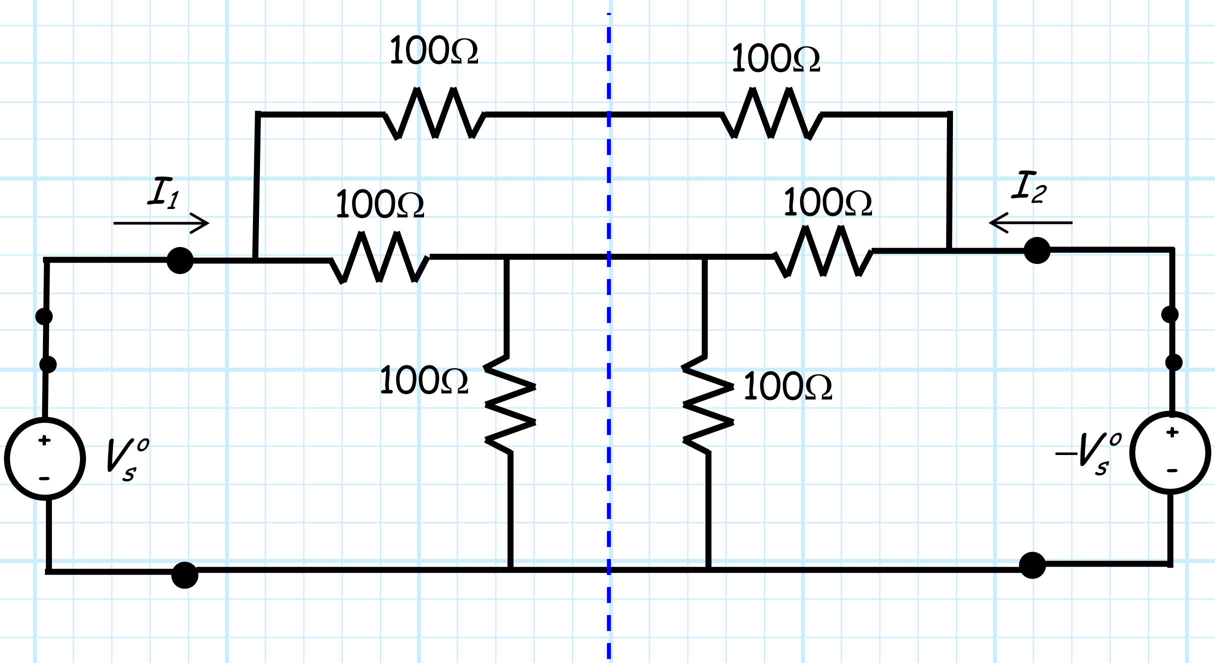


$$V_{s2} = V_s^e - V_s^o$$

We then can analyze the **even mode** circuit:



And then the **odd mode** circuit:

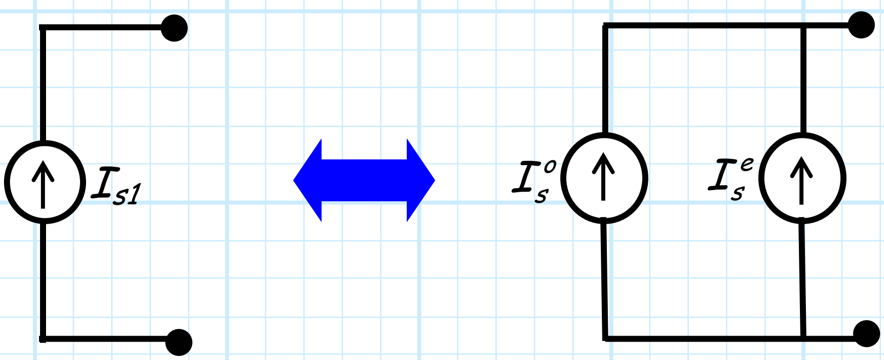


And then combine these results in a **linear superposition**!

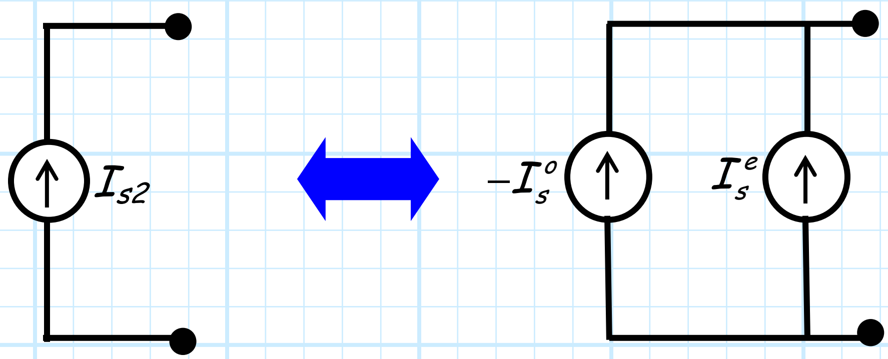
Q: *What about* ***current sources****? Can I likewise consider them to be a* ***sum*** *of an odd mode source and an even mode source?*

A: Yes, but be **very** careful! The current of two source will add if they are placed in **parallel—not** in series! Therefore:

$$I_s^e = \frac{I_{s1} + I_{s2}}{2} \qquad\qquad I_s^o = \frac{I_{s1} - I_{s2}}{2}$$



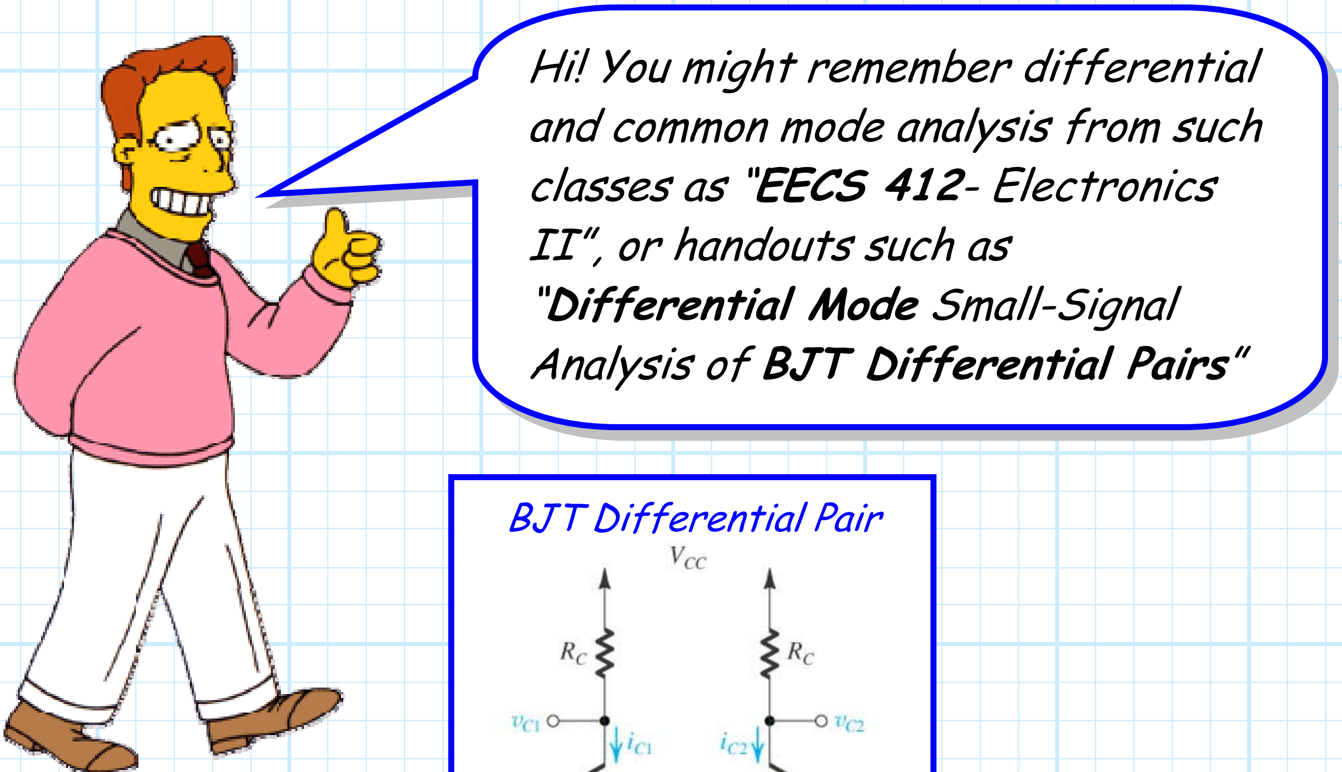
$$I_{s1} = I_s^e + I_s^o$$

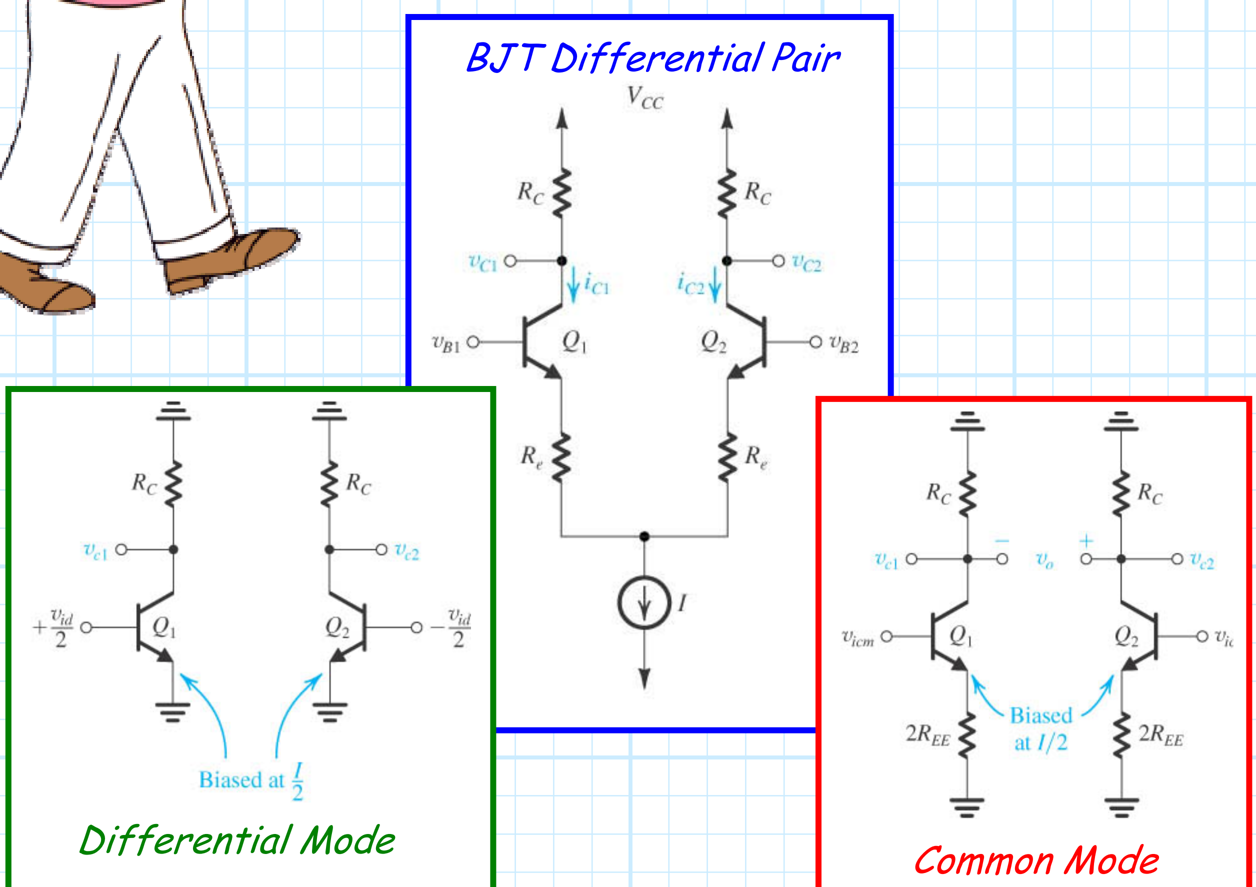


$$I_{s2} = I_s^e - I_s^o$$

One **final** word (I promise!) about circuit symmetry and even/odd mode analysis: **precisely** the **same** concept exits in **electronic circuit** design!

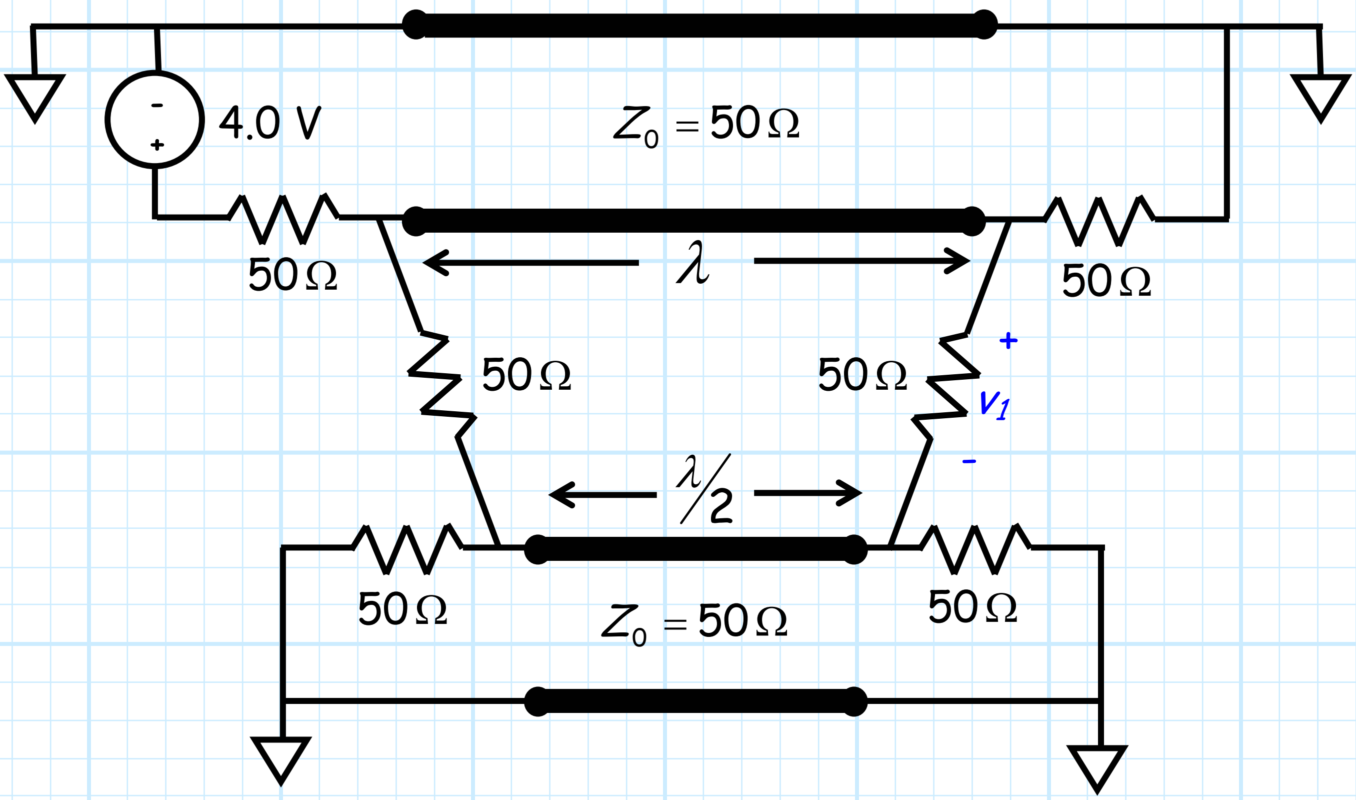
Specifically, the **differential** (odd) and **common** (even) **mode** analysis of bilaterally symmetric electronic circuits, such as **differential amplifiers**!





Example: Odd-Even Mode Circuit Analysis

Carefully (**very** carefully) consider the **symmetric** circuit below.

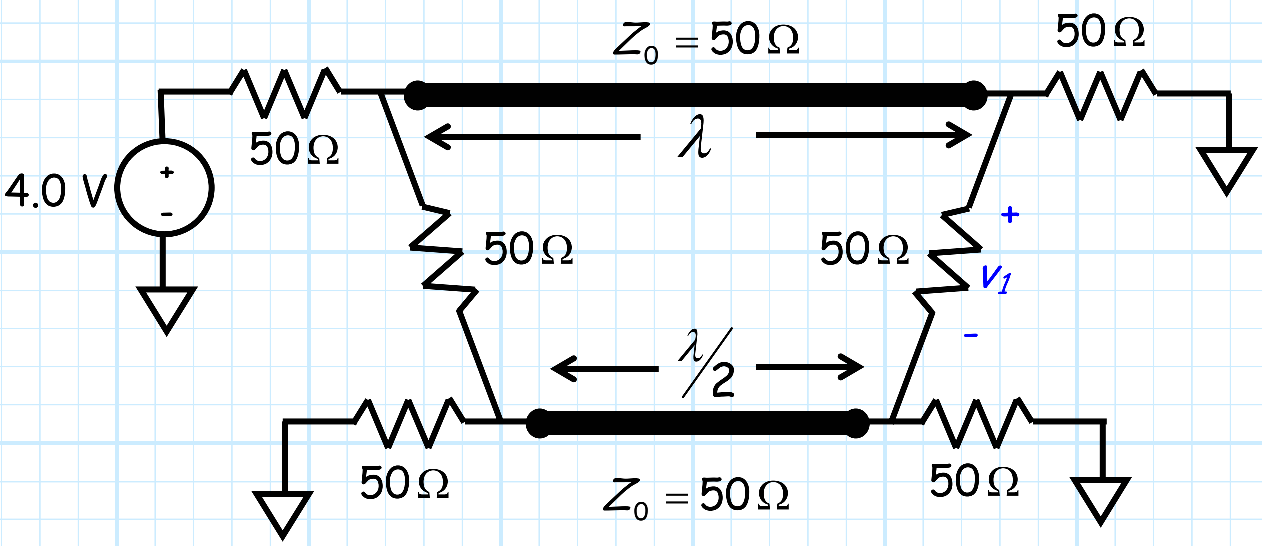


The two transmission lines each have a characteristic impedance of .

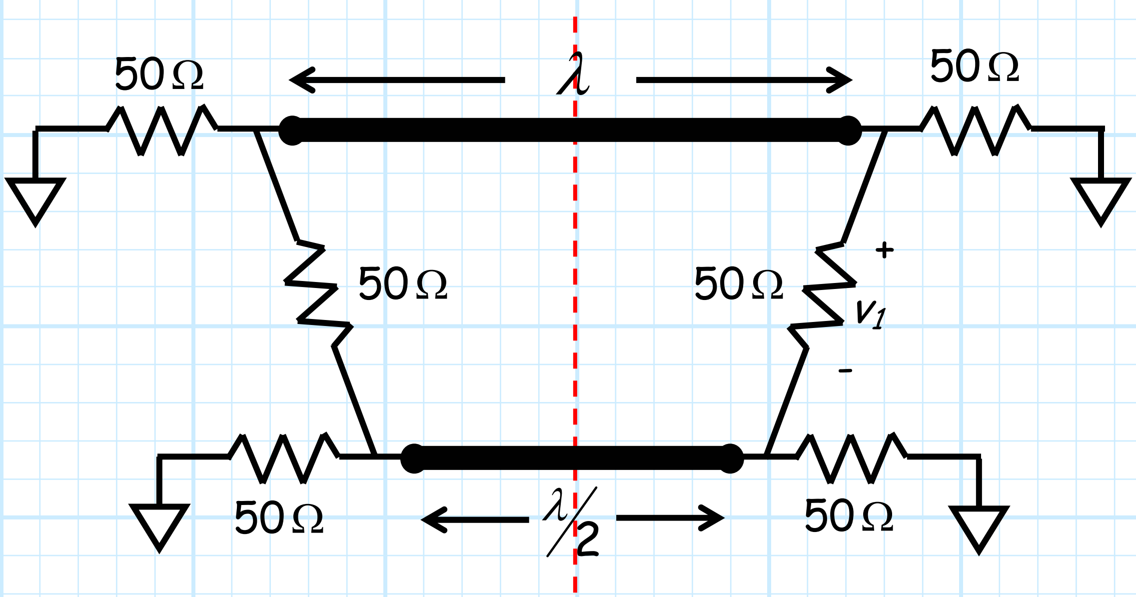
> Use **odd-even mode analysis** to determine the value of voltage v_1.

Solution

To simplify the circuit schematic, we first remove the bottom (i.e., ground) conductor of each transmission line:

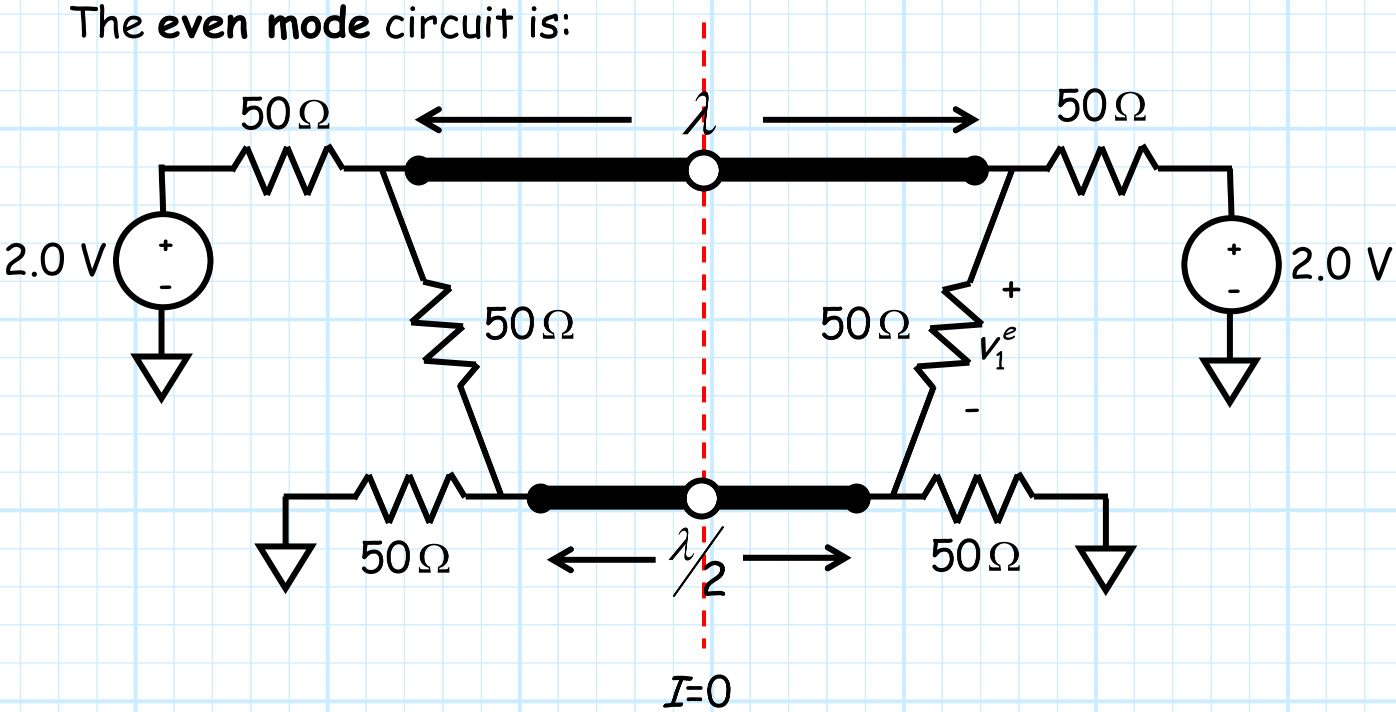


Note that the circuit has one plane of **bilateral symmetry**:

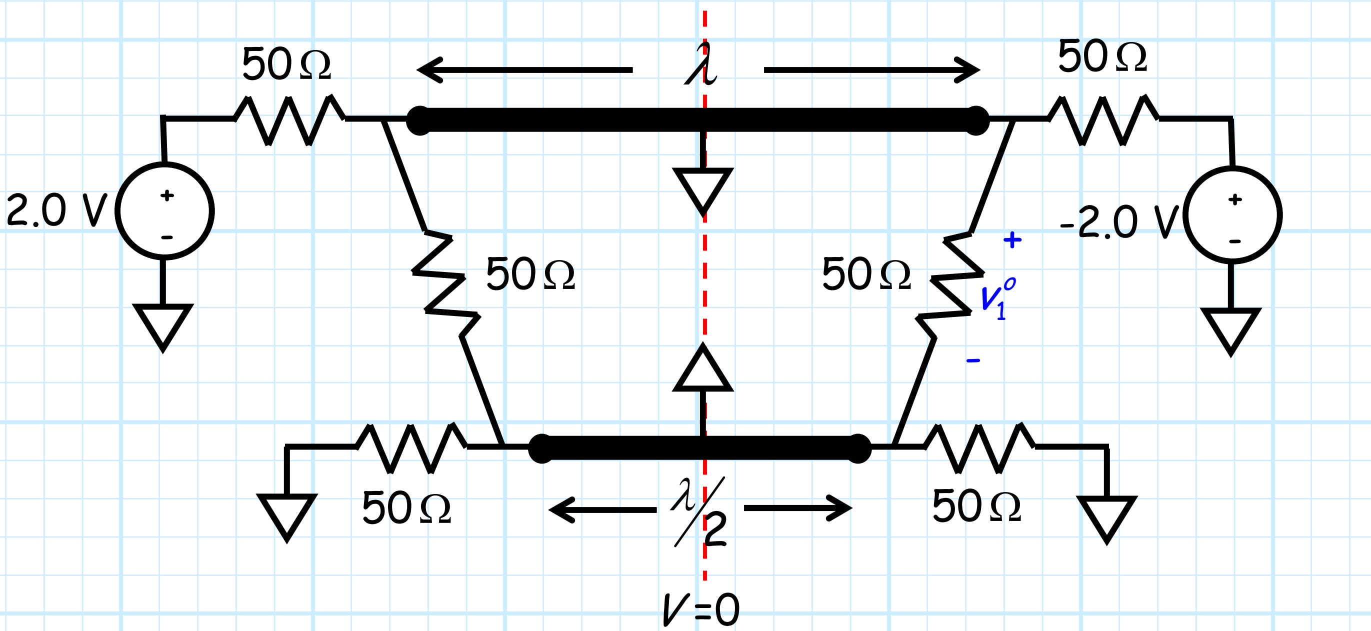


Thus, we can analyze the circuit using **even/odd mode analysis** (Yeah!).

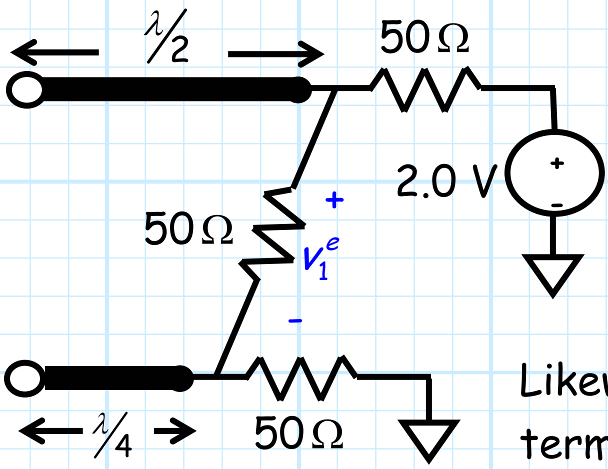
The **even mode** circuit is:



Whereas the **odd mode** circuit is:

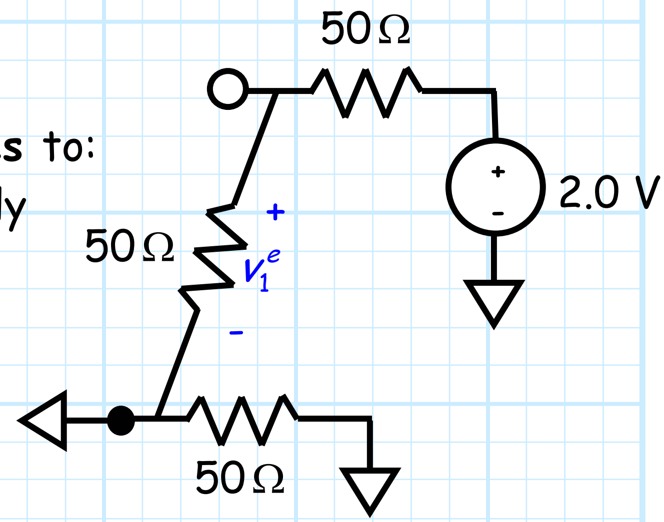


We split the modes into half-circuits from which we can determine voltages v_1^e and v_1^o:



Recall that a $\ell = \lambda/2$ transmission line terminated in an open circuit has an input impedance of $Z_{in} = \infty$—an **open** circuit!

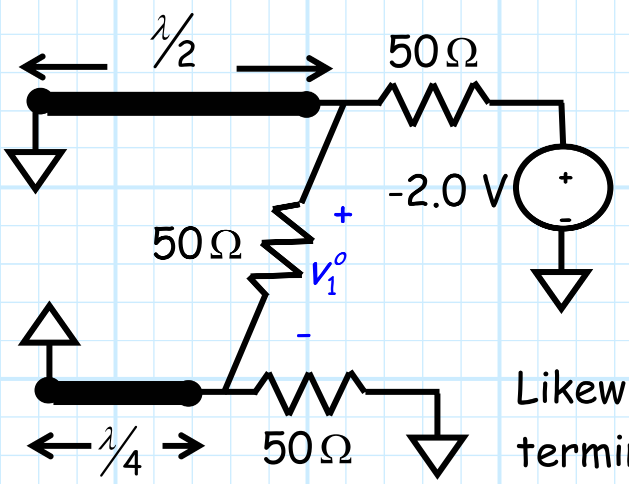
Likewise, a transmission line $\ell = \lambda/4$ terminated in an open circuit has an input impedance of $Z_{in} = 0$—a **short** circuit!



Therefore, this half-circuit **simplifies** to:
And therefore the voltage v_1^e is easily determined via voltage division:

$$v_1^e = 2\left(\frac{50}{50+50}\right) = 1.0\ V$$

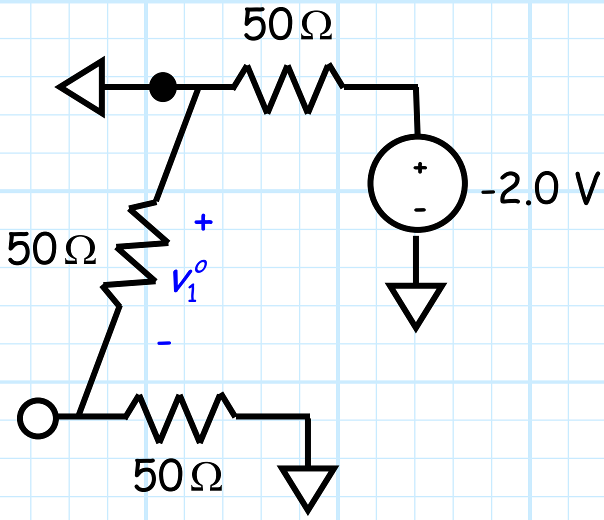
Now, examine the right half-circuit of the **odd mode**:



Recall that a $\ell = \lambda/2$ transmission line terminated in a short circuit has an input impedance of $Z_{in} = 0$— a **short** circuit!

Likewise, a transmission line $\ell = \lambda/4$ terminated in an short circuit has an input impedance of $Z_{in} = \infty$— an **open** circuit!

This half-circuit simplifies to →



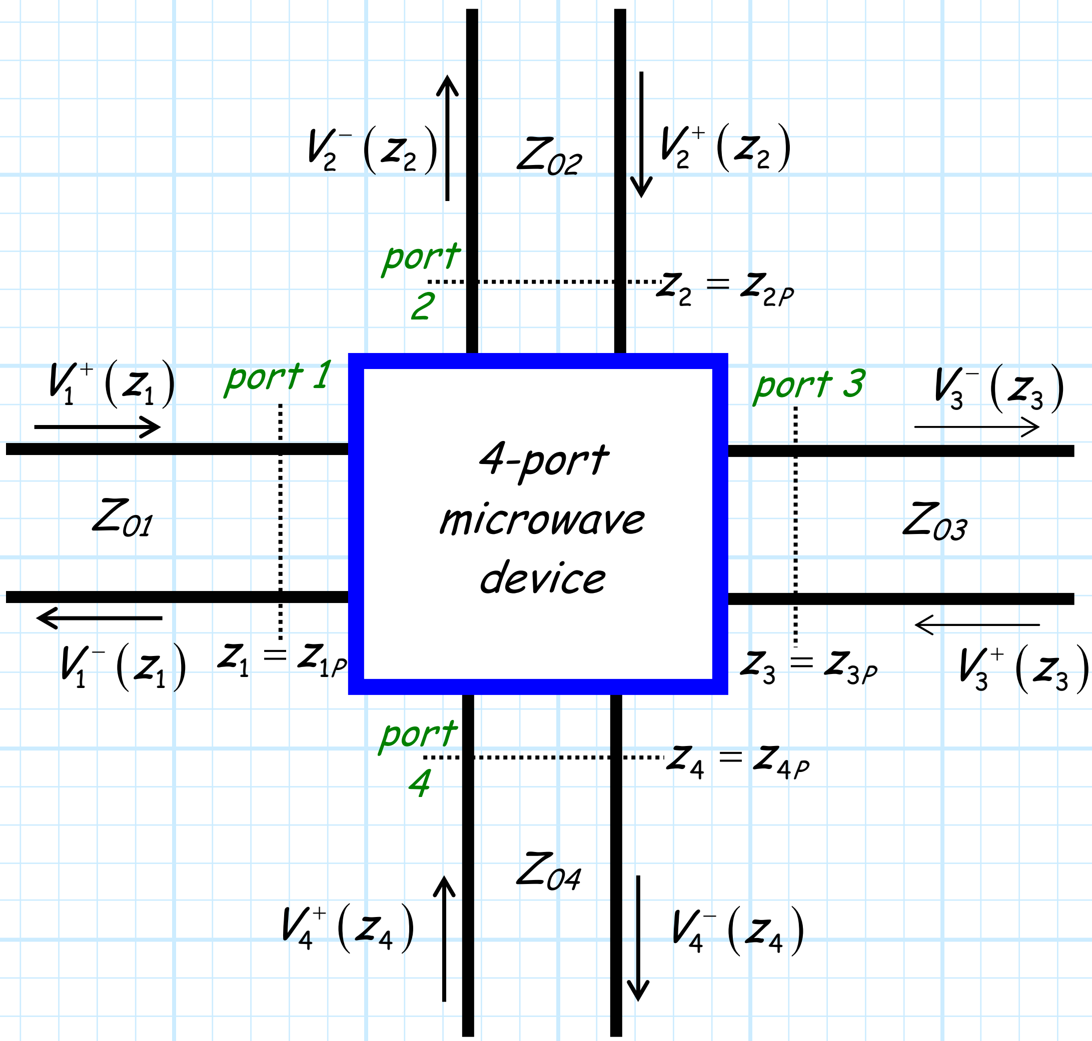
It is apparent from the circuit that the voltage $V_1^o = 0$!

Thus, the superposition of the odd and even modes leads to the result:

$$\underline{\underline{V_1}} = V_1^e + V_1^o = 1.0 + 0 = \underline{\underline{1.0\ V}}$$

Generalized Scattering Parameters

Consider now **this** microwave network:



Q: ***Boring!*** *We studied this* ***before****; this will lead to the definition of scattering parameters, right?*

A: Not exactly. For this network, the **characteristic impedance** of each transmission line is **different** (i.e., $Z_{01} \neq Z_{02} \neq Z_{03} \neq Z_{04}$)!

Q: *Yikes! You said scattering parameters are **dependent** on transmission line characteristic impedance Z_0. If these values are **different** for each port, **which** Z_0 do we use?*

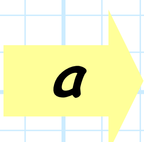
A: For this **general** case, we must use **generalized scattering parameters**! First, we define a slightly new form of complex wave amplitudes:

$$a_n = \frac{V_{0n}^{+}}{\sqrt{Z_{0n}}} \qquad b_n = \frac{V_{0n}^{-}}{\sqrt{Z_{0n}}}$$

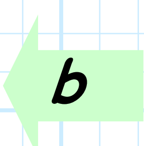
So for example:

$$a_1 = \frac{V_{01}^{+}}{\sqrt{Z_{01}}} \qquad b_3 = \frac{V_{03}^{-}}{\sqrt{Z_{03}}}$$

The key things to note are:



A **variable** ***a*** (e.g., $a_1, a_2, \cdots$) denotes the complex amplitude of an **incident (i.e., plus)** wave.



A **variable** ***b*** (e.g., $b_1, b_2, \cdots$) denotes the complex amplitude of an **exiting (i.e., minus)** wave.

We now get to **rewrite** all our transmission line knowledge in terms of these generalized complex amplitudes!

First, our two propagating wave amplitudes (i.e., plus and minus) are **compactly** written as:

$$V_{0n}^{+} = a_n \sqrt{Z_{0n}} \qquad\qquad V_{0n}^{-} = b_n \sqrt{Z_{0n}}$$

And so:

$$V_n^{+}(z_n) = a_n \sqrt{Z_{0n}}\; e^{-j\beta z_n}$$

$$V_n^{-}(z_n) = b_n \sqrt{Z_{0n}}\; e^{+j\beta z_n}$$

$$\Gamma(z_n) = \frac{b_n}{a_n}\, e^{+j2\beta z_n}$$

Likewise, the total voltage, current, and impedance are:

$$V_n(z_n) = \sqrt{Z_{0n}}\left(a_n\, e^{-j\beta z_n} + b_n\, e^{+j\beta z_n}\right)$$

$$I_n(z_n) = \frac{a_n\, e^{-j\beta z_n} - b_n\, e^{+j\beta z_n}}{\sqrt{Z_{0n}}}$$

$$Z(z_n) = \frac{a_n\, e^{-j\beta z_n} + b_n\, e^{+j\beta z_n}}{a_n\, e^{-j\beta z_n} - b_n\, e^{+j\beta z_n}}$$

Assuming that our port planes are defined with $z_{nP} = 0$, we can determine the total voltage, current, and impedance **at port *n*** as:

$$V_n \doteq V_n\left(z_n = 0\right) = \sqrt{Z_{0n}}\left(a_n + b_n\right) \qquad I_n \doteq I_n\left(z_n = 0\right) = \frac{a_n - b_n}{\sqrt{Z_{0n}}}$$

$$Z_n \doteq Z\left(z_n = 0\right) = \frac{a_n + b_n}{a_n - b_n}$$

Likewise, the **power** associated with each wave is:

$$P_n^+ = \frac{\left|V_{0n}^+\right|^2}{2Z_{0n}} = \frac{\left|a_n\right|^2}{2} \qquad P_n^- = \frac{\left|V_{0n}^-\right|^2}{2Z_{0n}} = \frac{\left|b_n\right|^2}{2}$$

As such, the power **delivered to** port n (i.e., the power **absorbed by** port n) is:

$$P_n = P_n^+ - P_n^- = \frac{\left|a_n\right|^2 - \left|b_n\right|^2}{2}$$

This result is also **verified**:

$$\begin{aligned}
P_n &= \frac{1}{2}\text{Re}\left\{V_n I_n^*\right\} \\
&= \frac{1}{2}\text{Re}\left\{\left(a_n + b_n\right)\left(a_n^* - b_n^*\right)\right\} \\
&= \frac{1}{2}\text{Re}\left\{a_n a_n^* + b_n a_n^* - a_n b_n^* - b_n b_n^*\right\} \\
&= \frac{1}{2}\text{Re}\left\{\left|a_n\right|^2 + b_n a_n^* - \left(b_n a_n^*\right)^* - \left|b_n\right|^2\right\} \\
&= \frac{1}{2}\text{Re}\left\{\left|a_n\right|^2 + j\,\text{Im}\left\{b_n a_n^*\right\} - \left|b_n\right|^2\right\} \\
&= \frac{\left|a_n\right|^2 - \left|b_n\right|^2}{2}
\end{aligned}$$

Q: *So what's the* ***big deal****? This is yet* ***another way*** *to express transmission line activity. Do we* ***really*** *need to know this, or is this simply a strategy for making the next exam* ***even harder****?*

$$Z_1 = \frac{a_1 + b_1}{a_1 - b_1}$$

A: You may have noticed that this notation (a_n, b_n) provides descriptions that are a bit **"cleaner"** and more symmetric between current and voltage.

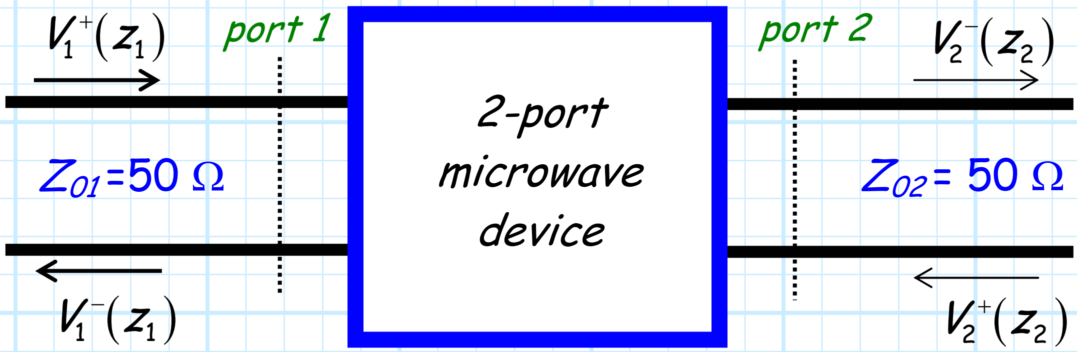
However, the **main reason** for this notation is for evaluating the **scattering parameters** of a device with **dissimilar** transmission line impedance (e.g., $Z_{01} \neq Z_{02} \neq Z_{03} \neq Z_{04}$).

For these cases we must use **generalized scattering parameters**:

$$S_{mn} = \frac{V_{0m}^{-}}{V_{0n}^{+}} \frac{\sqrt{Z_{0n}}}{\sqrt{Z_{0m}}} \qquad \text{(when } V_k^{+}(z_k) = 0 \text{ for all } k \neq n\text{)}$$

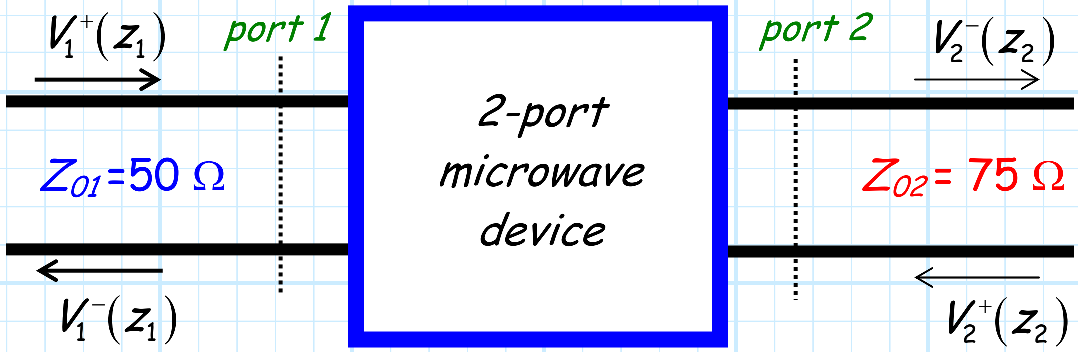
Note that **if** the transmission lines at each port are identical ($Z_{0m} = Z_{0n}$), the scattering parameter definition "reverts back" to the original (i.e., $S_{mn} = V_{0m}^{-}/V_{0n}^{+}$ **if** $Z_{0m} = Z_{0n}$). E.G.:

$$S_{21} = \frac{V_{02}^{-}}{V_{01}^{+}} \quad \text{when } V_{02}^{+} = 0$$



But, **if** the transmission lines at each port are **dissimilar** ($Z_{0m} \neq Z_{0n}$), our original scattering parameter definition is **not correct** (i.e., $S_{mn} \neq V_{0m}^{-}/V_{0n}^{+}$ **if** $Z_{0m} \neq Z_{0n}$)! E.G.:

$$S_{21} \neq \frac{V_{02}^{-}}{V_{01}^{+}} \quad \text{when } V_{02}^{+} = 0$$



$$S_{21} = \frac{V_{02}^{-}}{V_{01}^{+}} \frac{\sqrt{50}}{\sqrt{75}} \quad \text{when } V_{02}^{+} = 0$$

Note that the generalized scattering parameters can be more **compactly** written in terms of our **new** wave amplitude notation:

$$S_{mn} = \frac{V_{0m}^{-}}{V_{0n}^{+}} \frac{\sqrt{Z_{0n}}}{\sqrt{Z_{0m}}} = \frac{b_m}{a_n} \qquad \text{(when } a_k = 0 \text{ for all } k \neq n\text{)}$$

Remember, this is the **generalized** form of scattering parameter—it **always** provides the correct answer, **regardless** of the values of Z_{0m} or Z_{0n}!

Q: *But* ***why can't*** *we define the scattering parameter as* $S_{mn} = V_{0m}^{-}/V_{0n}^{+}$, ***regardless*** *of* Z_{0m} *or* Z_{0n}*??* ***Who says*** *we must define it with those* ***awful*** $\sqrt{Z_{0n}}$ *values in there?*

A: Good question! Recall that a lossless device is will **always** have a **unitary** scattering matrix. As a result, the scattering parameters of a lossless device will **always** satisfy, for example:

$$1 = \sum_{m=1}^{M} \left| S_{mn} \right|^2$$

This is true **only** if the scattering parameters are **generalized**!

The scattering parameters of a lossless device will form a unitary matrix **only** if defined as $S_{mn} = b_m / a_n$. If we use $S_{mn} = V_{0m}^- / V_{0n}^+$, the matrix will be unitary **only** if the connecting transmission lines have the **same** characteristic impedance.

Q: *Do we really **care** if the matrix of a lossless device is unitary or not?*

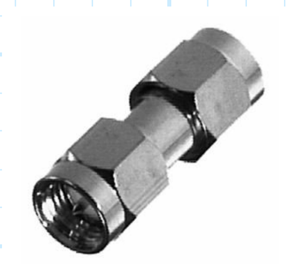
A: **Absolutely** we do! The:

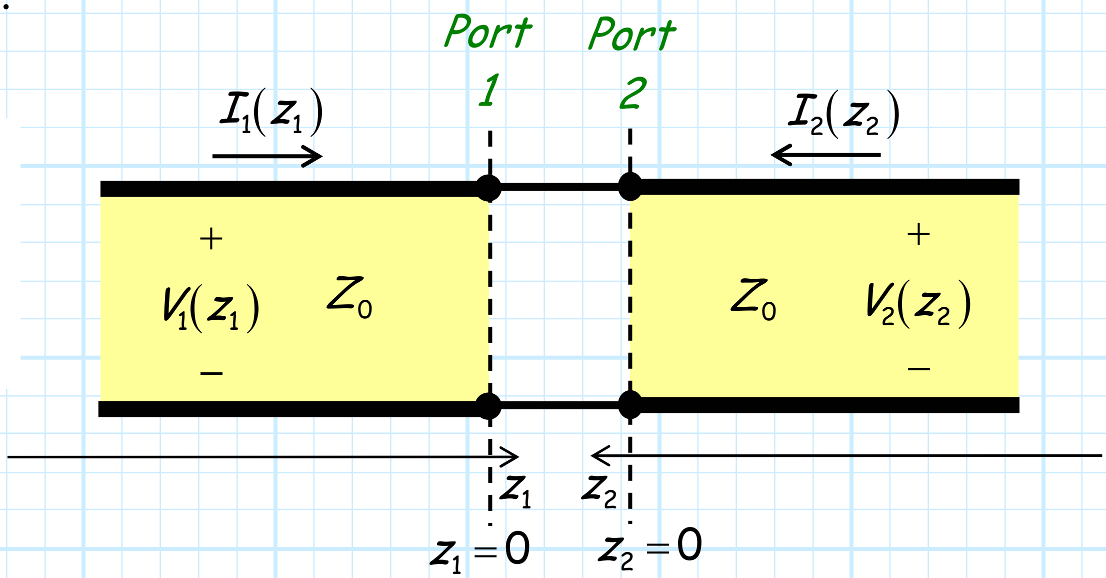
lossless device $\Leftrightarrow$ unitary scattering matrix

relationship is a very powerful one. It allows us to **identify** lossless devices, and it allows us to determine **if** specific lossless devices are **even possible**!

Example: The Scattering Matrix of a Connector

First, let's consider the scattering matrix of a **perfect connector**—an electrically **very small** two-port device that allows us to connect the ends of different transmission lines together.





If the connector is ideal, then it will exhibit **no** series inductance **nor** shunt capacitance, and thus from KVL and KCL:

$$V_1(z_1 = 0) = V_2(z_2 = 0) \qquad\qquad I_1(z_1 = 0) = -I_2(z_2 = 0)$$

Terminating **port 2 in a matched load**, and then analyzing the resulting circuit, we find that (not surprisingly!):

$$V_{01}^{-} = 0 \quad \text{and} \quad V_{02}^{-} = V_{01}^{+}$$

From this we conclude that (since $V_{02}^{+} = 0$):

$$S_{11} = \frac{V_{01}^{-}}{V_{01}^{+}} = \frac{0}{V_{01}^{+}} = 0.0 \qquad\qquad S_{21} = \frac{V_{02}^{-}}{V_{01}^{+}} = \frac{V_{01}^{+}}{V_{01}^{+}} = 1.0$$

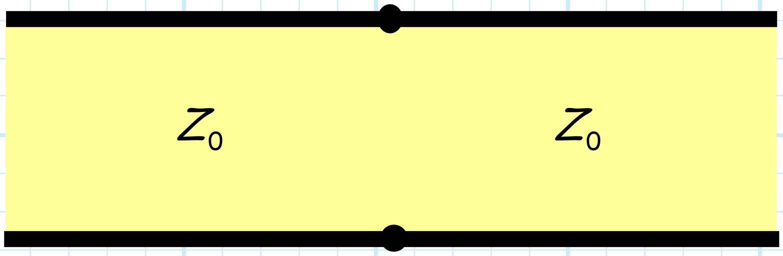
This two-port device has $\mathbf{D_2}$ symmetry (a plane of bilateral symmetry), meaning:

$$S_{22} = S_{11} = 0.0 \qquad \text{and} \qquad S_{21} = S_{12} = 1.0$$

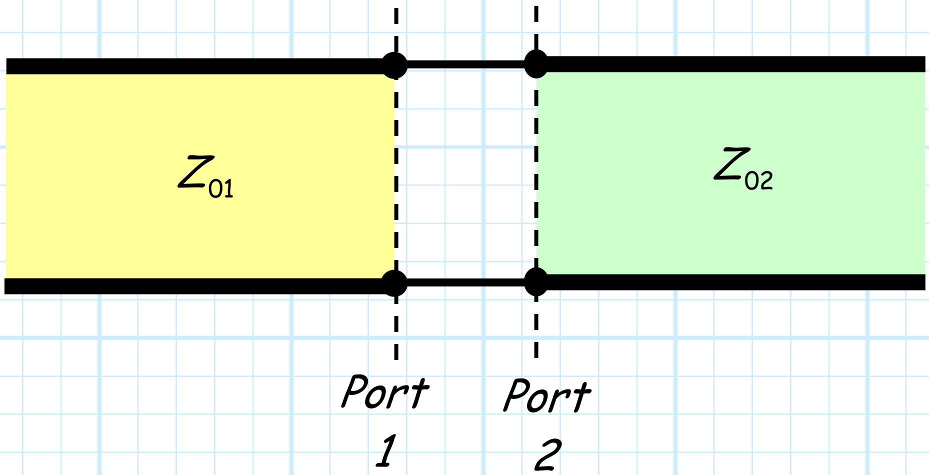
The **scattering matrix** for such this ideal connector is therefore:

$$\mathcal{S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

As a result, the perfect connector allows two transmission lines of **identical characteristic impedance** to be connected together into **one "seamless" transmission line**.



Now, however, consider the case where the transmission lines connected together have **dissimilar** characteristic impedances (i.e., $Z_0 \neq Z_1$):



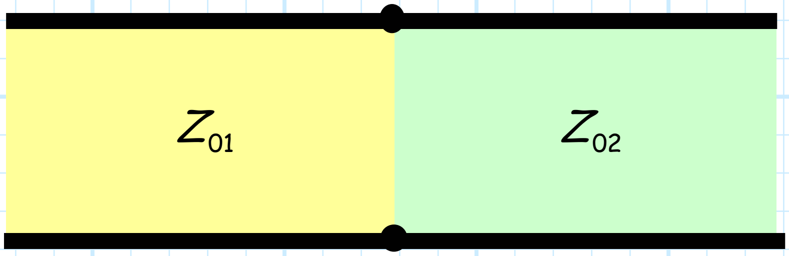
Q: *Won't the scattering matrix of this ideal connector remain the **same**? After all, the **device itself** has not changed!*

A: The impedance, admittance, and transmission matrix **will** remained unchanged—these matrix quantities **do not** depend on the characteristics of the transmission lines connected to the device.

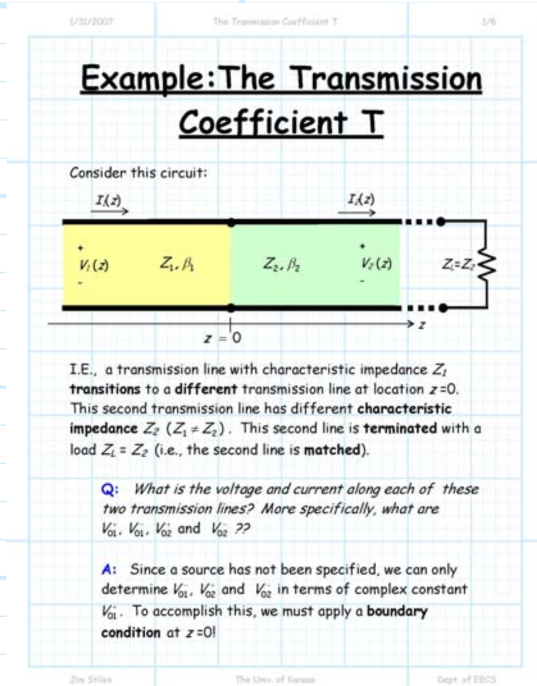
But remember, the **scattering matrix** depends on **both** the device **and** the characteristic impedance of the transmission lines attached to it.

After all, the **incident** and **exiting** waves are traveling on these transmission lines!

The ideal connector in this case establishes a "seamless" **interface** between two **dissimilar** transmission lines.



Remember, this is the **same** structure that we evaluated in an **earlier** handout!


1/31/2007 The Transmission Coefficient T 1/6

Example: The Transmission Coefficient T

Consider this circuit:

$I_1(z)$ $I_2(z)$ $V_1(z)$ Z_1, β_1 Z_2, β_2 $V_2(z)$ $Z_L=Z_2$ $z=0$ z

I.E., a transmission line with characteristic impedance Z_1 **transitions** to a **different** transmission line at location z=0. This second transmission line has different **characteristic impedance** Z_2 ($Z_1 \neq Z_2$). This second line is **terminated** with a load $Z_L = Z_2$ (i.e., the second line is **matched**).

Q: *What is the voltage and current along each of these two transmission lines? More specifically, what are* V_{01}^+, V_{01}^-, V_{02}^+ *and* V_{02}^- *??*

A: Since a source has not been specified, we can only determine V_{01}^-, V_{02}^+ and V_{02}^- in terms of complex constant V_{01}^+. To accomplish this, we must apply a **boundary condition** at z=0!

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In that analysis we found that—when $V_{02}^+ = 0$:

$$\frac{V_{01}^-}{V_{01}^+} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \qquad \text{and} \qquad \frac{V_{02}^-}{V_{01}^+} = \frac{2Z_{02}}{Z_{02} + Z_{01}}$$

And so the (generalized) scattering parameters S_{11} and S_{21} are:

$$S_{11} = \frac{V_{01}^-}{V_{01}^+}\frac{\sqrt{Z_{01}}}{\sqrt{Z_{01}}} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \qquad \text{and} \qquad S_{21} = \frac{V_{02}^-}{V_{01}^+}\frac{\sqrt{Z_{01}}}{\sqrt{Z_{02}}} = \frac{2\sqrt{Z_{01}Z_{02}}}{Z_{02} + Z_{01}}$$

As a result we can conclude that the **scattering matrix** of the ideal connector (when connecting dissimilar transmission lines) is:

$$\mathcal{S} = \begin{bmatrix} \dfrac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} & \dfrac{2\sqrt{Z_{01}Z_{02}}}{Z_{01} + Z_{02}} \\[2ex] \dfrac{2\sqrt{Z_{01}Z_{02}}}{Z_{01} + Z_{02}} & \dfrac{Z_{01} - Z_{02}}{Z_{01} + Z_{02}} \end{bmatrix}$$