M/G/1
Problem Definition
and
Analysis

Some Notation

- $\lambda =$ Mean Customer Arrival Rate (Poission Process)
- $\mu =$ Mean Customer Service Rate (General Distribution)
- $\rho = \frac{\lambda}{\mu} =$ Average System Utilization
- M= Service time
- $f_M(m)$ = pdf of service time (sec)
- $E[M]=1/\mu$
- E[n]=Expected number of packets in system
- E[T]=Expected delay (sec)

Goal: Find E[n] and using Little's result $\lambda E[T] = E[n]$ find average delay

Modified from: Communications Networks "Theory and Analysis" by Prof. A. Bruce McDonald

- Consider the system *"state"* at the departure instants;
- Define n_{i+1} as the number in the system immediately following the $(i+1)$ st departure;

  1. n_{i+1} enumerates the customers in the queue *and* in service;
  2. n_{i+1} is also equal to the number of customers in the system after the ith departure minus 1 (since there **must** have been one-and-only-one departure, plus the number of customer arrivals between the ith and the $(i+1)$ st departure;

- Define a_{i+1} as the number of customers that arrive to the system between the i th and the $(i+1)$ st departure instants.

$$n_{i+1} = n_i - 1 + a_{i+1}\ , \qquad n_i > 0 \tag{1}$$

$$n_{i+1} = a_{i+1}\ , \qquad n_i = 0$$

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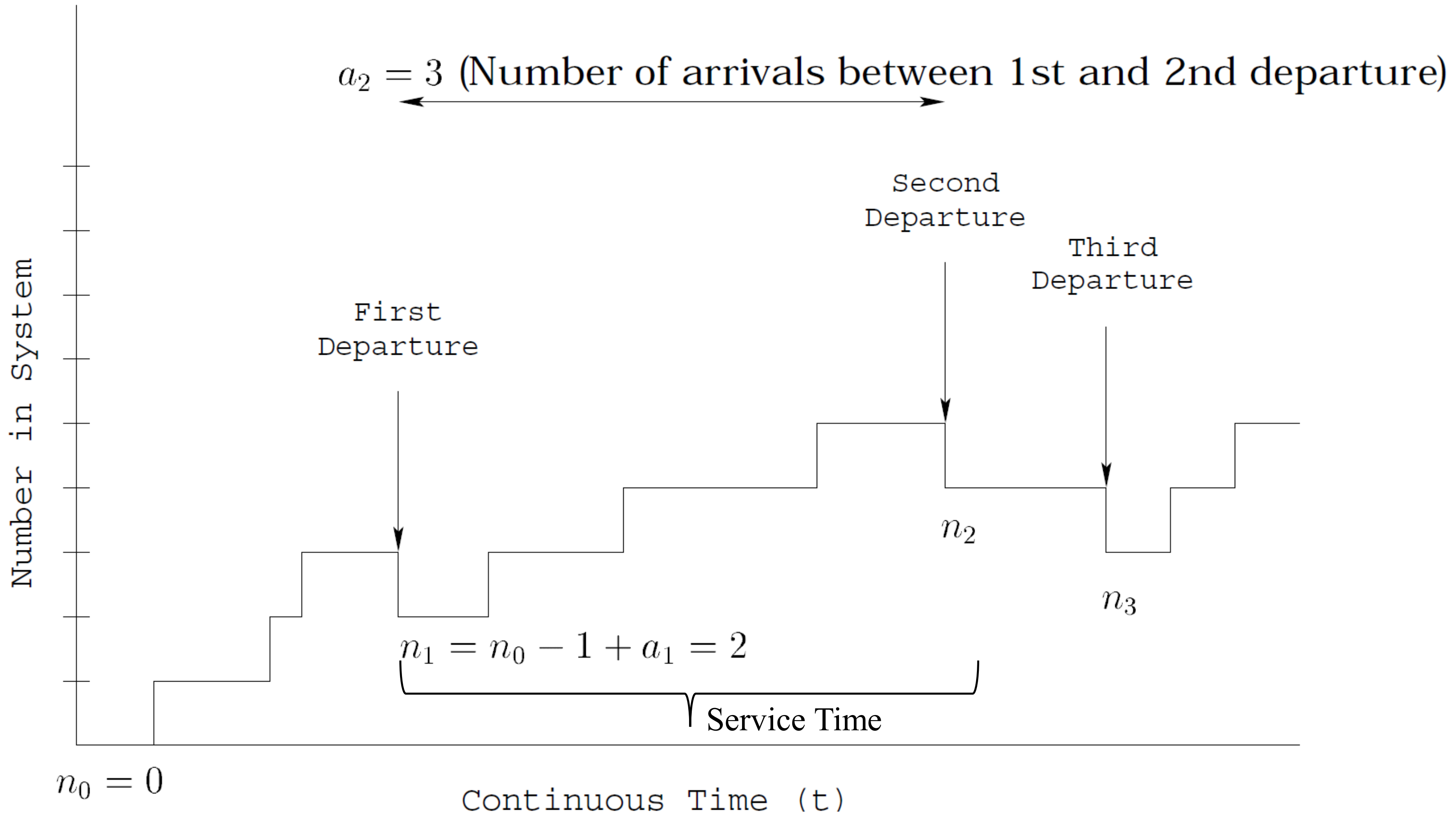

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$$n_{i+1} = n_i - u(n_i) + a_{i+1}$$

$$u(n_i) = \begin{cases} 1, & n_i > 0 \\ 0, & n_i = 0 \end{cases}$$

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Is n_i a Markov Chain?

- a_{i+1} is statistically independent of the n_i from nonoverlapping intervals.
- And n_{i+1} only depends on n_i
- Yes! n_i is a Markov chain
- The process for n_i is an example of an imbedded Markov chain
- If $\rho<1$ then this Markov chain is
  - Irreducible
  - Positive Recurrent
  - Ergodic
  - Equilibrium Probabilities (π_i) of i packets in the system exists

Define MGFs

$$n_i = \{0, 1, 2 \ldots \infty\}$$

$$P_i(z) = E[z^{n_i}] = \sum_{k=0}^{\infty} z^k \text{Prob}(n_i = k)$$

$$P_{i+1}(z) = E[z^{n_{i+1}}] = \sum_{k=0}^{\infty} z^k \text{Prob}(n_{i+1} = k)$$

$$a_{i+1} = \{0, 1, 2 \ldots \infty\}$$

$$A_{i+1}(z) = E[z^{a_{i+1}}] = \sum_{k=0}^{\infty} z^k \text{Prob}(a_{i+1} = k)$$

$A_{i+1}(z)$ is the MGF for the number of arrivals in a service time

Analysis

$P_{i+1}(z) = E[z^{n_{i+1}}] = E[z^{n_i - u(n_i) + a_{i+1}}]$

a_{i+1} is statistically independent of $n_i - u(n_i)$ so

$P_{i+1}(z) = E[z^{n_i - u(n_i)}]E[z^{a_{i+1}}] = E[z^{n_i - u(n_i)}]A_{i+1}(z)$

First focus on finding $E[z^{n_i - u(n_i)}]$

$$E[z^{n_i - u(n_i)}] = \sum_{k=0}^{\infty} z^k \text{Prob}(n_i - u(n_i) = k)$$

For k=0, $\text{Prob}(n_i - u(n_i) = 0) = \text{Prob}(n_i = 0) + \text{P rob}(n_i = 1)$ then

$$E[z^{n_i - u(n_i)}] = z^0(\text{Prob}(n_i = 0) + \text{Prob}(n_i = 1)) + \sum_{k=1}^{\infty} z^k \text{Prob}(n_i - u(n_i) = k)$$

For k=1, $\text{Prob}(n_i - u(n_i) = 1) = \text{Prob}(n_i = 2)$

For k=2, $\text{Prob}(n_i - u(n_i) = 2) = \text{Prob}(n_i = 3)$

$$E[z^{n_i - u(n_i)}] = \text{Prob}(n_i = 0) + \text{Prob}(n_i = 1) + \sum_{k=1}^{\infty} z^k \text{Prob}(n_i) = k + 1)$$

$$E[z^{n_i - u(n_i)}] = \text{Prob}(n_i = 0) + \sum_{j=1}^{\infty} z^{j-1} \text{Prob}(n_i) = j)$$

Analysis (continued)

$\text{Prob}(n_i = 0) = \text{Probability the system is empty} = 1\text{-}\rho$

$$E[z^{n_i - u(n_i)}] = 1\text{-}\rho + \sum_{j=1}^{\infty} z^{j-1}\text{Prob}(n_i) = j$$

Next focus on $\sum_{j=1}^{\infty} z^{j-1}\text{Prob}(n_i) = j$

$$\sum_{j=1}^{\infty} z^{j-1}\text{Prob}(n_i) = j) = z^{-1}\sum_{j=1}^{\infty} z^{j}\text{Prob}(n_i = j)$$

$$= z^{-1}(\sum_{j=0}^{\infty} z^{j}\text{Prob}(n = j) - \text{Prob}(n_i = 0))$$

$P_i(z) = E[z^{n_i}] = \sum_{k=0}^{\infty} z^{k}\text{Prob}(n_i = k)$ then

$$E[z^{n_i - u(n_i)}] = 1\text{-}\rho + z^{-1}(P_i(z) - (1\text{-}\rho))$$

$$P_{i+1}(z) = 1\text{-}\rho + z^{-1}(P_i(z) - (1\text{-}\rho))A_{i+1}(z)$$

But the system is in steady state so

$P_{i+1}(z) = P_i(z) = P(z)$ and $A_{i+1}(z) = A(z)$

$$P(z) = 1\text{-}\rho + z^{-1}(P(z) - (1\text{-}\rho))A(z)$$

Solving for $P(z)$

$$P(z) = \frac{(1-\rho)A(z)}{A(z) - z}$$

Analysis (continued)

$A(z) = \text{MGF for number of arrivals in a service time M (sec)} = \sum_{a=0}^{\infty} z^a \text{Prob[A=a]}$

With Poisson arrivals Prob[A=a]=$\int_0^{\infty} \frac{(\lambda m)^a e^{-\lambda m}}{a!} f_M(m)dm$

$A(z) = \sum_{a=0}^{\infty} z^a \int_0^{\infty} \frac{(\lambda m)^a e^{-\lambda m}}{a!} f_M(m)dm = \text{LaplaceTransform of } f_M(m) \text{ with } s = \lambda(1-z)$

Note

$\frac{dA(z)}{dz} \textit{ evaluated at } z = 1 = A'(1) = \lambda E[M] = \rho$

$\frac{d^2 A(z)}{dz^2} \textit{ evaluated at } z = 1 = A''(1) = \lambda^2 E[M^2]$

Analysis (continued)

$P(z) = \frac{(1-\rho)A(z)}{A(z)-z} =$ MGF for number in the system.

$\frac{dP(z)}{dz}$ *evaluated at* $z = 1 = P'(1) = E[n]$

$$E[n] = P'(1) = \frac{(1-\rho)A'(1)}{1-A'(1)} + \frac{A''(1)}{2(1-A'(1))}$$

Using $A'(1) = \lambda E[M] = \rho$ and $A''(1) = \lambda^2 E[M^2]$

Average number in the system

$$E[n] = \rho + \frac{\lambda^2 E[M^2]}{2(1-\rho)}$$

Average delay

$\mathrm{E[T]} = \frac{E[n]}{\lambda}$ from Little's result

$$E[T] = \frac{\rho}{\lambda} + \frac{\lambda E[M^2]}{2(1-\rho)} = E[M] + \frac{\lambda E[M^2]}{2(1-\rho)}$$

$$E[T] = E[M] + \frac{\lambda(Var[M] + (E[M])^2)}{2(1-\rho)}$$

Observations

- Average delay a function of first two moments of the message length distribution.
- M/D/1 is a system with deterministic (fixed) message lengths
  - E[T] for M/M/1 > E[T] for M/D/1
- Average waiting $E[W] = \frac{\lambda(Var[M] + (E[M])^2)}{2(1-\rho)}$
- With E[M] fixed as Var[M] increases average delay increases

Examples

- C=1Mb/s and λ=600 packets/sec
- M/M/1 with E[L]=1000 bits
  - E[M]=1ms
  - ρ=0.6
  - E[T]=2.5ms
- M/D/1 with E[L]=1000 bits
  - E[M]=1ms
  - ρ=0.6
  - Var[M]=0
  - E[T]=1.75ms
- M/G/1 with L~ Uniform[500,1500]
  - E[L]=1000 & E[M]=1ms
  - ρ=0.6
  - E[T]=1.81ms

General solution for pmf of the number in the system

- From $f_M(m)$ find A(z)
- Solve for $P(z) = \frac{(1-\rho)A(z)}{A(z)-z}$
- Take the inverse z-transform of P(z)