5.4 – The Quarter-Wave Transformer

Reading Assignment: ***pp. 73-76, 240-243***

By now you've noticed that a **quarter-wave length** of transmission line ($\ell = \lambda/4$, $2\beta\ell = \pi$) appears **often** in microwave engineering problems.

Another application of the $\ell = \lambda/4$ transmission line is as an **impedance matching network**.

HO: THE QUARTER-WAVE TRANSFORMER

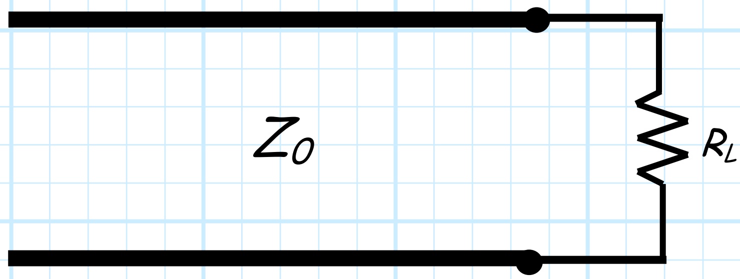
HO: THE SIGNAL-FLOW GRAPH OF A QUARTER-WAVE TRANSFORMER

Q: ***Why*** *does the quarter-wave matching network work—after all, the quarter-wave line is* ***mismatched*** *at both ends?*

A: HO: MULTIPLE REFLECTION VIEWPOINT

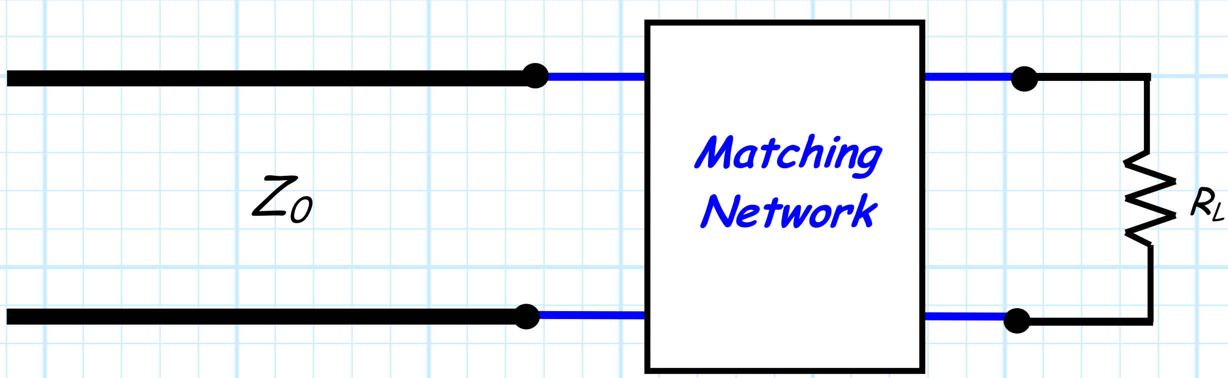
The Quarter-Wave Transformer

Say the end of a transmission line with characteristic impedance Z_0 is terminated with a **resistive** (i.e., real) load.



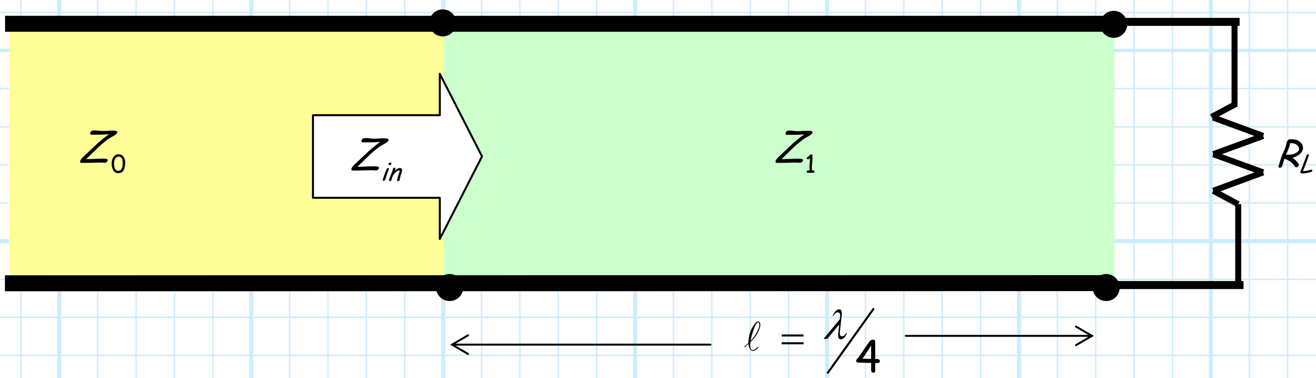
Unless $R_L = Z_0$, the resistor is **mismatched** to the line, and thus some of the incident power will be **reflected**.

We can of course correct this situation by placing a matching network between the line and the load:



In addition to the designs we have just studied (e.g., L-networks, stub tuners), one of the simplest matching network designs is the **quarter-wave transformer**.

The quarter-wave transformer is simply a transmission line with characteristic impedance Z_1 and length $\ell = \lambda/4$ (i.e., a quarter-wave line).



The $\lambda/4$ line **is** the **matching network**!

Q: *But what about the characteristic impedance Z_1; what **should** its value be??*

A: Remember, the quarter wavelength case is one of the **special** cases that we studied. We know that the **input** impedance of the quarter wavelength line is:

$$Z_{in} = \frac{(Z_1)^2}{Z_L} = \frac{(Z_1)^2}{R_L}$$

Thus, if we wish for Z_{in} to be numerically equal to Z_0, we find:

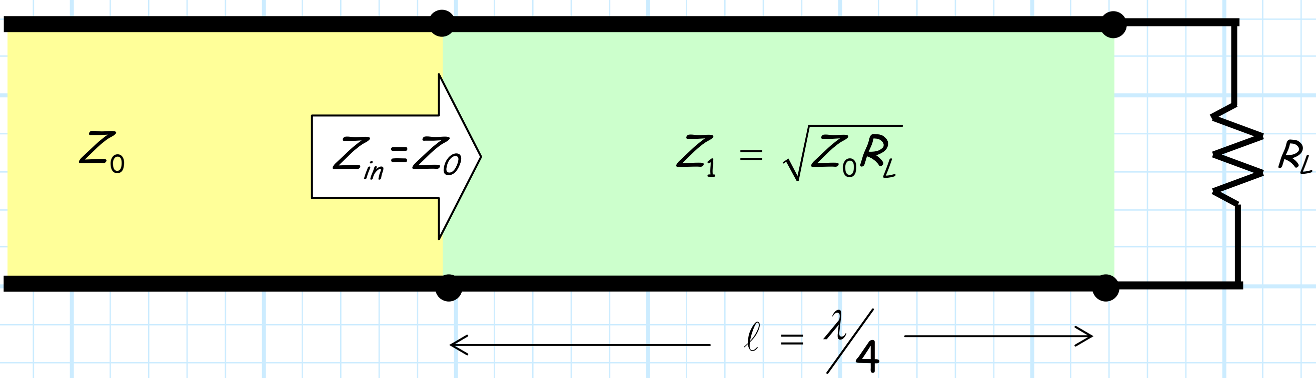
$$Z_{in} = \frac{(Z_1)^2}{R_L} = Z_0$$

Solving for Z_1, we find its **required** value to be:

$$(Z_1)^2 / R_L = Z_0$$
$$(Z_1)^2 = Z_0 R_L$$
$$Z_1 = \sqrt{Z_0 R_L}$$

In other words, the characteristic impedance of the quarter wave line is the **geometric average** of Z_0 and R_L!

Therefore, a $\lambda/4$ line with characteristic impedance $Z_1 = \sqrt{Z_0 R_L}$ will **match** a transmission line with characteristic impedance Z_0 **to** a resistive load R_L.



Thus, **all power** is delivered to load R_L !

Alas, the quarter-wave transformer (like all our designs) has a few problems!

Problem #1

The matching **bandwidth** is **narrow** !

In other words, we obtain a **perfect** match at precisely the frequency where the length of the matching transmission line is a **quarter**-wavelength.

→ But remember, this length can be a quarter-wavelength at just **one** frequency!

Remember, **wavelength** is related to **frequency** as:

$$\lambda = \frac{v_p}{f} = \frac{1}{f\sqrt{LC}}$$

where v_p is the **propagation velocity** of the wave .

For **example**, assuming that $v_p = c$ (c = the speed of light in a vacuum), one wavelength at 1 GHz is 30 cm ($\lambda = 0.3\ m$), while one wavelength at 3 GHz is 10 cm ($\lambda = 0.1\ m$). As a result, a transmission line length $\ell = 7.5\ cm$ is a quarter wavelength for a signal at 1GHz **only**.

Thus, a quarter-wave transformer provides a **perfect** match ($\Gamma_{in} = 0$) at **one** and **only one** signal frequency!

As the signal frequency (i.e., wavelength) changes, the **electrical** length of the matching transmission line changes. It will **no longer** be a **quarter** wavelength, and thus we **no longer** will have a **perfect** match.

We find that the **closer** R_L (R_{in}) is to characteristic impedance Z_0, the **wider** the bandwidth of the quarter wavelength transformer.

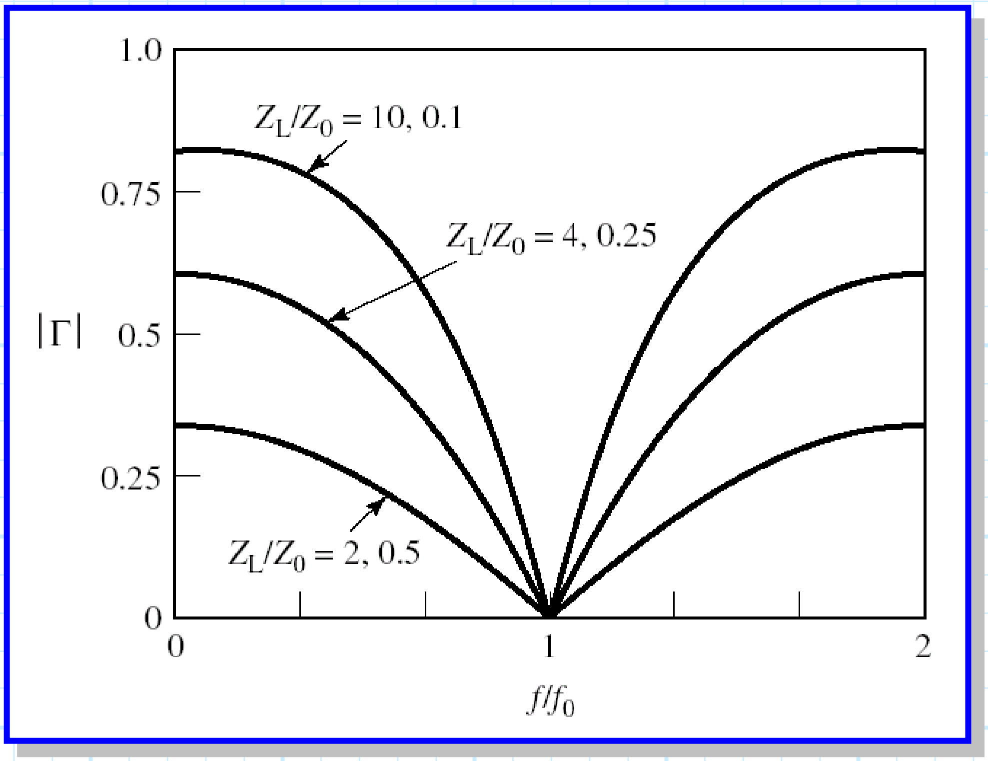


Figure 5.12 (p. 243) *Reflection coefficient magnitude versus frequency for a single-section quarter-wave matching transformer with various load mismatches.*

We will find that the bandwidth can be **increased** by adding **multiple** $\lambda/4$ sections!

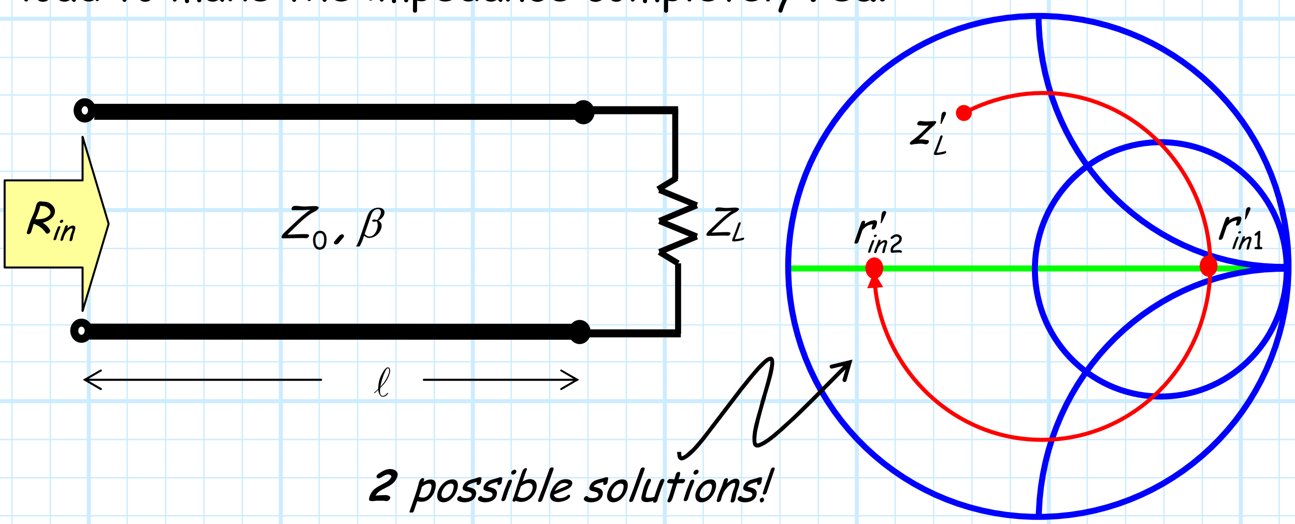
Problem #2

Recall the matching solution was limited to loads that were **purely real**! I.E.:

$$Z_L = R_L + j0$$

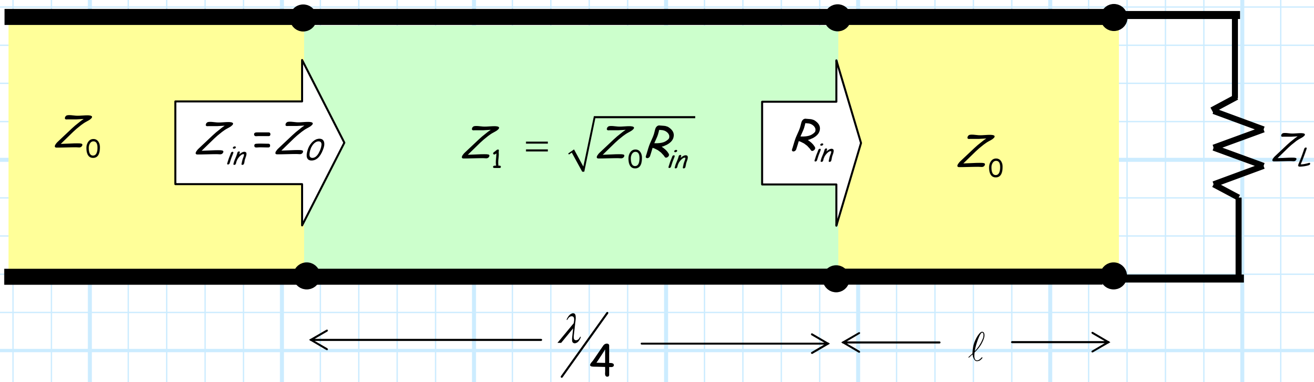
Of course, this is a BIG problem, as most loads will have a **reactive** component!

Fortunately, we have a relatively easy **solution** to this problem, as we can always add some **length** ℓ of transmission line to the load to make the impedance completely **real**:



However, remember that the input impedance will be purely real at only **one** frequency!

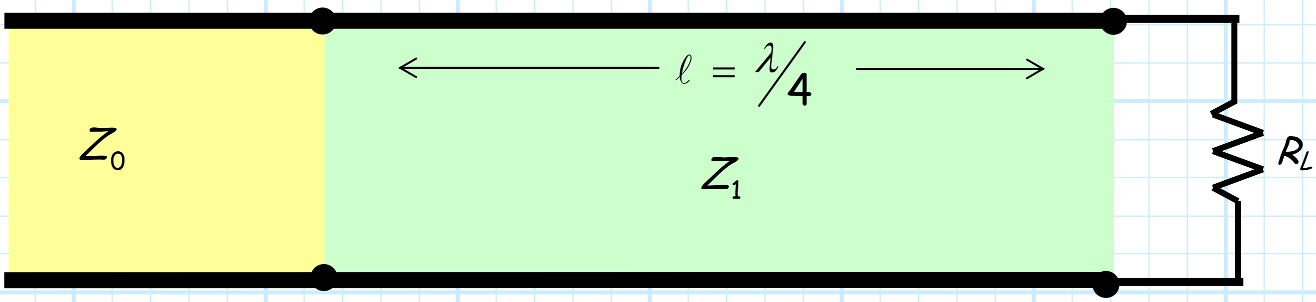
We can then build a quarter-wave transformer to **match** the line Z_0 to resistance R_{in}:

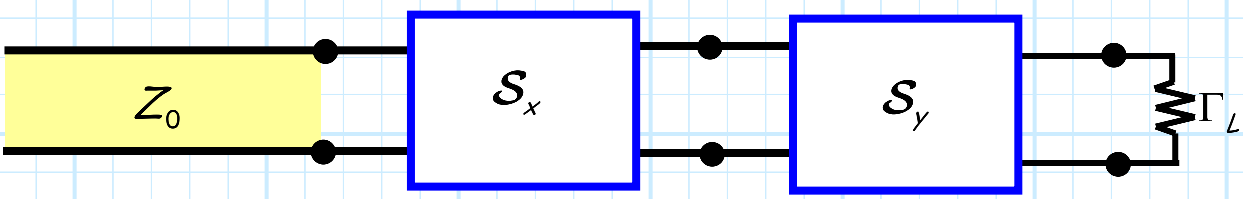


Again, since the transmission lines are lossless, **all** of the incident power is delivered to the **load** Z_L.

The Signal Flow Graph of a Quarter-Wave Transformer

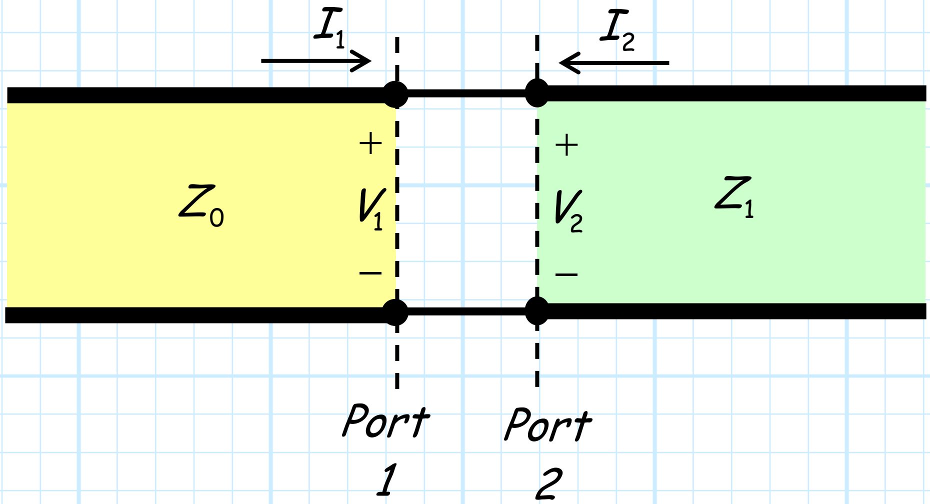
A quarter wave transformer can be thought of as a cascaded series of two two-port devices, terminated with a load R_L:





Q: ***Two*** *two-port devices? It appears to me that a quarter-wave transformer is* ***not*** *that complex. What* ***are*** *the two two-port devices?*

A: The **first** is a "**connector**". Note a connector is the interface between one transmission line (characteristic impedance Z_0) to a second transmission line (characteristic impedance Z_1).



Recall that we **earlier** determined the scattering matrix of this two-port device:

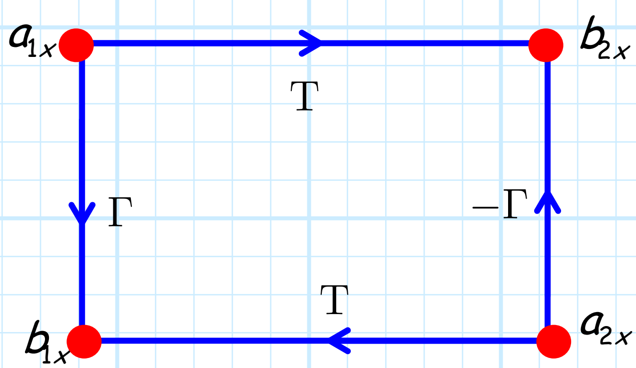
$$\mathcal{S}_x = \begin{bmatrix} \dfrac{Z_1 - Z_0}{Z_1 + Z_0} & \dfrac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1} \\ \dfrac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1} & \dfrac{Z_0 - Z_1}{Z_0 + Z_1} \end{bmatrix}$$

This result can be more **compactly** stated as:

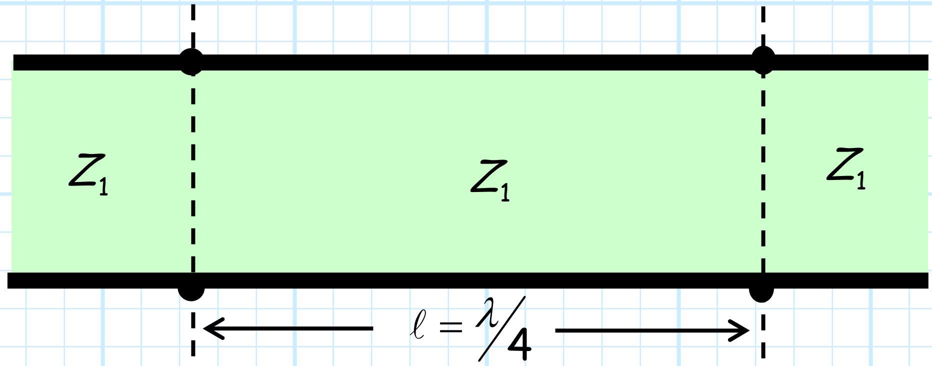
$$\mathcal{S} = \begin{bmatrix} \Gamma & \mathrm{T} \\ \mathrm{T} & -\Gamma \end{bmatrix}$$

where $\Gamma \doteq \dfrac{Z_1 - Z_0}{Z_1 + Z_0}$ and $\mathrm{T} \doteq \dfrac{2\sqrt{Z_0 Z_1}}{Z_0 + Z_1}$

The signal flow graph of this device is therefore:

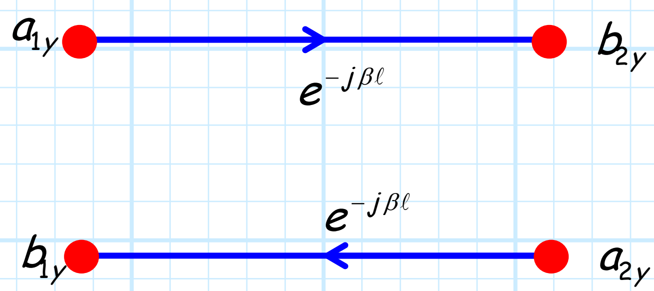


Now, the second two-port device is a quarter wavelength of **transmission line.**

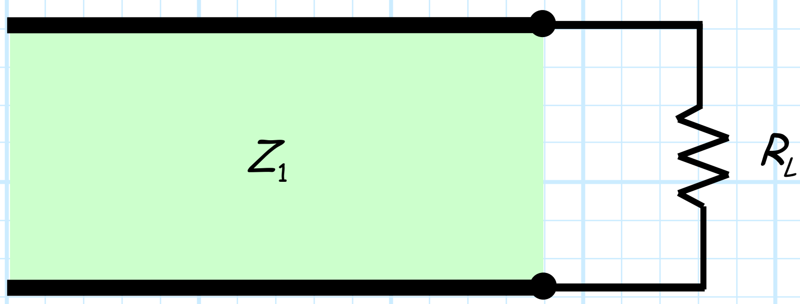


We know that it has the scattering matrix:

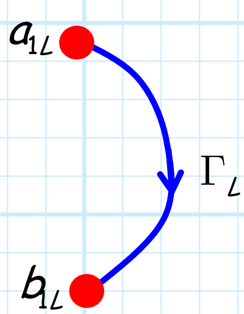
$$\mathcal{S}_y = \begin{bmatrix} 0 & e^{-j\beta\ell} \\ e^{-j\beta\ell} & 0 \end{bmatrix}$$



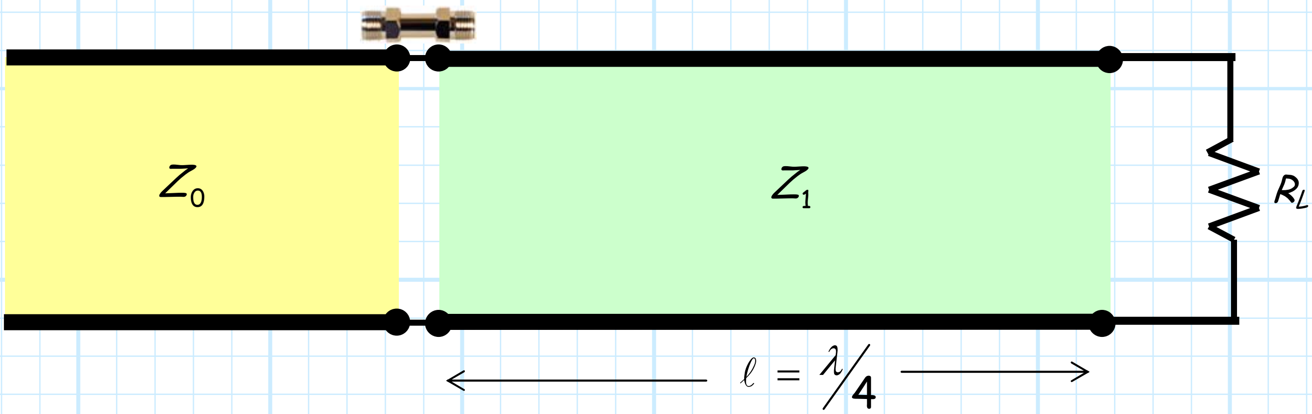
Finally, a **load** has a "scattering matrix" of:



$$\mathcal{S} = \left[\frac{R_L - Z_1}{R_L + Z_1}\right] = \Gamma_L$$



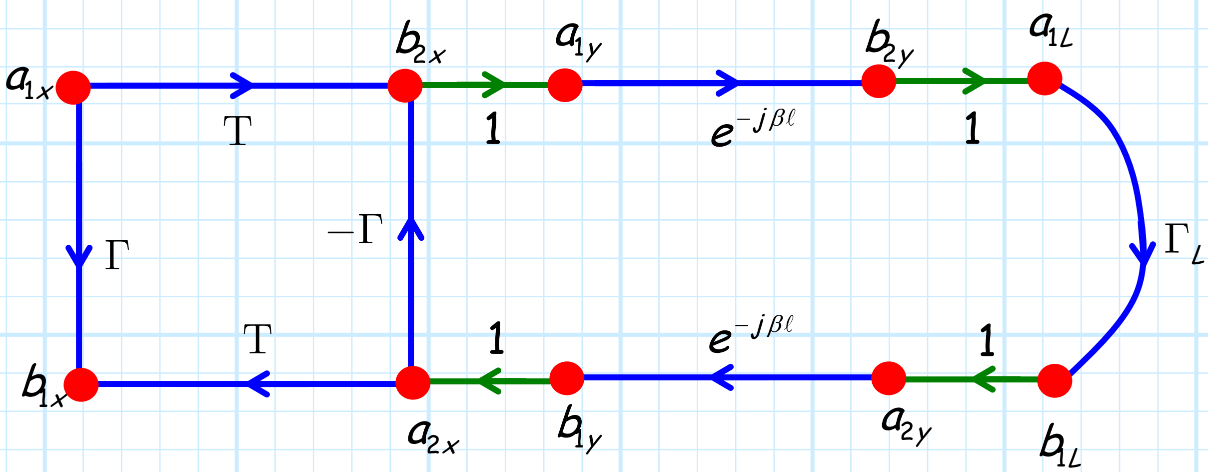
Of course, if we connect the ideal connector to a quarter wavelength of transmission line, and terminate the whole thing with load R_L, we have formed a **quarter wave matching network**!



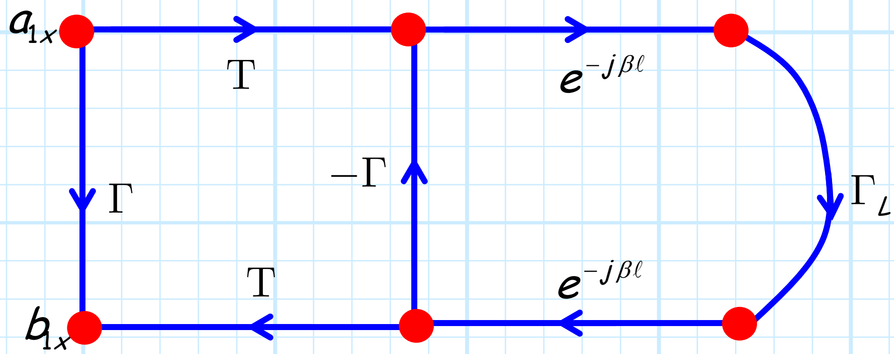
The boundary conditions associated with these connections are likewise:

$$a_{1y} = b_{2x} \qquad a_{2x} = b_{1y} \qquad a_{1L} = b_{2y} \qquad a_{2y} = b_{1L}$$

We can thus put the signal-flow graph pieces together to form the **signal-flow graph** of the quarter wave network:



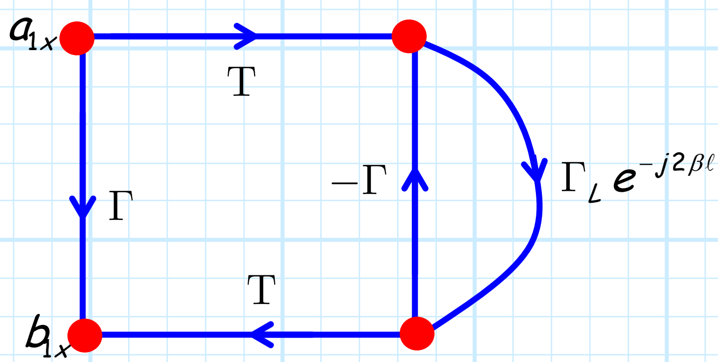
And simplifying:



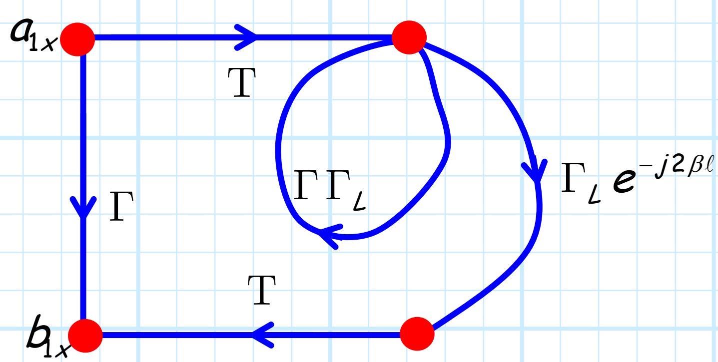
Now, let's see if we can **reduce** this graph to determine:

$$\Gamma_{in} \doteq \frac{b_{1x}}{a_{1x}}$$

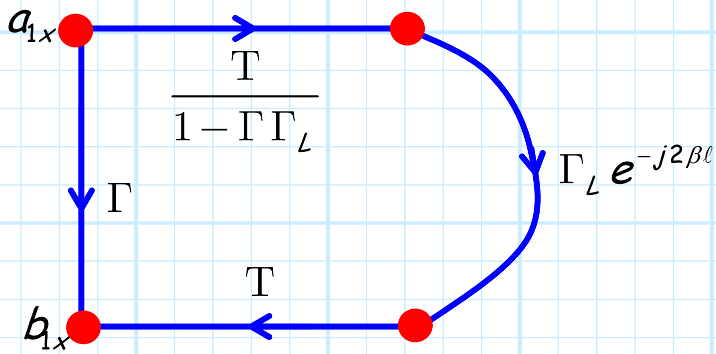
From the **series** rule:



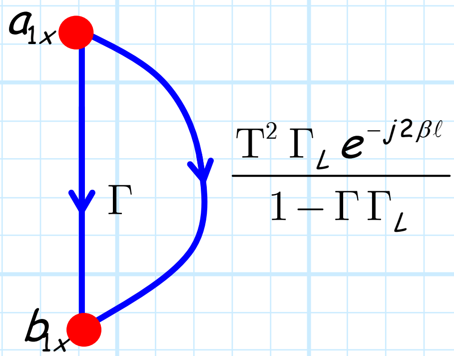
From the **splitting** rule:



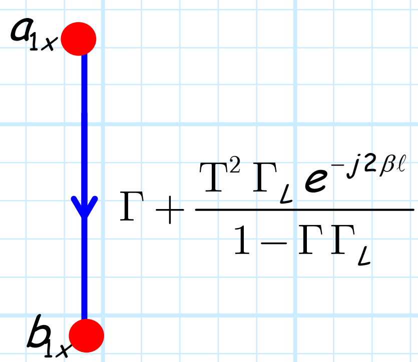
From the **self-loop** rule:



Again with the **series** rule:



And finally with the **parallel** rule:



So that:

$$\Gamma_{in} \doteq \frac{b_{1x}}{a_{1x}} = \Gamma + \frac{\text{T}^2 \Gamma_L e^{-j2\beta\ell}}{1-\Gamma\Gamma_L}$$

Q: *Hey wait! If the quarter-wave transformer is a **matching network**, shouldn't $\Gamma_{in} = 0$??*

A: Who says it isn't! Consider now **three important facts**.

For a **quarter wave transformer**, we set Z_1 such that:

$$Z_1^2 = Z_0 \, R_L \qquad \Rightarrow \qquad Z_0 = Z_1^2 / R_L$$

Inserting this into the scattering parameter S_{11} of the connector, we find:

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{Z_1 - {Z_1^2}/{R_L}}{Z_1 + {Z_1^2}/{R_L}} = \frac{R_L - Z_1}{R_L + Z_1}$$

Look at this result! For the quarter-wave transformer, the **connector** S_{11} value (i.e., Γ) is the **same** as the **load** reflection coefficient Γ_L:

$$\Gamma = \frac{R_L - Z_1}{R_L + Z_1} = \Gamma_L \qquad \leftarrow \textbf{Fact 1}$$

Since the connector is **lossless** (unitary scattering matrix!), we can conclude (and likewise show) that:

$$1 = |S_{11}|^2 + |S_{21}|^2 = |\Gamma|^2 + |T|^2$$

Since Z_0, Z_1, and R_L are all real, the values Γ and T are also **real valued**. As a result, $|\Gamma|^2 = \Gamma^2$ and $|T|^2 = T^2$, and we can likewise conclude:

$$\Gamma^2 + \mathrm{T}^2 = 1 \qquad \leftarrow \textbf{ Fact 2}$$

Likewise, the Z1 transmission line has $\ell = \lambda/4$, so that:

$$2\beta\ell = 2\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{4} = \pi$$

where **you** of course recall that $\beta = 2\pi/\lambda$! Thus:

$$e^{-j2\beta\ell} = e^{-j\pi} = -1 \qquad \leftarrow \textbf{ Fact 3}$$

As a result:

$$\Gamma_{in} = \Gamma + \frac{\mathrm{T}^2\,\Gamma_L\, e^{-j2\beta\ell}}{1-\Gamma\,\Gamma_L} = \Gamma - \frac{\mathrm{T}^2\,\Gamma_L}{1-\Gamma\,\Gamma_L}$$

And using the **newly discovered** fact that (for a correctly designed transformer) $\Gamma_L = \Gamma$:

$$\Gamma_{in} = \Gamma - \frac{\mathrm{T}^2\,\Gamma_L}{1-\Gamma\,\Gamma_L} = \Gamma - \frac{\mathrm{T}^2\,\Gamma}{1-\Gamma^2}$$

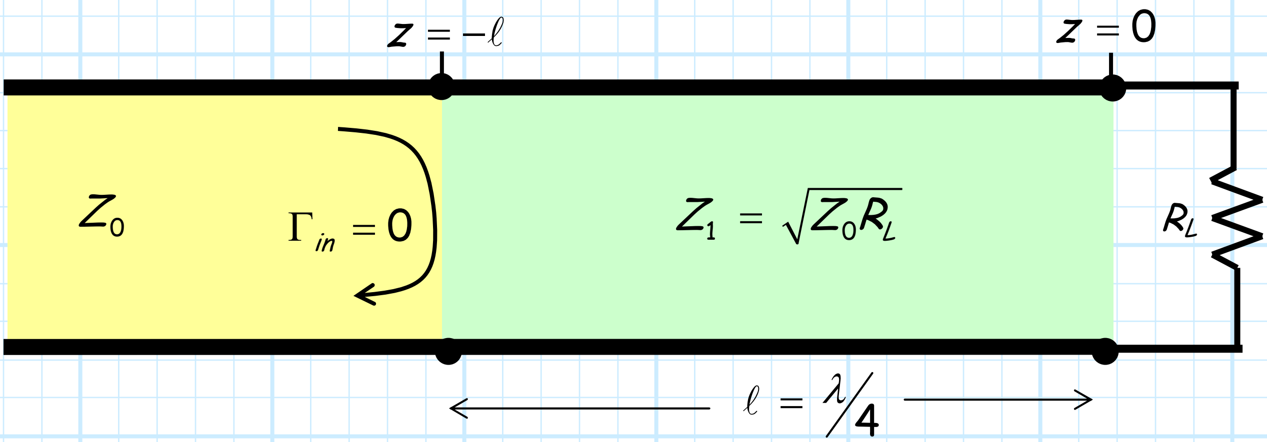
And also are **recent** discovery that $\mathrm{T}^2 = 1-\Gamma^2$:

$$\Gamma_{in} = \Gamma - \frac{\mathrm{T}^2\,\Gamma}{1-\Gamma^2} = \Gamma - \frac{\mathrm{T}^2\,\Gamma}{\mathrm{T}^2} = 0$$

A **perfect match**! The quarter-wave transformer does indeed work!

Multiple Reflection Viewpoint

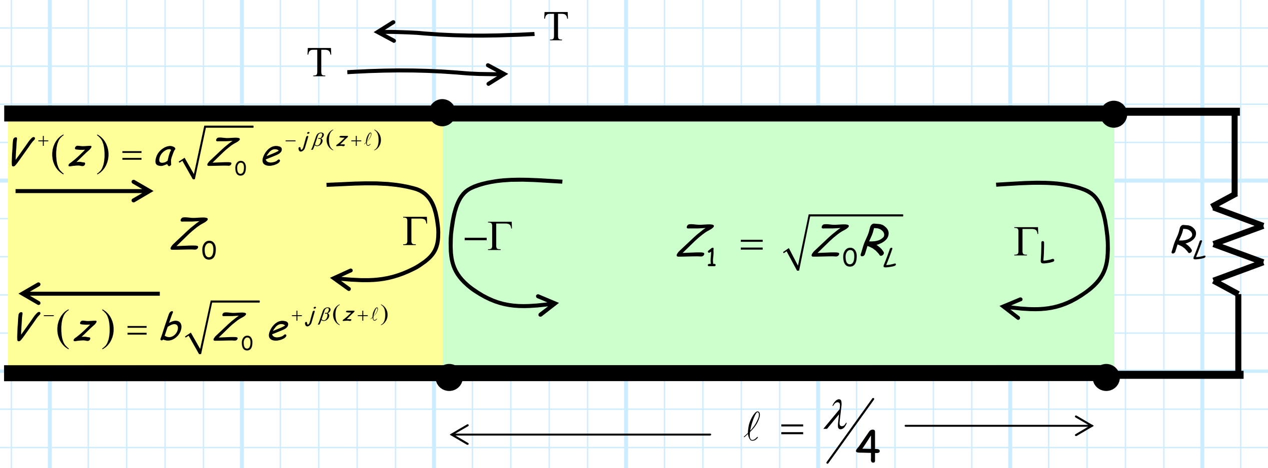
The **quarter-wave** transformer brings up an interesting question in μ-wave engineering.



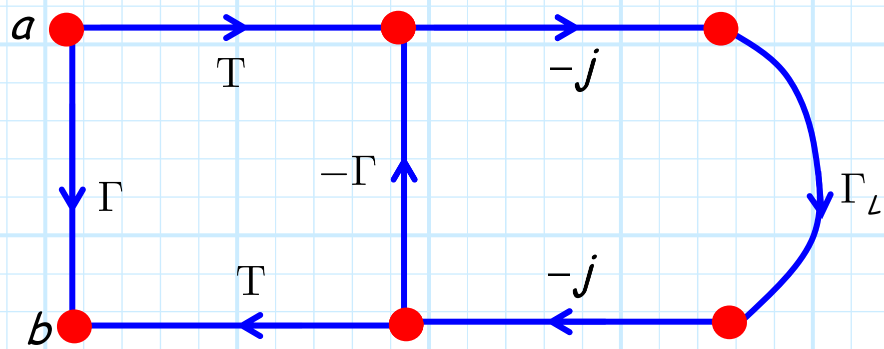
Q: ***Why*** *is there* ***no*** *reflection at $z = -\ell$? It appears that the line is* ***mismatched*** *at both $z = 0$ and $z = -\ell$.*

A: In fact there **are** reflections at these mismatched interfaces—an **infinite** number of them!

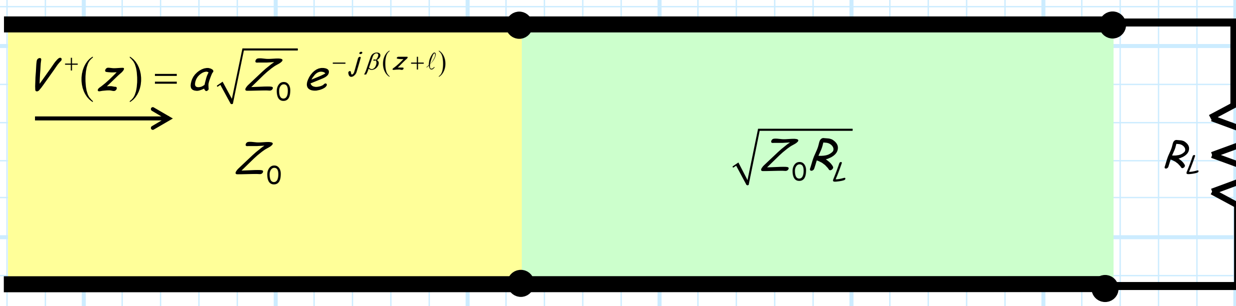
We can use our **signal flow graph** to determine the propagation series, once we determine all the **propagation paths** through the quarter-wave transformer.



$$b = a\sum_{n=1}^{\infty} p_n$$

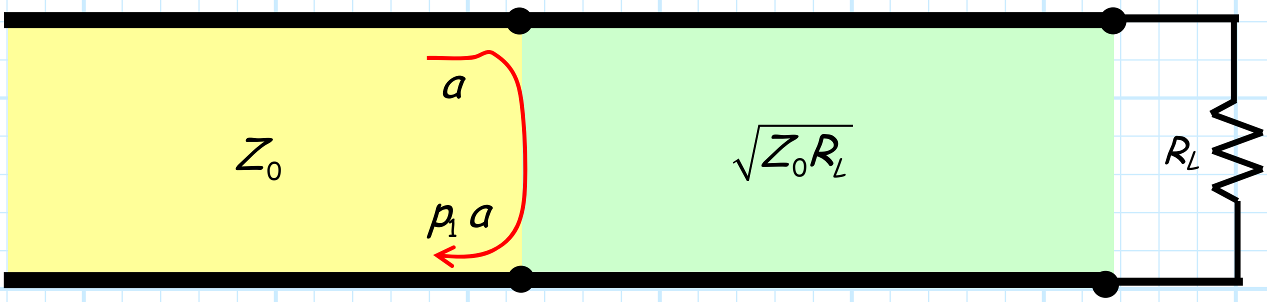


Now let's try to interpret what **physically** happens when the **incident** voltage wave:

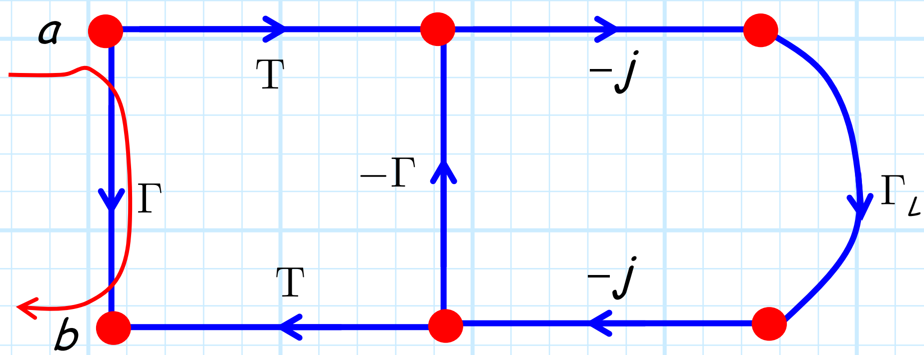


reaches the interface at $z = -\ell$. We find that there are **two forward paths** through the quarter-wave transformer signal flow graph.

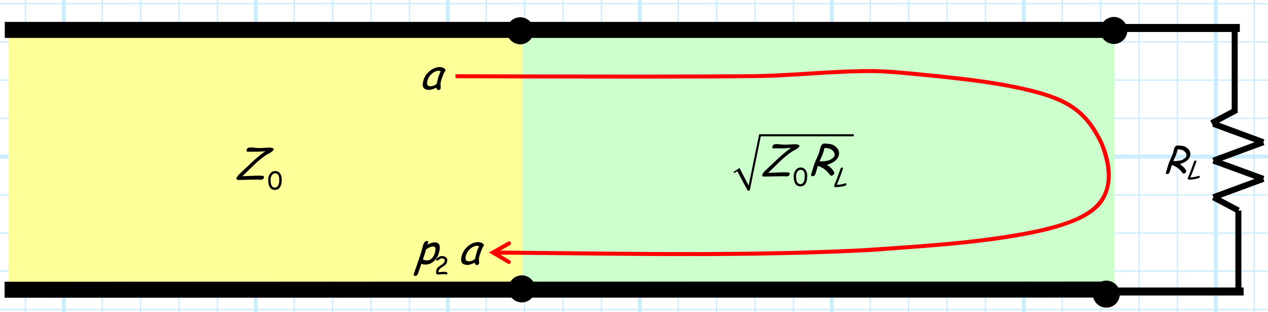
Path 1. At $z=-\ell$, the characteristic impedance of the transmission line changes from Z_0 to Z_1. This mismatch creates a **reflected** wave, with complex amplitude $p_1 a_1$:



So, $p_1 = \Gamma$.



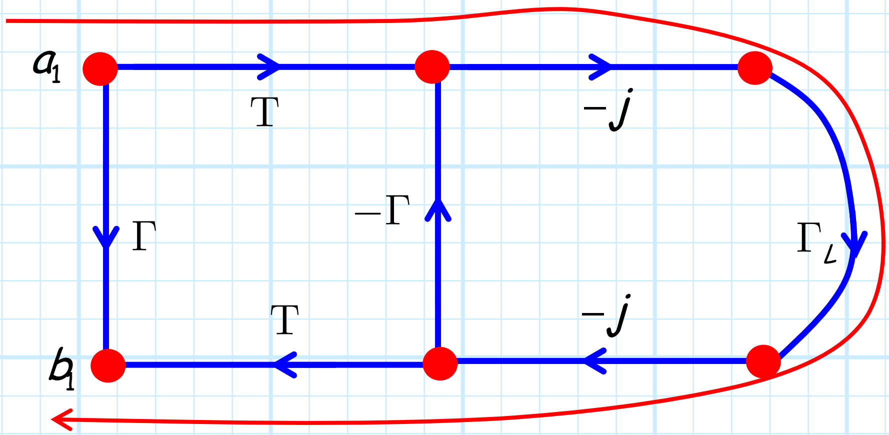
Path 2. However, a **portion** of the incident wave is transmitted (T) across the interface at $z=-\ell$, this wave travels a distance of $\beta\ell = 90^\circ$ to the load at $z=0$, where a portion of it is reflected (Γ_L). This wave travels back $\beta\ell = 90^\circ$ to the interface at $z=-\ell$, where a portion is again transmitted (T) across into the Z_0 transmission line—**another** reflected wave !



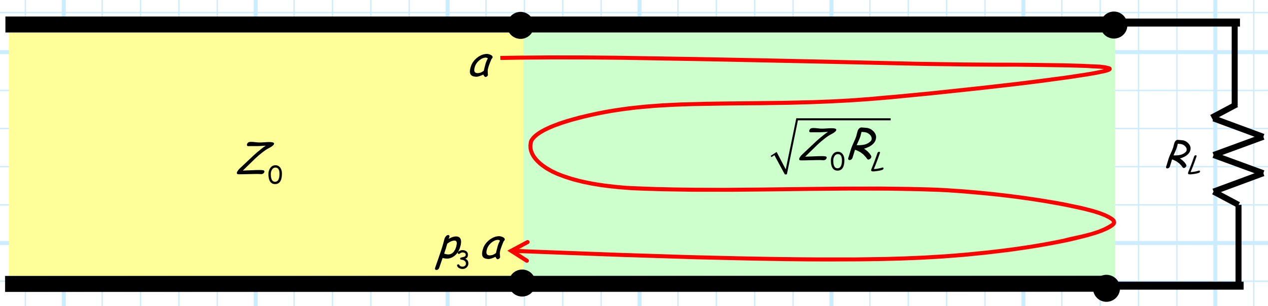
So the **second direct path** is

$$p_2 = \mathrm{T}\ e^{-j90^\circ}\ \Gamma_L\ e^{-j90^\circ}\ \mathrm{T} = -\mathrm{T}^2\ \Gamma_L$$

note that traveling $2\beta\ell = 180^\circ$ has produced a **minus** sign in the result.

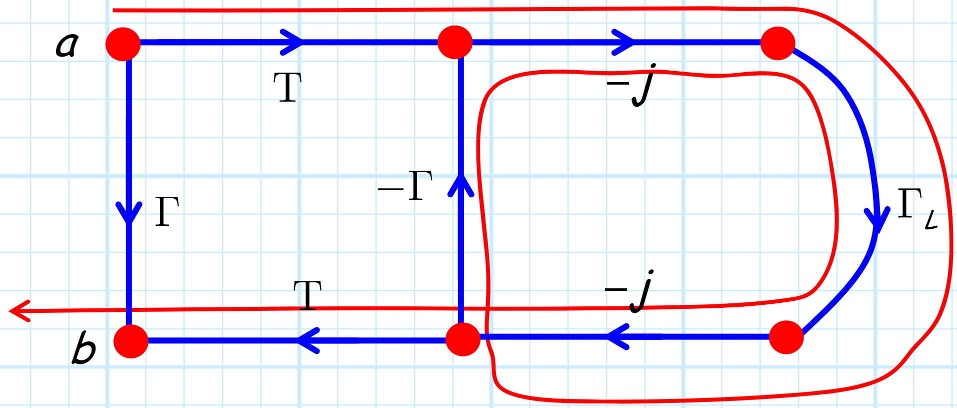


Path 3. However, a **portion** of this **second** wave is also **reflected** (Γ) back into the Z_1 transmission line at $z = -\ell$, where it again travels to $\beta\ell = 90^\circ$ the load, is partially reflected (Γ_L), travels $\beta\ell = 90^\circ$ back to $z = -\ell$, and is partially transmitted into Z_0 (T)—our **third** reflected wave!



where:

$$p_3 = \mathrm{T}\, e^{-j90^\circ}\ \Gamma_L\, e^{-j90^\circ}\ (-\Gamma)\, e^{-j90^\circ}\ \Gamma_L\, e^{-j90^\circ}\,\mathrm{T}$$
$$= -\mathrm{T}^2 \left(\Gamma_L\right)^2 \Gamma$$



Note that path 3 is **not** a direct path!

Path *n*. We can see that this "bouncing" back and forth can go on **forever**, with each trip launching a **new** reflected wave into the Z_0 transmission line.

Note however, that the **power** associated with each successive reflected wave is **smaller** than the previous, and so eventually, the power associated with the reflected waves will **diminish** to insignificance!

Q: *But, why then is* $\Gamma = 0$ *?*

A: Each reflected wave is a **coherent** wave. That is, they all oscillate at same frequency ω; the reflected waves differ only in terms of their **magnitude** and **phase**.

Therefore, to determine the **total** reflected wave, we must perform a **coherent summation** of each reflected wave—this summation of course results in our **propagation series**, a series that must converge for passive devices.

$$b = a \sum_{n=1}^{\infty} p_n$$

It can be shown that the infinite propagation series for **this** quarter-wavelength structure **converges** to the closed-form expression:

$$\frac{b}{a} = \sum_{n=1}^{\infty} p_n = \frac{\Gamma - \Gamma^2\,\Gamma_L - \mathrm{T}^2\,\Gamma_L}{1-\Gamma^2}$$

Thus, the **input** reflection coefficient is:

$$\Gamma_{in} = \frac{b}{a} = \frac{\Gamma - \Gamma^2\,\Gamma_L - \mathrm{T}^2\,\Gamma_L}{1-\Gamma^2}$$

Using our definitions, it can likewise be shown that the **numerator** of the above expression is:

$$\Gamma - \Gamma^2\,\Gamma_L - \mathrm{T}^2\,\Gamma_L = \frac{2\left(Z_1^2 - Z_0\,R_L\right)}{\left(Z_1 + Z_0\right)\left(R_L + Z_1\right)}$$

It is evident that the numerator (and therefore Γ) will be **zero if**:

$$Z_1^2 - Z_0\,R_L = 0 \qquad \Rightarrow \qquad Z_1 = \sqrt{Z_0\,R_L}$$

Just as we expected!

Physically, this results insures that all the reflected waves add coherently together to produce a **zero value**!

Note **all** of our transmission line analysis has been **steady-state** analysis. We assume our signals are **sinusoidal**, of the form $\exp(j\omega t)$. Note this signal exists for **all time** t—the signal is assumed to have been "on" **forever**, and assumed to continue on forever.

In other words, in steady-state analysis, **all** the multiple reflections have long since occurred, and thus have reached a steady state—the reflected wave is **zero**!