EECS 861

Homework #4

1. (Concept: Transformations of random variables)
X is a Gaussian RV with zero mean and variance of σ^2; find and plot (with σ^2=1) the pdf of Y where

a. Y=3X+5

b. Y=2|X|

c. Y=4 X^2

d. Y=2r(x) where r(x) is a ramp function, i.e., Y = 0 for X<0 and Y=X for X>0

2. (Concept: calculating probabilities of transformed RV's)
X is a Gaussian RV with zero mean and variance of σ^2=1; find the probabilities indicated below directly from $f_x(x)$, i.e., not integrating the pdf's found in problem 1.

a. Y=3X+5; find P(Y>8)

b. Y=2|X|; find P(Y>2)

c. Y=4 X^2; find P(Y>2)

3. Let X_1 and X_2 be statistically independent identically distributed (iid) random variables with a common pdf of $f_X(x) = \frac{1}{4} e^{-x/4}$u(x). Find the pdf of Y=$X_1 + X_2$. Plot the pdf of Y.

4. (Concept: A method of generating correlated RV's with specified correlation coefficient.)
Let X_1 and X_2 be uncorrelated random variables with zero means and a common variance of σ^2 and define the RV Y as $Y = aX_1 + \sqrt{1 - a^2}\ X_2$

a. Find E[Y]

b. Find $E\left[Y^2\right]$

c. Find the correlation coefficient between Y and X_1.

d. Discuss how you could use the results of this problem to generate correlated pseudo-random variables with a specified correlation coefficient.

5. (Concept: Properties of the sample mean)
Let $X_1, \cdots, X_n$ be n independent zero mean Gaussian random variables with equal variances, σ^2. $Y = \frac{1}{N} \sum_{i=1}^{n} X_i$

a. Find E[Y].

b. Find Var[Y]

c. Find P(Y>$\frac{\sigma}{\sqrt{N}}$)

d. Y is the mean of a sample of n observations of X, that is, the sample mean. Comment on the relationship between the original variance, i.e, Var[X_i], and the variance of the sample mean, Var[Y].

6. (Concept: Properties of sums of RV's and central limit theorem)
A RV X_i is uniformly distributed between 100 and 200 and the $X_1 \ldots X_{10}$ are 10 i.i.d random variables. Let N=10 and $Y = \frac{1}{N} \sum_{i=1}^{N} X_i$.

a. Find E[Y]

b. Find Var[Y]

c. Find P(Y>205) with no approximation.

d. Approximate P(Y>159.1)

7. (Concept: bivariate Gaussian pdf.)

X is a multivariate Gaussian random vector with

$\mu_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\Sigma_X = \begin{pmatrix} 1 & -0.75 \\ -0.75 & 1 \end{pmatrix}$

a. Find Var[X_1]

b. Find $\rho_{X_1 X_2}$

c. Find $P(X_1 > 1)$

d. First find $E[X_1 \mid X_2 = 1]$ and Var[$X_1 \mid X_2 = 1$] then find $P(X_1 > 1 \mid X_2 = 1)$

8. (Concept: Linear combinations of correlated Gaussian RV's)

X is a multivariate Gaussian random vector with

$$\mu_X = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \Sigma_X = \begin{pmatrix} 1.5 & 0.75 & 0.5 \\ 0.75 & 1.5 & 0.75 \\ 0.5 & 0.75 & 1.5 \end{pmatrix}$$

a. Find Var[x_1] and Var[x_2] .

b. Find covariance x_1 and x_2

c. Find correlation coefficient for x_1 and x_2

d. Given a transformation between X and Y as

$Y_1 = X_1 + 2 X_2 + 3 X_3$

$Y_2 = X_1 + X_3$

$Y_3 = 2 X_2 + 3 X_3$

Find μ_Y and Σ_Y

e. Given Σ_Y find μ_{Y3} and Var[Y_3] and the pdf of $Y_{3.}$

f. Let Z=AX with

$$A = \begin{pmatrix} -0.328989 & -0.368347 & -0.328989 \\ -0.707107 & 0. & 0.707107 \\ 0.540154 & -0.964877 & 0.540154 \end{pmatrix}$$

Find Find μ_Z and Σ_Z

g. Find $P(Z_1 > 1 | Z_3 = 0)$

See: **Whitening Bivariate and 3-dimensional Gaussian Random Vectors**

9. (Concept: Whitening a bivariate Gaussian random vector)

For the bivariate Gaussian random vector X with $\mu_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\Sigma_X = \begin{pmatrix} 1 & -0.75 \\ -0.75 & 1 \end{pmatrix}$

Find a transformation T, where Z=TX, such that Z_1 and Z_2 are identically distributed and statistically independent (i.i.d.) with unit variance.

10. (Concept: Scatter plots and calculating estimators)

a. Create a scatter plot for the data in Sheet labeled Data 1, Data 2, and Data 3 in:

http://www.ittc.ku.edu/~frost/EECS_861/EECS_861_HW_Fall_2025/data_HW_4_prob_9.xls

b. What can you say about from visual examination of the scatter plot of the data in Data in Sheet labeled Sheet1, Sheet2, and Sheet3?

c. Apply the estimators defined below to each data set and report the estimated means, variances and correlation coefficient for each data set.

For estimators use:

$$\mu_X = E[X] \;\&\; \mu_y = E[Y]$$

Their estimates are:

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{n} x_i \;\&\; \bar{Y} = \frac{1}{N}\sum_{i=1}^{n} y_i$$

$$\sigma_X^2 = E[(X-\mu_X)^2] = E[X^2]\text{-}(E[X])^2 \;\&\; \sigma_Y^2 = E[(X-\mu_Y)^2] = E[Y^2]\text{-}(E[Y])^2$$

Their estimates are

$$s_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{X})^2 \;\&\; s_Y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})^2$$

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = E[XY]-\mu_X\mu_y = Covariance$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_x\sigma_y}$$

Estimate of the correlation coefficient is:

$$\bar{\rho}_{XY} = \frac{\frac{1}{N}\sum_{i=1}^{N} x_i y_i - \bar{X}\bar{Y}}{s_x s_Y}$$

10. (Concept: Bounds)

For $f_x(x)$ = u(x) e^{-x} approximate P(X> a) with a>0; plot the Tchebycbeff (Chebyshev) and Chernoff bounds and the exact probability as a function of a for 0.5<a<3.