Volume Scattering

In many radar remote sensing applications, the illuminated target is a **random** collection of **irregular** particles or elements, dispersed throughout some 3-dimensional **volume**.

Our challenge is to determine how a monochromatic plane wave might **scatter** and **propagate** through such a medium.

I. Electromagnetic Scattering

First, let's review some **background** material, involving the *Electromagnetic Scattering* from a single object or particle.

* We find that the scattered **far-fields** from any object can be expressed in terms of its *Scattering Matrix*.

* We can likewise describe the scattered **power density** (or intensity) from an object in terms of its *Scattering Cross Section.*

II. Scattering from a Random Media

Now, let's consider the case where **multiple** scatterers are present.

* With knowledge of the scattering matrix and **location** of each particle, we can, provided we consider all *Multiple Scattering Mechanisms*, determine the far-field scattering from this collection.

* For many (or most!) applications, our scattering volume consists of a **random** collection of scatterers, described with **statistical** measures. Thus, we must likewise characterize the *Scattering from Random Media* with **statistical** measures.

* Although a scattering volume can consist many **different** types, shapes, and sizes of random scatterers, we find that the statistics of the scattered fields can usually be well described in terms of *Rayleigh Fading Statistics*.

III. Propagation through a Random Media

When a plane wave interacts with a particle, it **reduces** the energy in the wave.

* The rate at which a **single** particle removes energy from a plane wave is specified by its *Extinction Cross-Section*.

* The wave attenuation exhibited by a **collection** of particles is described by an *Extinction Coefficient*.

* Using the *Optical Theorem*, we can likewise determine the **effective propagation constant** for a collection of random particles.

* We must account for extinction if we wish to **accurately** determine the average scattering from a large volume of scattering particles. We can accomplish this by implementing the *Distorted Born Approximation.*

IV. Volume Scattering from Collections of Rayleigh Scatterers

* Scattering from particles that are **small** with respect to a wavelength is described by a theory known as *Rayleigh Scattering*.

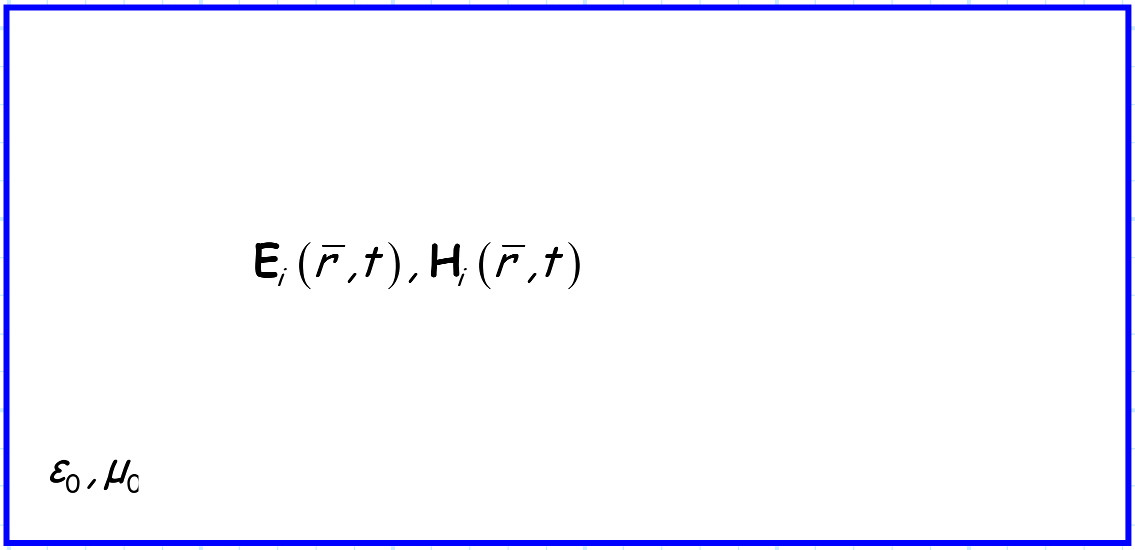
* A Rayleigh Scatterer is **completely** characterized by a *Polarizability Tensor.*

* The propagation through a random volume of Rayleigh scatterers can be described in terms of an **equivalent dielectric constant** derived from *Mixing Models.*

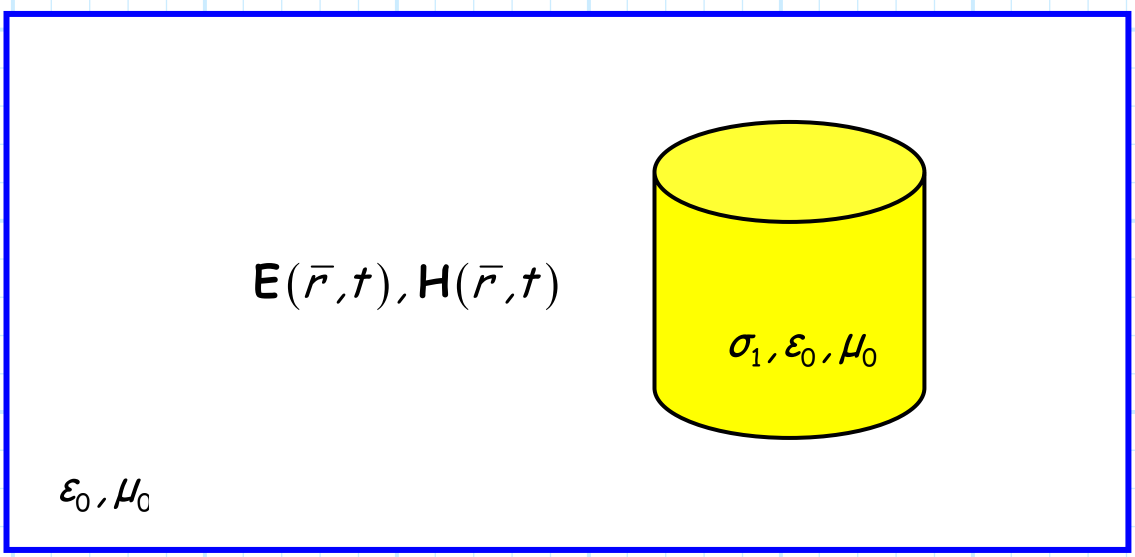
* We can apply **all** of our acquired knowledge to determine the *Volume Scattering from a Layer of Rayleigh Scatterers.*

Electromagnetic Scattering

Consider some **known** electromagnetic fields, existing throughout empty space:



If we insert some object into this space, the electromagnetic fields are **modified**:



The **difference** between the initial fields $\mathbf{E}_i(\bar{r},t), \mathbf{H}_i(\bar{r},t)$, and the modified fields $\mathbf{E}(\bar{r},t), \mathbf{H}(\bar{r},t)$, are defined as the **scattered fields** $\mathbf{E}_s(\bar{r},t), \mathbf{H}_s(\bar{r},t)$:

$$\mathbf{E}_s(\bar{r},t) \doteq \mathbf{E}(\bar{r},t) - \mathbf{E}_i(\bar{r},t)$$

$$\mathbf{H}_s(\bar{r},t) \doteq \mathbf{H}(\bar{r},t) - \mathbf{H}_i(\bar{r},t)$$

Rearranging, we can alternatively state that:

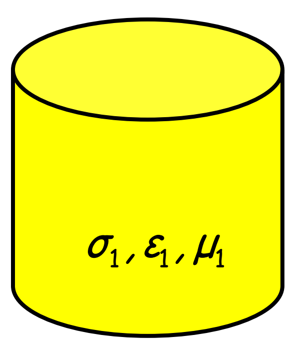
$$\mathbf{E}(\bar{r},t) = \mathbf{E}_i(\bar{r},t) + \mathbf{E}_s(\bar{r},t)$$

$$\mathbf{H}(\bar{r},t) = \mathbf{H}_i(\bar{r},t) + \mathbf{H}_s(\bar{r},t)$$

Thus, the modified fields are formed when the scattered fields $\mathbf{E}_s(\bar{r},t), \mathbf{H}_s(\bar{r},t)$ are added to the original (incident) fields $\mathbf{E}_i(\bar{r},t), \mathbf{H}_i(\bar{r},t)$. As a result, these modified fields are most often referred to as **total fields** $\mathbf{E}(\bar{r},t), \mathbf{H}(\bar{r},t)$:

$$\mathbf{E}(\bar{r},t) = \mathbf{E}_i(\bar{r},t) + \mathbf{E}_s(\bar{r},t)$$

$$\mathbf{H}(\bar{r},t) = \mathbf{H}_i(\bar{r},t) + \mathbf{H}_s(\bar{r},t)$$



ε_0, μ_0

Q: *Why are the incident fields modified? Where do the scattered fields come from?*

A: The incident fields **induce currents** (conduction, polarization, and/or magnetization) within the **object**—these currents in turn create new (i.e., scattered) fields!

We find that the **total** fields can be related to the **scattered** fields as:

$$\mathbf{E}_s(\bar{r},t) = \iint \overleftrightarrow{G}_e(\bar{r},t;\bar{r}',t') \cdot \mathbf{E}(\bar{r}',t')\, dv'\, dt'$$

$$\mathbf{H}_s(\bar{r},t) = \iint \overleftrightarrow{G}_m(\bar{r},t;\bar{r}',t') \cdot \mathbf{H}(\bar{r}',t')\, dv'\, dt'$$

where $\overleftrightarrow{G}_e(\bar{r},t;\bar{r}',t')$ and $\overleftrightarrow{G}_m(\bar{r},t;\bar{r}',t')$ are called **dyadic Green's functions** (not important!).

Note in these expressions, the scattered fields are **dependent** on the **total** fields (the charges and dipoles within the object **can't** tell the difference between a scattered and incident field!).

But recall we likewise determined that total fields are **dependent** on the **scattered** fields:

$$\mathbf{E}(\bar{r},t) = \mathbf{E}_i(\bar{r},t) + \mathbf{E}_s(\bar{r},t)$$

$$\mathbf{H}(\bar{r},t) = \mathbf{H}_i(\bar{r},t) + \mathbf{H}_s(\bar{r},t)$$

This **circular logic** is expressed when we combine these results:

$$\mathbf{E}(\bar{r},t) = \mathbf{E}_i(\bar{r},t) + \iint \overleftrightarrow{G}_e(\bar{r},t;\bar{r}',t') \cdot \mathbf{E}(\bar{r}',t')\, dv'\, dt'$$

$$\mathbf{H}(\bar{r},t) = \mathbf{H}_i(\bar{r},t) + \iint \overleftrightarrow{G}_m(\bar{r},t;\bar{r}',t') \cdot \mathbf{H}(\bar{r}',t')\, dv'\, dt'$$

These equations demonstrate the **difficulty** in finding microwave scattering solutions—note the **unknown** **total fields** are likewise part of the integration!

These expressions are therefore known as **integral equations**, and finding there solutions are **exceedingly** difficult. In fact, humans have found solutions **only** for the simplest of objects (e.g., spheres, cylinders)!

Q: *So then, how* ***do*** *we determine electromagnetic scattering?*

A: There are basically **three** methods: numerical approximation, asymptotic approximation, and direct measurement.

Numerical Approximations - Given the computational resources available today, this method has become the most popular method. Techniques such as the moment method allow

us to numerically approximation the integration (e.g., as a summation).
The results are often very accurate, although since they are numeric, they do not provide much parametric insight (e.g., the scattering variation with respect to dielectric constant), unless we run multiple cases.

Asymptotic Approximations - Often, we find that we can solve the integral equation if one or more of the physical characteristics of the object takes some **extreme** value. For example, if the volume of the scattering object is zero, then $\mathbf{E}_s(\bar{r}) = 0$ and $\mathbf{E}(\bar{r}) = \mathbf{E}_i(\bar{r})$.

Although these exact solutions (like this example) are generally not particularly useful, we can often determine from them **approximate** solutions, valid when the scattering problem **approaches** these extreme cases. Accordingly, from the above example, we might determine an approximation that is accurate when the scattering object is **very small**.

Direct Measurement - Some scattering objects are so complex (e.g. aircraft) that scattering solutions can only be determined by **direct measurement** in an anechoic chamber.

Additionally, we often make direct measurement of simpler objects, so as to **validate** a numeric or asymptotic approximation.

The Scattering Matrix

The incident field for most scattering problems is assumed to be a **monochromatic plane wave** of the form:

$$\mathbf{E}_i(\bar{r},t) = Re\left\{\mathbf{E}_i(\bar{r})\, e^{+j\omega_0 t}\right\}$$

where

$$\begin{aligned}\mathbf{E}_i(\bar{r}) &= \mathbf{E}_0^i\, e^{-j\bar{k}_i\cdot\bar{r}} \\ &= \left(E_v^i\,\hat{\mathbf{v}}_i + E_h^i\,\hat{\mathbf{h}}_i\right) e^{-j\bar{k}_i\cdot\bar{r}}\end{aligned}$$

and:

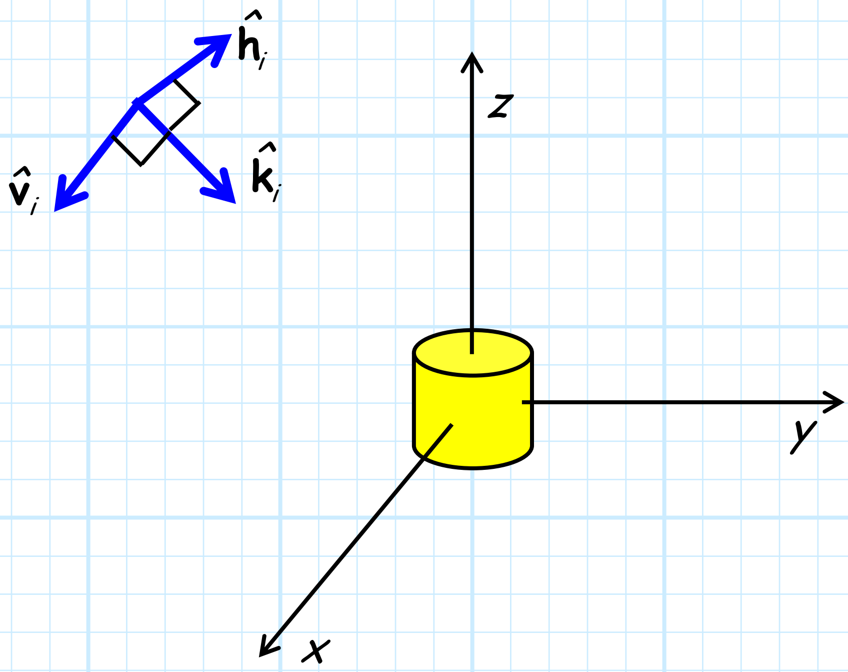
$$\bar{k}_i = \frac{2\pi}{\lambda_0}\,\hat{\mathbf{k}}_i = k_0\,\hat{\mathbf{k}}_i$$

$$\hat{\mathbf{v}}_i \times \hat{\mathbf{h}}_i = \hat{\mathbf{k}}_i$$

$$\hat{\mathbf{v}}_i \cdot \hat{\mathbf{h}}_i = 0$$

The complex values E_v^i and E_h^i thus define **wave polarization**, and the unit vector $\hat{\mathbf{k}}_i$ describes the **propagation direction**.

Say now we place a **finite object** in this incident field and locate it at the **origin**. A **scattered** field must be generated!



If the scattering object is made of **simple** (i.e., linear) material, and is **time invariant** (i.e., it's not moving!), then the scattered field will have the form:

$$\mathbf{E}_s(\bar{r},t) = Re\left\{\mathbf{E}_s(\bar{r})\, e^{+j\omega_0 t}\right\}$$

where:

$$\mathbf{E}_s(\bar{r}) = \int \tilde{\mathbf{E}}_s\left(\bar{\mathbf{k}}'_s\right) e^{-j\,\bar{k}'_s\cdot\bar{r}}\, d\,\bar{\mathbf{k}}'_s$$

This simply states that the scattered field is a **superposition of plane waves**, propagating in all possible directions $\hat{\mathbf{k}}'_s$. The scatterer simultaneously scatters in **all** directions!

However, the scatterer will **not** scatter **equally** in all directions, **nor** with the same polarization. The **distribution** of scattered energy across direction and polarization is described by the **scattering spectrum** $\tilde{\mathbf{E}}_s\left(\bar{\mathbf{k}}_s'\right)$. The scattering spectrum complete describes the scattered field $\mathbf{E}_s(\bar{r})$, as it is essentially the **Fourier transform** of $\mathbf{E}_s(\bar{r})$ (using the basis functions $e^{-j\bar{k}_s \cdot \bar{r}}$).

Note that the **incident** field can likewise be described in terms of a scattering spectrum. Since:

$$\mathbf{E}_i(\bar{r}) = \int \tilde{\mathbf{E}}_i\left(\bar{\mathbf{k}}_i'\right) e^{-j\bar{k}_i' \cdot \bar{r}} d\,\bar{\mathbf{k}}_i' = \left(E_v^i\,\hat{\mathbf{v}}_i + E_h^i\,\hat{\mathbf{h}}_i\right) e^{-j\bar{k}_i \cdot \bar{r}}$$

it is apparent that:

$$\begin{aligned}\tilde{\mathbf{E}}_i\left(\bar{\mathbf{k}}_i'\right) &= \left(E_v^i\,\hat{\mathbf{v}}_i + E_h^i\,\hat{\mathbf{h}}_i\right)\delta\left(\hat{\mathbf{k}}_i' - \hat{\mathbf{k}}_i\right) \\ &= \mathbf{E}_0^i\,\delta\left(\hat{\mathbf{k}}_i' - \hat{\mathbf{k}}_i\right)\end{aligned}$$

This of course simple states what we already know: the incident field "spectrum" consists of precisely **one** plane wave!

Q: *So we can express the **incident** field as* $\tilde{\mathbf{E}}_i\left(\bar{\mathbf{k}}_i'\right)$, *and the **scattered** field as* $\tilde{\mathbf{E}}_s\left(\bar{\mathbf{k}}_s'\right)$. *Is there some way to **relate** these two functions?*

A: Yes! They are related by the **scattering tensor** $\overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s', \bar{\mathbf{k}}_i'\right)$:

$$\tilde{\mathbf{E}}_s\left(\bar{\mathbf{k}}_s'\right) = \int \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s', \bar{\mathbf{k}}_i'\right) \cdot \tilde{\mathbf{E}}_i\left(\bar{\mathbf{k}}_i'\right) d\,\bar{\mathbf{k}}_i'$$

Using the spectrum of our **single incident plane wave**, this becomes:

$$\begin{aligned}\tilde{\mathbf{E}}_s\left(\bar{\mathbf{k}}_s'\right) &= \int \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s' ; \bar{\mathbf{k}}_i'\right) \cdot \tilde{\mathbf{E}}_i\left(\bar{\mathbf{k}}_i'\right) d\,\bar{\mathbf{k}}_i' \\ &= \int \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s' ; \bar{\mathbf{k}}_i'\right) \cdot \mathbf{E}_0^i\, \delta\left(\bar{\mathbf{k}}_i - \bar{\mathbf{k}}_i'\right) d\,\bar{\mathbf{k}}_i' \\ &= \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s' ; \bar{\mathbf{k}}_i\right) \cdot \mathbf{E}_0^i\end{aligned}$$

The scattering tensor $\overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s' ; \bar{\mathbf{k}}_i\right)$ is **entirely** dependent on the scattering object (but this includes size, shape, orientation and material!).

Thus, **if** we know the scattering tensor $\overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s' ; \bar{\mathbf{k}}_i\right)$, we can "simply" find the scattered field as:

$$\begin{aligned}\mathbf{E}_s(\bar{r}) &= \int \tilde{\mathbf{E}}_s\left(\bar{\mathbf{k}}_s'\right) e^{-j\,\bar{k}_s' \cdot \bar{r}}\, d\,\bar{\mathbf{k}}_s' \\ &= \int \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s' ; \hat{\mathbf{k}}_i\right) \cdot \mathbf{E}_0^i\, e^{-j\,\bar{k}_s' \cdot \bar{r}}\, d\,\bar{\mathbf{k}}_s'\end{aligned}$$

Q: *Yikes! This* ***doesn't*** *look simple at all!*

A: Actually, evaluating this integral is typically **impossible**—at least without the aid of either numeric or asymptotic approximations!

So here we will apply our **first** asymptotic approximation. Consider the scattered field at a point denoted as $\bar{r} = r\,\hat{\mathbf{k}}_s$ (i.e., at a distance r from the origin, in the direction $\hat{\mathbf{k}}_s$).

We will determine the scattered field as this point approaches an **infinite distance** from the scattering object (i.e., as $|\bar{r}| = r$ approaches ∞):

$$\lim_{r\rightarrow\infty} \mathbf{E}_s\left(\bar{r} = r\,\hat{\mathbf{k}}_s\right) = \lim_{r\rightarrow\infty} \int \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s', \bar{\mathbf{k}}_i\right) \cdot \mathbf{E}_0^i \, e^{-j\,\bar{k}_s' \cdot \hat{\mathbf{k}}_s r} \, d\,\hat{\mathbf{k}}_s'$$

$$= \frac{e^{-j\,\bar{\mathbf{k}}_s \cdot \bar{r}}}{r} \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s, \bar{\mathbf{k}}_i\right) \cdot \mathbf{E}_0^i$$

where $\bar{\mathbf{k}}_s = k_0\,\hat{\mathbf{k}}_s$.

We use this result to **approximately** determine the scattered field at points a **significant distance** from the scattering object. Note this approximation says that the scattered field at this distant point **appears** to be a plane wave of the form:

$$\mathbf{E}_s(\bar{r}) \approx \frac{e^{-j\,\bar{k}_s \cdot \bar{r}}}{r} \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s, \bar{\mathbf{k}}_i\right) \cdot \mathbf{E}_0^i$$

$$= \frac{e^{-j\,\bar{k}_s \cdot \bar{r}}}{r} \mathbf{E}_0^s$$

This is approximation is simply the **far-field approximation**!

We can further express this scattered field as:

$$\begin{aligned}\mathbf{E}_s(\bar{r}) &= \frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r}\mathbf{E}_0^s \\ &= \frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r}\left(E_v^s\,\hat{\mathbf{v}}_s + E_h^s\,\hat{\mathbf{h}}_s\right)\end{aligned}$$

where $\hat{\mathbf{v}}_s \times \hat{\mathbf{h}}_s = \hat{\mathbf{k}}_s$ and $\hat{\mathbf{v}}_s \cdot \hat{\mathbf{h}}_s = 0$.

Therefore:

$$\mathbf{E}_0^s = \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s;\bar{\mathbf{k}}_i\right)\cdot\mathbf{E}_0^i$$

$$\left(E_v^s\,\hat{\mathbf{v}}_s + E_h^s\,\hat{\mathbf{h}}_s\right) = \overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s;\bar{\mathbf{k}}_i\right)\cdot\left(E_v^s\,\hat{\mathbf{v}}_s + E_h^s\,\hat{\mathbf{h}}_s\right)$$

It is evident that this **far-field scattering tensor** can be expressed as:

$$\begin{aligned}\overleftrightarrow{\mathcal{S}}\left(\bar{\mathbf{k}}_s;\bar{\mathbf{k}}_i\right) &= \mathcal{S}_{vh}\left(\bar{\mathbf{k}}_s;\bar{\mathbf{k}}_i\right)\hat{\mathbf{v}}_s\hat{\mathbf{h}}_i + \mathcal{S}_{hh}\left(\bar{\mathbf{k}}_s;\bar{\mathbf{k}}_i\right)\hat{\mathbf{h}}_s\hat{\mathbf{h}}_i \\ &\quad + \mathcal{S}_{vv}\left(\bar{\mathbf{k}}_s;\bar{\mathbf{k}}_i\right)\hat{\mathbf{v}}_s\hat{\mathbf{v}}_i + \mathcal{S}_{hv}\left(\bar{\mathbf{k}}_s;\bar{\mathbf{k}}_i\right)\hat{\mathbf{h}}_s\hat{\mathbf{v}}_i\end{aligned}$$

Q: *Huh?*

A: At this point, it's simpler to just to use **matrix** notation. We first define the **scattering matrix**:

$$\mathcal{S}\left(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i\right) \doteq \begin{bmatrix}\mathcal{S}_{vv}\left(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i\right) & \mathcal{S}_{vh}\left(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i\right) \\ \mathcal{S}_{vh}\left(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i\right) & \mathcal{S}_{hh}\left(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i\right)\end{bmatrix}$$

where the values $\mathcal{S}_{vv}, \mathcal{S}_{vh}, \mathcal{S}_{hh}, \mathcal{S}_{hv}$ are **complex** scattering coefficients. These coefficients completely describe the far-field scattering in direction $\hat{\mathbf{k}}_s$, given some incident field of direction $\hat{\mathbf{k}}_i$.

Now, expressing $\mathbf{E}_0^i$ and $\mathbf{E}_0^s$ as vectors:

$$\mathbf{E}_0^i = \begin{bmatrix} E_v^i \\ E_h^i \end{bmatrix} \qquad \text{and} \qquad \mathbf{E}_0^s = \begin{bmatrix} E_v^s \\ E_h^s \end{bmatrix}$$

we can say:

$$\begin{bmatrix} E_v^s \\ \\ E_h^s \end{bmatrix} = \begin{bmatrix} \mathcal{S}_{vv}\left(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i\right) & \mathcal{S}_{vh}\left(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i\right) \\ \mathcal{S}_{vh}\left(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i\right) & \mathcal{S}_{hh}\left(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i\right) \end{bmatrix} \begin{bmatrix} E_v^i \\ \\ E_h^i \end{bmatrix}$$

$$\mathbf{E}_0^s = \mathcal{S}\,\mathbf{E}_0^i$$

Therefore, the **far-field** scattered field is expressed as:

$$\begin{aligned} \mathbf{E}_s(\bar{r}) &= \frac{e^{-j\bar{k}_s \cdot \bar{r}}}{r}\,\mathbf{E}_0^i \\ &= \frac{e^{-j\bar{k}_s \cdot \bar{r}}}{r}\,\mathcal{S}\,\mathbf{E}_0^i \end{aligned}$$

Q: *What if the scatterer is **not** located at the origin? Say it's located at the point denoted by $\bar{r}_s$?*

A: In that case, the scattered far-field is:

$$\mathbf{E}_s(\bar{r}) = \frac{e^{-j\bar{k}_s\cdot(\bar{r}-\bar{r}_s)}}{r}\left(\mathbf{E}_0^i\, e^{-j\bar{k}_i\cdot\bar{r}_s}\right)$$

$$= \frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r}\left(\mathcal{S}\, e^{-j\left(\bar{k}_i-\bar{k}_s\right)\cdot\bar{r}_s}\right)\mathbf{E}_0^i$$

Radar Cross Section

From the Poynting **vector**, we can show that the power density of the **incident** wave is in free-space is:

$$\mathbf{W}_i(\hat{\mathbf{k}}_i) = \frac{\left|\mathbf{E}_0^i\right|^2}{\eta_0}\hat{\mathbf{k}}_i \qquad \left[\frac{W}{m^2}\right]$$

while the power density of a **scattered** field is:

$$\begin{aligned}\mathbf{W}_s(\hat{\mathbf{k}}_s) &= \frac{1}{r^2}\frac{\left|\mathbf{E}_0^s\right|^2}{\eta_0}\hat{\mathbf{k}}_s \\ &= \frac{1}{r^2}\frac{\left|\mathcal{S}\,\mathbf{E}_0^i\right|^2}{\eta_0}\hat{\mathbf{k}}_s \qquad \left[\frac{W}{m^2}\right]\end{aligned}$$

We can likewise define the scattered power density of one **polarization component** $\hat{\mathbf{p}}_s$ (e.g., $\hat{\mathbf{p}}_s = \hat{\mathbf{h}}_s$ or $\hat{\mathbf{p}}_s = \hat{\mathbf{v}}_s$) as:

$$\mathbf{W}_s^p = \frac{1}{r^2}\frac{\left|\hat{\mathbf{p}}_s^T\,\mathcal{S}\,\mathbf{E}_0^i\right|^2}{\eta_0}\hat{\mathbf{k}}_s$$

The **scattering cross section** σ of an object can be defined as:

$$\sigma(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) \doteq \lim_{r\to\infty} r^2\frac{\left|\mathbf{W}_s(\hat{\mathbf{k}}_s)\right|}{\left|\mathbf{W}_i(\hat{\mathbf{k}}_i)\right|} = \frac{\left|\mathcal{S}\,\mathbf{E}_0^i\right|^2}{\left|\mathbf{E}_0^i\right|^2} \qquad \left[m^2\right]$$

Note this value is dependent on the **incident wave polarization**, as well as the scattering object.

Accordingly, we typically define scattering cross section in terms of an **explicit** incident wave polarization, as well as one polarization **component** of scattered field. For example, the **four standard** cross-section values are:

$$\sigma_{vv}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) \doteq \lim_{r\to\infty} r^2 \frac{\left|\mathbf{W}_s^v(\hat{\mathbf{k}}_s)\right|}{\left|\mathbf{W}_i(\hat{\mathbf{k}}_i)\right|} = \frac{\left|\hat{\mathbf{v}}_s^T \mathcal{S}\, \mathbf{E}_0^i\right|^2}{\left|\mathbf{E}_0^i\right|^2} \qquad \left(\text{where } \mathbf{E}_0^i = E_v^i\, \hat{\mathbf{v}}_i\right)$$

$$\sigma_{vh}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) \doteq \lim_{r\to\infty} r^2 \frac{\left|\mathbf{W}_s^v(\hat{\mathbf{k}}_s)\right|}{\left|\mathbf{W}_i(\hat{\mathbf{k}}_i)\right|} = \frac{\left|\hat{\mathbf{v}}_s^T \mathcal{S}\, \mathbf{E}_0^i\right|^2}{\left|\mathbf{E}_0^i\right|^2} \qquad \left(\text{where } \mathbf{E}_0^i = E_h^i\, \hat{\mathbf{h}}_i\right)$$

$$\sigma_{hv}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) \doteq \lim_{r\to\infty} r^2 \frac{\left|\mathbf{W}_s^v(\hat{\mathbf{k}}_s)\right|}{\left|\mathbf{W}_i(\hat{\mathbf{k}}_i)\right|} = \frac{\left|\hat{\mathbf{h}}_s^T \mathcal{S}\, \mathbf{E}_0^i\right|^2}{\left|\mathbf{E}_0^i\right|^2} \qquad \left(\text{where } \mathbf{E}_0^i = E_v^i\, \hat{\mathbf{v}}_i\right)$$

$$\sigma_{hh}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) \doteq \lim_{r\to\infty} r^2 \frac{\left|\mathbf{W}_s^v(\hat{\mathbf{k}}_s)\right|}{\left|\mathbf{W}_i(\hat{\mathbf{k}}_i)\right|} = \frac{\left|\hat{\mathbf{h}}_s^T \mathcal{S}\, \mathbf{E}_0^i\right|^2}{\left|\mathbf{E}_0^i\right|^2} \qquad \left(\text{where } \mathbf{E}_0^i = E_h^i\, \hat{\mathbf{h}}_i\right)$$

Of particular relevance to radar problems is the **backscattering cross-section**. The back scattering cross-section is simply the scattering cross-section evaluated for the case when:

$$\hat{\mathbf{k}}_s = -\hat{\mathbf{k}}_i \qquad \text{(backscattering condition)}$$

In other words, the case when the scattered wave is traveling **back** toward the source of the incident wave.

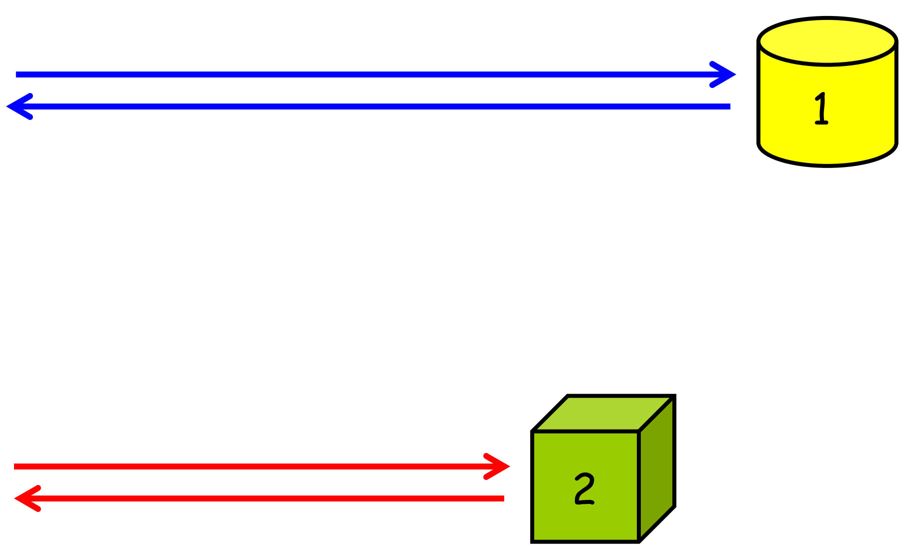
Because of its relevance to the **radar** problem, the backscattering cross-section is often referred to as the **radar cross section**.

Multiple Scattering and the Born Approximation

Now, let's consider the case where we have **two** scattering objects (at locations $\bar{r}_1$ and $\bar{r}_2$), each illuminated by the same **incident wave**.

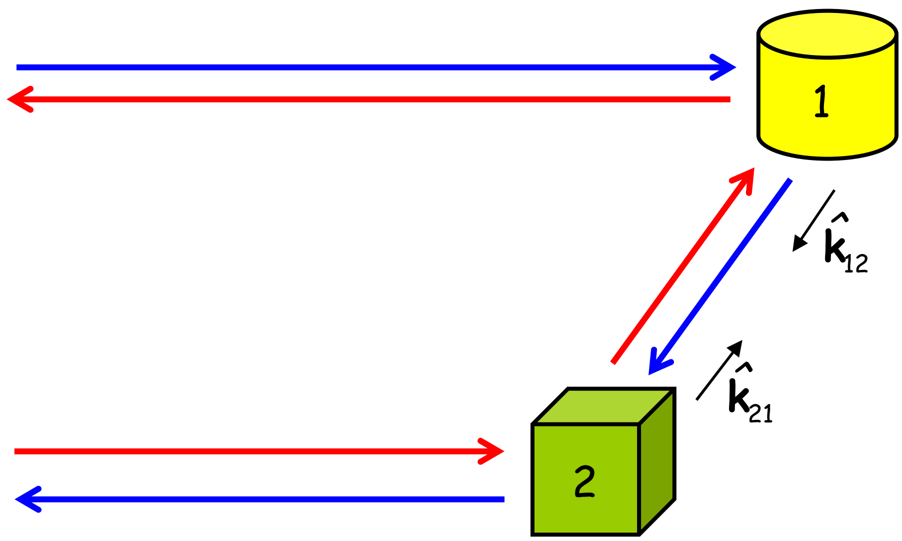
Q: *So isn't the resulting scattered field just the **sum** of the scattered field from each:*

$$\mathbf{E}_s(\bar{r}) = \frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r}\left(\mathcal{S}_1(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\,e^{-j(\bar{k}_i-\bar{k}_s)\cdot\bar{r}_1} + \mathcal{S}_2(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\,e^{-j(\bar{k}_i-\bar{k}_s)\cdot\bar{r}_2}\right)\mathbf{E}_0^i\ ???$$



1st-order scattering mechanisms

A: **NO!** If it were **only** that easy! The problem is that the scattered field from object 1 creates a **second** incident field at object 2, and the scattered field from object 2 creates a **second** incident field at object 1.



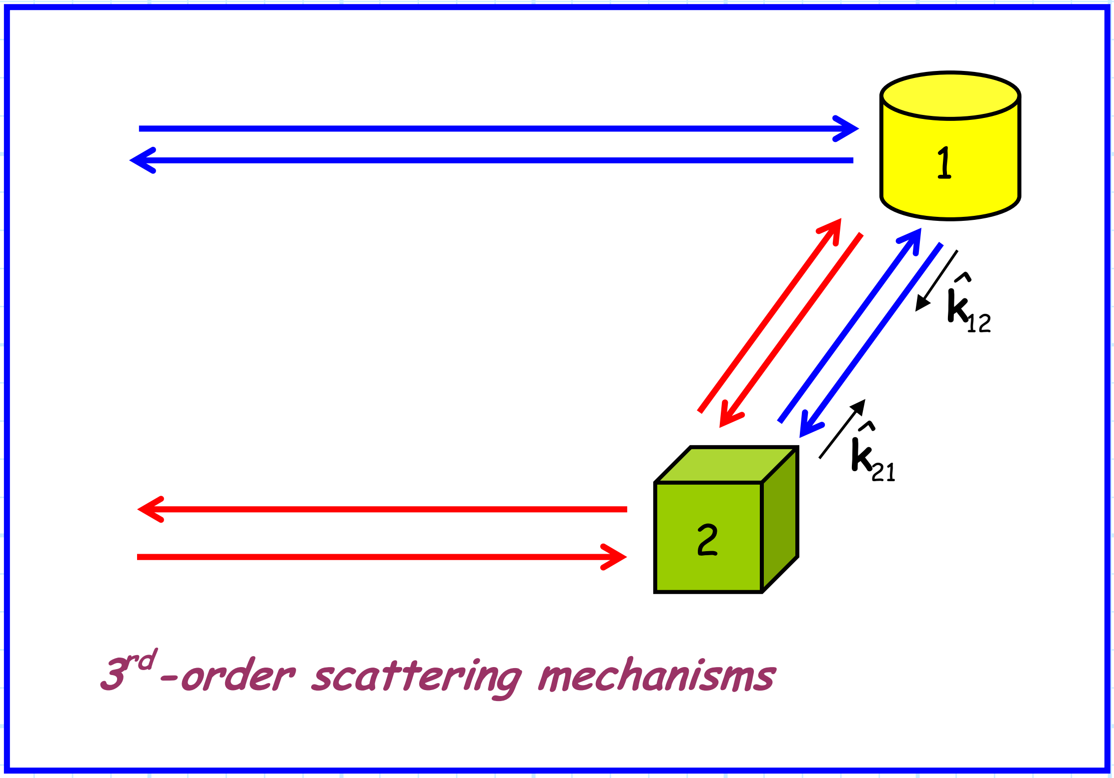
2^{nd}-order scattering mechanisms

$$\mathbf{E}_s(\bar{r}) = \frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r}\left(\mathcal{S}_1(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\,e^{-j(\bar{k}_i-\bar{k}_s)\cdot\bar{r}_1} + \mathcal{S}_2(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\,e^{-j(\bar{k}_i-\bar{k}_s)\cdot\bar{r}_2}\right)\mathbf{E}_0^i$$

$$+\frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r}\left(e^{+j\bar{k}_s\cdot\bar{r}_2}\mathcal{S}_2(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_{12})\frac{e^{-jk_0\hat{\mathbf{k}}_{12}\cdot(\bar{r}_2-\bar{r}_1)}}{|\bar{r}_2-\bar{r}_1|}\mathcal{S}_1(\hat{\mathbf{k}}_{12};\hat{\mathbf{k}}_i)\,e^{-j\bar{k}_i\cdot\bar{r}_1}\right.$$

$$\left.+\,e^{+j\bar{k}_s\cdot\bar{r}_1}\mathcal{S}_1(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_{21})\frac{e^{-jk_0\hat{\mathbf{k}}_{21}\cdot(\bar{r}_2-\bar{r}_1)}}{|\bar{r}_1-\bar{r}_2|}\mathcal{S}_2(\hat{\mathbf{k}}_{21};\hat{\mathbf{k}}_i)\,e^{-j\bar{k}_i\cdot\bar{r}_2}\right)\mathbf{E}_0^i + \cdots$$

But wait—that's not all! The new scattered field from object 1 creates a **third** incident field on object 2, and vice versa.



$$\mathbf{E}_s(\bar{r}) =$$

$$\frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r}\left(\mathcal{S}_1(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\,e^{-j(\bar{k}_i-\bar{k}_s)\cdot\bar{r}_1}+\mathcal{S}_2(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\,e^{-j(\bar{k}_i-\bar{k}_s)\cdot\bar{r}_2}\right)\mathbf{E}_0^i$$

$$+\frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r}\left(e^{+j\bar{k}_s\cdot\bar{r}_2}\mathcal{S}_2(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_{12})\frac{e^{-jk_0\hat{\mathbf{k}}_{12}\cdot(\bar{r}_2-\bar{r}_1)}}{|\bar{r}_2-\bar{r}_1|}\mathcal{S}_1(\hat{\mathbf{k}}_{12};\hat{\mathbf{k}}_i)\,e^{-j\bar{k}_i\cdot\bar{r}_1}\right.$$

$$\left.+e^{+j\bar{k}_s\cdot\bar{r}_1}\mathcal{S}_1(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_{21})\frac{e^{-jk_0\hat{\mathbf{k}}_{21}\cdot(\bar{r}_1-\bar{r}_2)}}{|\bar{r}_1-\bar{r}_2|}\mathcal{S}_2(\hat{\mathbf{k}}_{21};\hat{\mathbf{k}}_i)\,e^{-j\bar{k}_i\cdot\bar{r}_2}\right)\mathbf{E}_0^i$$

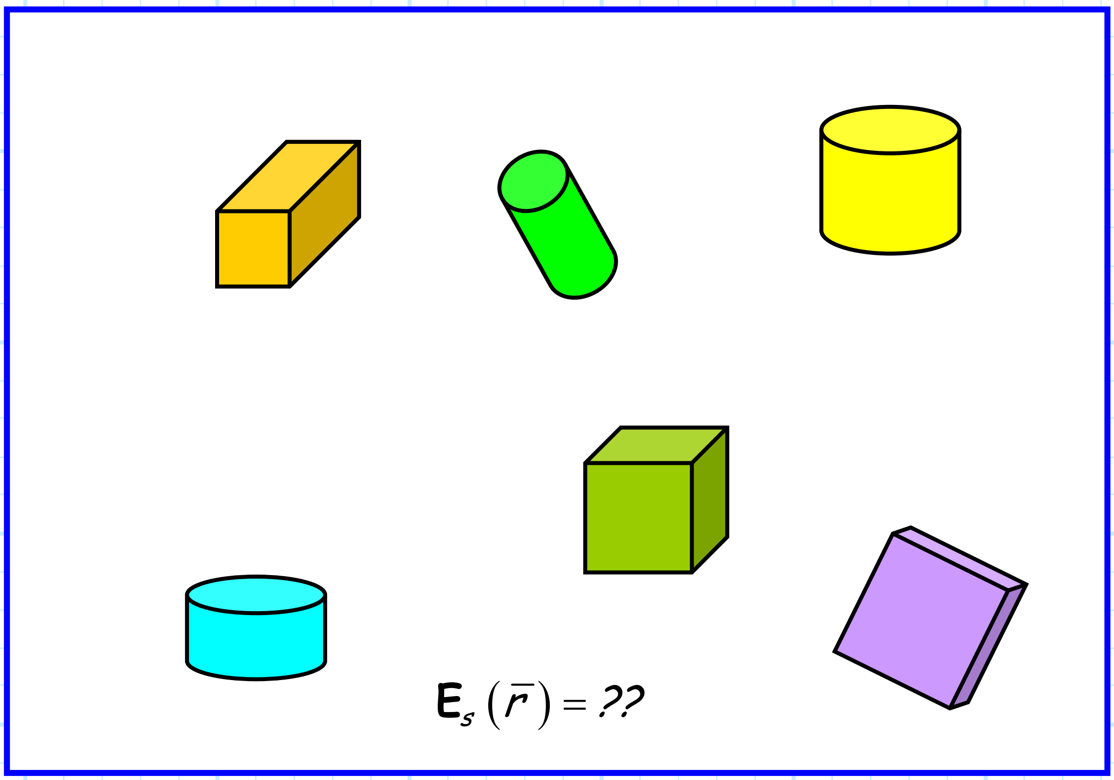
$$+\frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r}\left(e^{+j\bar{k}_s\cdot\bar{r}_1}\mathcal{S}_1(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_{21})\frac{e^{-jk_0\hat{\mathbf{k}}_{21}\cdot(\bar{r}_1-\bar{r}_2)}}{|\bar{r}_1-\bar{r}_2|}\mathcal{S}_2(\hat{\mathbf{k}}_{21};\hat{\mathbf{k}}_{12})\right.$$

$$\frac{e^{-jk_0\hat{\mathbf{k}}_{12}\cdot(\bar{r}_2-\bar{r}_1)}}{|\bar{r}_2-\bar{r}_1|}\mathcal{S}_1(\hat{\mathbf{k}}_{12};\hat{\mathbf{k}}_i)\,e^{-j\bar{k}_i\cdot\bar{r}_1}+e^{+j\bar{k}_s\cdot\bar{r}_2}\mathcal{S}_2(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_{12})\frac{e^{-jk_0\hat{\mathbf{k}}_{12}\cdot(\bar{r}_2-\bar{r}_1)}}{|\bar{r}_2-\bar{r}_1|}$$

$$\left.\mathcal{S}_1(\hat{\mathbf{k}}_{12};\hat{\mathbf{k}}_{21})\frac{e^{-jk_0\hat{\mathbf{k}}_{21}\cdot(\bar{r}_1-\bar{r}_2)}}{|\bar{r}_1-\bar{r}_2|}\mathcal{S}_2(\hat{\mathbf{k}}_{21};\hat{\mathbf{k}}_i)\,e^{-j\bar{k}_i\cdot\bar{r}_2}\right)\mathbf{E}_0^i+\cdots$$

Hopefully, you can see that this analysis can continue **forever**. There are an **infinite** number of scattering mechanisms, and **all** of them are required to provide the precise scattering solution.

Of course, this example included only **two** scatterers—imagine the **mess** we would create trying to determine **all** the scattering mechanisms associated with **multiple** scatterers!



Fortunately, we generally find that each **successive** scattering order (term) is **less significant** than the previous one. Therefore, we eventually find that we can **truncate** this infinite summation (called the **Born Series**), will **little** impact on the solution accuracy—we **don't** have to consider an infinite number of scattering terms!

In fact, we often need to consider only a **few** scattering terms to get acceptable accuracy. The number of required terms depends on several things, but mostly on the **scattering intensity** and **density** of the particles.

If the particles are **lightly** scattering and **sparsely** populated, then we can assume the **total** field at each particle is approximately that of the original **incident** field only.

This approximation is known as the **Born approximation**, and it results in the scattered field being approximated by the **first-order** scattering terms **only**. Thus, for a collection of N scatterers:

$$\begin{aligned} \mathbf{E}_s(\bar{r}) &\approx \sum_{n=1}^{N} \mathbf{E}_s^n(\bar{r}) \\ &= \frac{e^{-j\bar{k}_s\cdot\bar{r}}}{r} \sum_{n=1}^{N} e^{-j\left(\bar{k}_i-\bar{k}_s\right)\cdot\bar{r}_n} \mathcal{S}_n(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i)\,\mathbf{E}_0^i \end{aligned}$$

Q: *So, does this mean that the **scattering cross-section** of this collection of scatterers is **likewise** the summation of the cross-section of each scatterer:*

$$\sigma = \sum_{n=1}^{N} \sigma_n \;\; ???$$

A: NO! This is **definitely** not true. The **scattered power density** from this collection of objects is:

$$\left|\mathbf{W}_s(\hat{\mathbf{k}}_s)\right| = \frac{1}{r^2}\frac{1}{\eta_0}\left|\sum_{n=1}^{N} e^{-j\left(\bar{k}_i-\bar{k}_s\right)\cdot\bar{r}_n}\mathcal{S}_n(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\,\mathbf{E}_0^i\right|^2$$

$$= \frac{1}{r^2}\frac{1}{\eta_0}\sum_{n=1}^{N}\left|\mathcal{S}_n(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\mathbf{E}_0^i\right|^2$$

$$+\frac{2}{r^2}\frac{1}{\eta_0}Re\left\{\sum_{n=1}^{N}\sum_{m=1}^{n-1} e^{-j\left(\bar{k}_i-\bar{k}_s\right)\cdot(\bar{r}_n-\bar{r}_m)}\,\mathbf{E}_0^{iH}\mathcal{S}_n^H(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\mathcal{S}_m(\hat{\mathbf{k}}_s;\hat{\mathbf{k}}_i)\,\mathbf{E}_0^i\right\}$$

$$= \frac{1}{r^2}\frac{1}{\eta_0}\sum_{n=1}^{N}\left|\mathbf{E}_n^s\right|^2$$

$$+\frac{2}{r^2}\frac{1}{\eta_0}Re\left\{\sum_{n=1}^{N}\sum_{m=1}^{n-1} e^{-j\left(\bar{k}_i-\bar{k}_s\right)\cdot(\bar{r}_n-\bar{r}_m)}\,\mathbf{E}_n^{sH}\,\mathbf{E}_m^s\right\}$$

Note that the **first** term represents the sum of power density from each scatterer. However, there is **second** term in this expression! This term is known as the **coherent term**. It is a real value, but it can be **positive** or **negative**.

As a result, the total scattered power density can be much **greater**, or much **less**, than simply the sum of the scattered power from each object. In fact, the total power density can even be **zero**!

The total **scattering cross-section** for this collection of scatterers is therefore:

$$\sigma = \sum_{n=1}^{N}\sigma_n + \frac{2}{\left|\mathbf{E}_0^i\right|^2}Re\left\{\sum_{n=1}^{N}\sum_{m=1}^{n-1} e^{-j\left(\bar{k}_i-\bar{k}_s\right)\cdot(\bar{r}_n-\bar{r}_m)}\,\mathbf{E}_n^{sH}\,\mathbf{E}_m^s\right\}$$

Again, we see that this cross-section can be much **smaller**, or much **larger**, that simply the sum of individual scattering cross-sections.

Scattering from Random Media

* Typically, if we are interested in determining the scattering from a large collection of discrete objects, we describe the characteristics of the collection with **statistical measures**.

* That is, we treat some, most, or even all of the physical descriptors as **random variables**. These random variables can include particle location, orientation, material, size, and shape.

* We can explicitly describe these random variables with a joint **probability density function** (pdf), or more simply in terms of **statistical moments** such as mean, variance, and covariance.

Q: *How can we determine scattering from a collection of scatterers if we don't know **precisely** what the collection is??*

A: We have to describe the scattered field in the same way we describe the collection of scatters—using **statistical measures**! In other words, we must likewise treat the scattered field itself as a **random process** (over 3-dimensions of space).

Typically, we simply describe these random fields in terms of their first two **statistical moments**:

$$\langle \mathbf{E}(\bar{r}) \rangle \doteq \text{the } \textbf{mean} \text{ value of } \mathbf{E}(\bar{r})$$

$$\left\langle \left| \mathbf{E}(\bar{r}) \right|^2 \right\rangle \doteq \text{the } \textbf{variance} \text{ of } \mathbf{E}(\bar{r})$$

Note that the variance has a more **physical** interpretation, as it is proportional to the **average power density** of the wave.

For example, consider a collection of scatterers where the **Born approximation** is applicable. We know that the scattered field is approximately:

$$\begin{aligned}\mathbf{E}_s(\bar{r}) &\approx \sum_{n=1}^{N} \mathbf{E}_s^n(\bar{r}) \\ &= \frac{e^{-jk_0 \hat{k}_s \cdot \bar{r}}}{r} \sum_{n=1}^{N} e^{-jk_0 \left(\hat{k}_i - \hat{k}_s\right)\cdot \bar{r}_n} \mathcal{S}_n(\hat{\mathbf{k}}_s ; \hat{\mathbf{k}}_i)\mathbf{E}_0^i\end{aligned}$$

Assuming the random variables of dissimilar elements are **independent**, the **mean** scattered field is:

$$\begin{aligned}\langle \mathbf{E}_s(\bar{r}) \rangle &\approx \sum_{n=1}^{N} \langle \mathbf{E}_s^n(\bar{r}) \rangle \\ &= \frac{e^{-jk_0 \hat{k}_s \cdot \bar{r}}}{r} \sum_{n=1}^{N} \left\langle e^{-jk_0 \left(\hat{k}_i - \hat{k}_s\right)\cdot \bar{r}_n} \right\rangle \left\langle \mathcal{S}_n(\hat{\mathbf{k}}_s ; \hat{\mathbf{k}}_i) \right\rangle \mathbf{E}_0^i\end{aligned}$$

The value $\left\langle \mathcal{S}_n(\hat{\mathbf{k}}_s ; \hat{\mathbf{k}}_i) \right\rangle$ is the scattering matrix **averaged** across the distribution of particle size, shape, material, etc.

The value $\left\langle e^{-jk_0\left(\hat{k}_i-\hat{k}_s\right)\cdot\bar{r}_n}\right\rangle$ is dependent on the distribution of particle positions $\bar{r}_n$ (here it has been assumed that particle position is likewise **independent** of other parameters such as size and shape).

It turns out, if the particles are distributed throughout some volume that, in each dimension, is greater than a wavelength (i.e., $k_0^3 V \gg 1$), then:

$$\left\langle e^{-jk_0\left(\hat{k}_i-\hat{k}_s\right)\cdot\bar{r}_n}\right\rangle \approx 0$$

and therefore:

$$\boxed{\left\langle \mathbf{E}_s\left(\bar{r}\right)\right\rangle = 0}$$

In other words, the scattered field from a random scattering medium is typically—**on average**—zero. Note this does **not** mean that the scattered field **itself** is typically zero—it almost **never** is!

Note then that the average **total** field in/from a random scattering media is:

$$\begin{aligned}\left\langle \mathbf{E}\left(\bar{r}\right)\right\rangle &= \left\langle \mathbf{E}_i\left(\bar{r}\right)\right\rangle + \left\langle \mathbf{E}_s\left(\bar{r}\right)\right\rangle \\ &= \mathbf{E}_0^i e^{-jk_0\hat{k}_i\cdot\bar{r}} + 0 \\ &= \mathbf{E}_0^i e^{-jk_0\hat{k}_i\cdot\bar{r}}\end{aligned}$$

In other words, the average **total** field is simply equal to the **incident** field. Note that the **incident** field is **not** a random field!

Now let's consider the **variance** of the scattered field:

$$\begin{aligned}
\left\langle \left| \mathbf{E}_s(\bar{r}) \right|^2 \right\rangle &\approx \sum_{n=1}^{N} \sum_{m=1}^{N} \left\langle \mathbf{E}_s^{nH}(\bar{r}) \mathbf{E}_s^{m}(\bar{r}) \right\rangle \\
&= \frac{1}{r^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \left\langle e^{-jk_0\left(\hat{k}_i - \hat{k}_s\right)\cdot(\bar{r}_n - \bar{r}_m)} \right\rangle \left\langle \mathbf{E}_n^{sH} \mathbf{E}_m^{s} \right\rangle \\
&= \frac{1}{r^2} \sum_{n=1}^{N} \left\langle \left| \mathbf{E}_n^{s} \right|^2 \right\rangle \\
&\quad + \frac{1}{r^2} 2\,Re\left\{ \sum_{n=1}^{N} \sum_{m \neq n}^{n-1} \left\langle e^{-jk_0\left(\hat{k}_i - \hat{k}_s\right)\cdot\bar{r}_n} \right\rangle \left\langle e^{+jk_0\left(\hat{k}_i - \hat{k}_s\right)\cdot\bar{r}_m} \right\rangle \left\langle \mathbf{E}_n^{sH} \mathbf{E}_m^{s} \right\rangle \right\}
\end{aligned}$$

Here we have assumed that:

$$\left\langle e^{-jk_0\left(\hat{k}_i - \hat{k}_s\right)\cdot(\bar{r}_n - \bar{r}_m)} \right\rangle = \left\langle e^{-jk_0\left(\hat{k}_i - \hat{k}_s\right)\cdot\bar{r}_n} \right\rangle \left\langle e^{+jk_0\left(\hat{k}_i - \hat{k}_s\right)\cdot\bar{r}_m} \right\rangle \quad \text{if} \quad n \neq m$$

In other words, we have assumed that the locations of dissimilar particles are **independent**.

→ This actually **cannot** be true !

The reason for this is that two particles **cannot** occupy the **same** location. However, for **sparsely** distributed particles, we find that the **independent** assumption is **approximately** true.

At any rate, we have already determined that **if** the scattering volume is sufficiently large, then:

$$\left\langle e^{-jk_0\left(\hat{k}_i-\hat{k}_s\right)\cdot\bar{r}_n}\right\rangle = 0 = \left\langle e^{+jk_0\left(\hat{k}_i-\hat{k}_s\right)\cdot\bar{r}_m}\right\rangle$$

and thus:

$$\begin{aligned}\left\langle\left|\mathbf{E}_s\left(\bar{r}\right)\right|^2\right\rangle &= \frac{1}{r^2}\sum_{n=1}^{N}\left\langle\left|\mathbf{E}_n^s\right|^2\right\rangle \\ &\quad+\frac{1}{r^2}2\,Re\left\{\sum_{n=1}^{N}\sum_{m\neq n}^{n-1}\left\langle e^{-jk_0\left(\hat{k}_i-\hat{k}_s\right)\cdot\bar{r}_n}\right\rangle\left\langle e^{+jk_0\left(\hat{k}_i-\hat{k}_s\right)\cdot\bar{r}_m}\right\rangle\left\langle\mathbf{E}_n^{sH}\mathbf{E}_m^s\right\rangle\right\} \\ &= \frac{1}{r^2}\sum_{n=1}^{N}\left\langle\left|\mathbf{E}_n^s\right|^2\right\rangle\end{aligned}$$

This says that the total average scattered power density **is** simply the sum of the average scattered power from each particle.

This means that the **average** scattering cross-section from this collection of particles is likewise simply the sum of the **average** scattering cross-section of each particle:

$$\langle\sigma\rangle = \sum_{n=1}^{N}\langle\sigma_n\rangle$$

But be careful! These results are only true for a **large** volume of **sparse**, **independent** scatterers.

Rayleigh Fading Statistics

Q: *So we now know the average (i.e., mean) value of the scattering cross-section of a* ***collection*** *of scattering particles. But what about the variance of* σ *? Can we describe* σ *with more statistical* ***specificity****?*

A: As a matter of fact, we can determine (approximately) the entire **probability density function** (*pdf*) of σ!

Consider just one **scalar component** of the scattered field $\mathbf{E}_s(\bar{r})$(say $E_s^v(\bar{r})$):

$$\begin{aligned} E_s^v(\bar{r}) &= \hat{\mathbf{v}}_s \cdot \mathbf{E}(\bar{r}) \\ &= \hat{\mathbf{v}}_s \cdot \sum_{n=1}^{N} \mathbf{E}_s^n(\bar{r}) \\ &= \sum_{n=1}^{N} \hat{\mathbf{v}}_s \cdot \mathbf{E}_s^n(\bar{r}) \\ &= \sum_{n=1}^{N} E_s^{vn}(\bar{r}) \end{aligned}$$

Now, recall that $E_s^n(\bar{r})$ is a **complex** function, that can be expressed in terms of its **real** and **imaginary** parts:

$$\begin{aligned} E_s^v(\bar{r}) &= E_{vs}^r(\bar{r}) + jE_{vs}^i(\bar{r}) \\ &= \sum_{n=1}^{N} E_{vs}^{nr}(\bar{r}) + j\sum_{n=1}^{N} E_{vs}^{ni}(\bar{r}) \end{aligned}$$

In other words the **real** part of the scattered field is the sum of the **real** part of the scattered field from each individual scatterer! Oh, by the way, the **same** is true for the **imaginary** part.

→ Now watch out, here comes the **statistics**!

From the **central limit theorem**, if N (the number of scattering particles) is large, then the pdf*s* of $E_{vs}^{r}(\bar{r})$ and $E_{vs}^{i}(\bar{r})$ are **identical, independent Gaussian** probability density functions!

We have already determined the mean values are **zero** (recall $\langle \mathbf{E}_s(\bar{r}) \rangle = 0$), so the pdf*s* are:

$$p\left(E_s^{vr}\right) = \frac{1}{\sqrt{2\pi\sigma_e^2}} e^{-\left(\left(E_s^{vr}\right)^2 / 2\sigma_e^2\right)}$$

$$p\left(E_s^{vi}\right) = \frac{1}{\sqrt{2\pi\sigma_e^2}} e^{-\left(\left(E_s^{vi}\right)^2 / 2\sigma_e^2\right)}$$

where:

$$\left\langle \left(E_s^{vr}\right)^2 \right\rangle = \left\langle \left(E_s^{vr}\right)^2 \right\rangle = \sigma_e^2$$

Since the two distributions are **independent**, we can write the **joint** distribution as:

$$p\left(E_s^{vr}, E_s^{vi}\right) = p\left(E_s^{vr}\right) p\left(E_s^{vi}\right)$$
$$= \frac{1}{\sqrt{2\pi\sigma_e^2}} e^{-\left(\left(E_s^{vr}\right)^2 + \left(E_s^{vi}\right)^2\right)/2\sigma_e^2}$$

But recall:

$$\left|E_s^{v}\right|^2 = \left(E_s^{vr}\right)^2 + \left(E_s^{vi}\right)^2$$

and thus it can be shown that:

$$p\left(\left|E_s^{v}\right|^2\right) = \frac{1}{2\sigma_e^2} e^{-\left|E_s^{v}\right|^2/2\sigma_e^2}$$

From this pdf, we can determine that:

$$\left\langle \left|E_s^{v}\right|^2 \right\rangle = 2\sigma_e^2$$

so therefore:

$$\sigma_e^2 = \frac{\left\langle \left|E_s^{v}\right|^2 \right\rangle}{2}$$

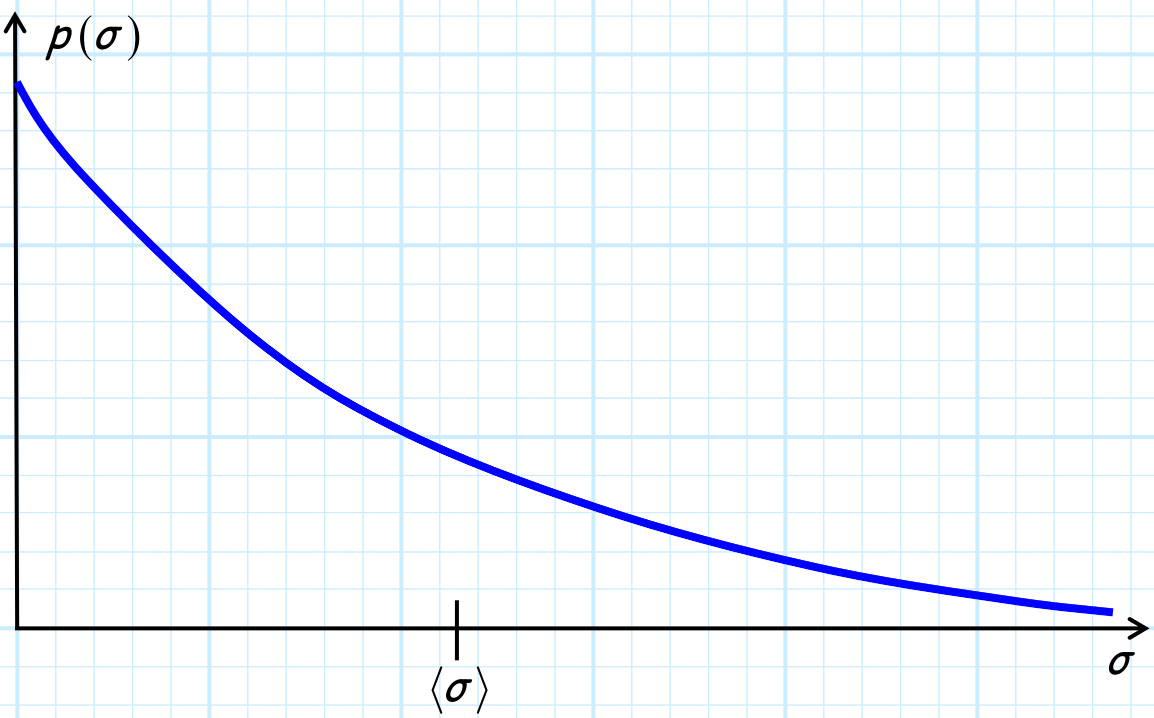
and thus:

$$p\left(\left|E_s^{v}\right|^2\right) = \frac{1}{\left\langle \left|E_s^{v}\right|^2 \right\rangle} e^{-\left|E_s^{v}\right|^2/\left\langle \left|E_s^{v}\right|^2 \right\rangle}$$

Since $\left\langle \left|E_s^{v}\right|^2 \right\rangle \propto \langle \sigma \rangle$, we can conclude that:

$$p(\sigma) = \frac{1}{\langle\sigma\rangle} e^{-\sigma/\langle\sigma\rangle}$$

This distribution is called the **exponential** distribution.



This distribution shows the **wide** variance in possible values of scattering cross-section σ.

Thus, although we might know the **average** value of the scattering cross-section of a random collection of scatterers, the **actual** value of σ is often **very** different than this mean value!

This statistical description is know as **Rayleigh Fading Statistics,** a name derived from the pdf of the value $\sqrt{\sigma}$:

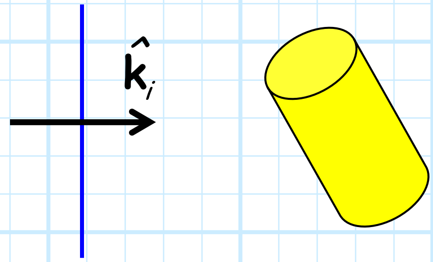
$$p\left(\sqrt{\sigma}\right)=\frac{2\sqrt{\sigma}}{\langle\sigma\rangle}e^{-\sqrt{\sigma}^{2}/\langle\sigma\rangle}$$

A probability density function known as the **Rayleigh** distribution.

The Extinction Cross-Section

Say a lossy particle is illuminated by a plane wave with power density $\mathbf{W}_i(\bar{r})$:

$$\mathbf{W}_i(\bar{r}) = \frac{\left|\mathbf{E}_0^i\right|^2}{\eta_0}\hat{\mathbf{k}}_i$$

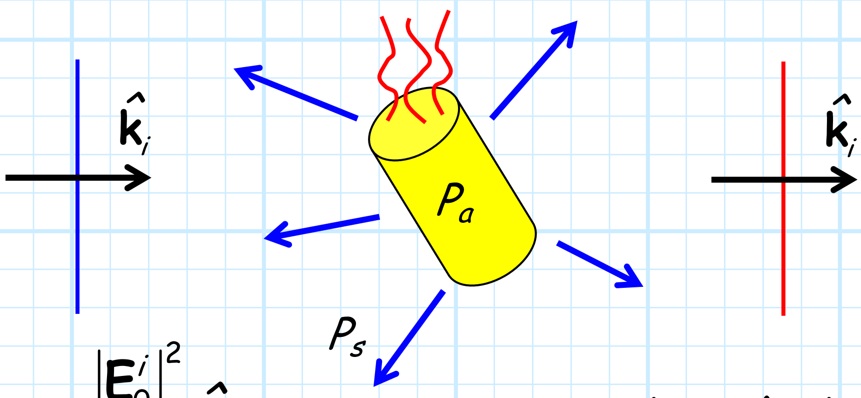


The particle will:

1. **Absorb** energy at a rate P_a Watts.

2. **Scatter** energy (in all directions) at a rate P_s Watts.

Because of conservation of energy, the power density of the incident wave must be diminished after interacting with the particle:



$$\mathbf{W}_i(\bar{r}) = \frac{\left|\mathbf{E}_0^i\right|^2}{\eta_0}\hat{\mathbf{k}}_i \qquad \mathbf{W}_i(\bar{r}+\Delta\bar{r}) < \frac{\left|\mathbf{E}_0^i\right|^2}{\eta_0}\hat{\mathbf{k}}_i$$

We can therefore define the **absorption cross-section** of this particle as:

$$\sigma_a \doteq \frac{P_a}{\left|\mathbf{W}_i(\bar{r})\right|} \qquad \left[m^2\right]$$

and likewise the **total scattering cross-section** as:

$$\sigma_s \doteq \frac{P_s}{\left|\mathbf{W}_i(\bar{r})\right|} \qquad \left[m^2\right]$$

Note that using these definitions we find that:

$$P_a = \sigma_a \left|\mathbf{W}_i(\bar{r})\right| \qquad \text{and} \qquad P_s = \sigma_s \left|\mathbf{W}_i(\bar{r})\right|$$

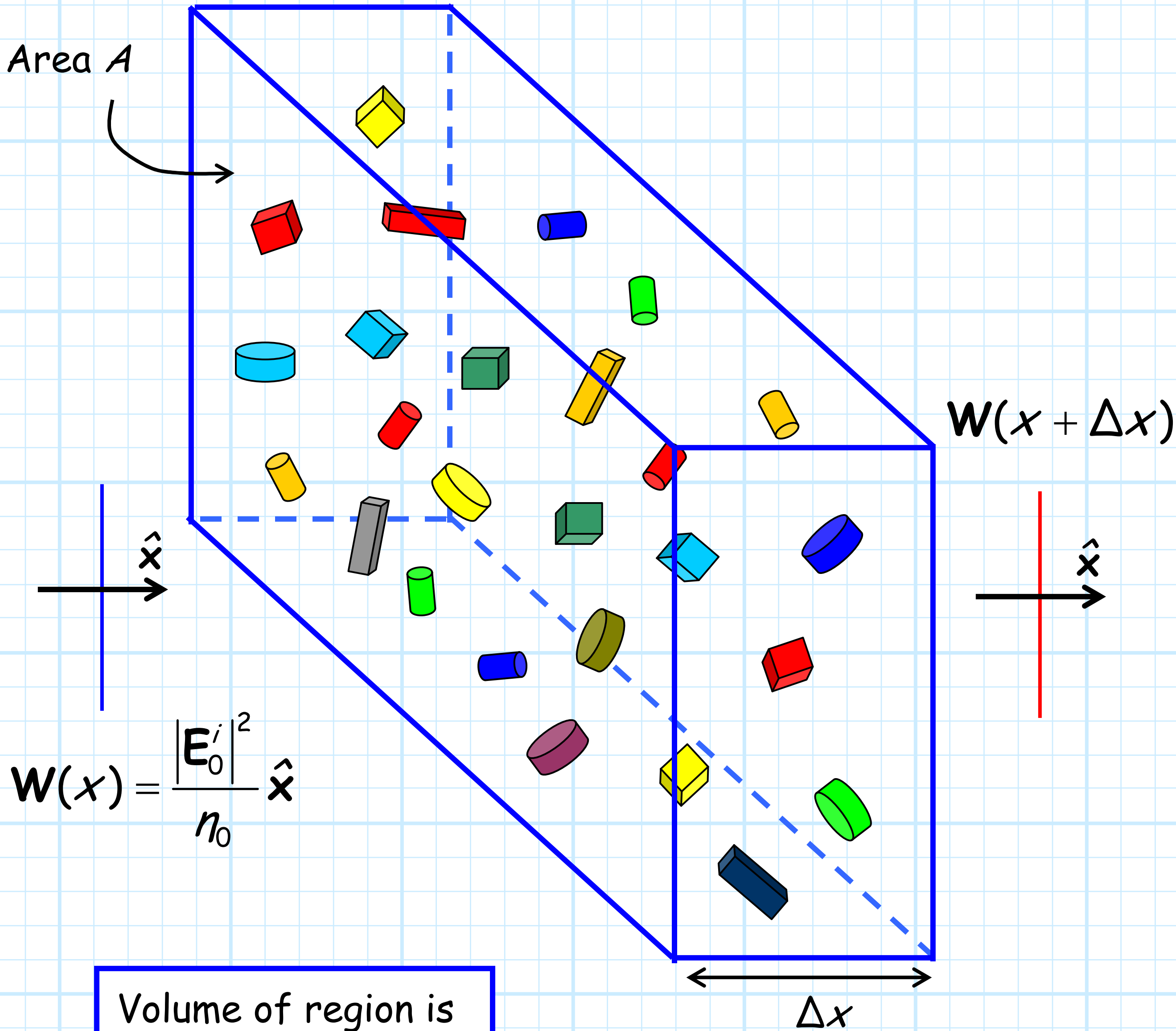
We can likewise define a particle **extinction cross-section** as:

$$\sigma_s \doteq \frac{P_a + P_s}{\left|\mathbf{W}_i(\bar{r})\right|} = \frac{P_a}{\left|\mathbf{W}_i(\bar{r})\right|} + \frac{P_s}{\left|\mathbf{W}_i(\bar{r})\right|} = \sigma_a + \sigma_e \qquad \left[m^2\right]$$

The extinction cross section (as well as σ_a and σ_s), depend on the physical properties (size, shape, material) of the particle.

The Extinction Coefficient

Consider a plane wave that propagates a distance Δx through a thin volume with cross-sectional area A.



$$\mathbf{W}(x) = \frac{\left|\mathbf{E}_0^i\right|^2}{\eta_0}\hat{\mathbf{x}}$$

Volume of region is

$V = A\,\Delta x$

Say that this volume is filled with N particles. The **particle density** n_o is therefore defined as:

$$n_o \doteq \frac{N}{V} = \frac{N}{A\,\Delta x} \qquad \left[\frac{particles}{m^3}\right]$$

Now, we know that energy is flowing into the **front** of this volume at a rate of:

$$P_{in} = \left|\mathbf{W}(x)\right| A \qquad [W]$$

while the power of the plane wave **exiting** the back of the volume is:

$$P_{in} = \left|\mathbf{W}(x+\Delta x)\right| A \qquad [W]$$

Of course, the particles within the volume will **extract power** from this plane wave (due to absorption and scattering). Thus, the power lost due to **extinction** can be written as:

$$\begin{aligned}\Delta P &= P_{out} - P_{in} \\ &= \left|\mathbf{W}(x+\Delta x)\right| A - \left|\mathbf{W}(x)\right| A \\ &= \Delta\left|\mathbf{W}\right| A\end{aligned}$$

Where $\Delta\left|\mathbf{W}\right| \doteq \left|\mathbf{W}(x+\Delta x)\right| - \left|\mathbf{W}(x)\right|$.

Note since $P_{out} < P_{in}$, both ΔP and $\Delta\left|\mathbf{W}\right|$ will be **negative** values (i.e., the power **decreases** as it passes through the volume).

Q: *What happened to the **missing energy**?*

A: The particles within the volume **extract** this energy by absorbing or scattering the incident plane wave. If P_{en} is the rate at which energy is extracted by the ***n*-th** particle, then the **total** rate of energy extinction by the entire collection is:

$$\begin{aligned} P_e &= \sum_{n=1}^{N} P_{en} \\ &= \sum_{n=1}^{N} \sigma_{en} \left| \mathbf{W}(x) \right| \end{aligned}$$

By **conservation of energy**, we can conclude that:

$$P_e = P_{in} - P_{out} = -\Delta P$$

And therefore:

$$\begin{aligned} \Delta P &= -P_e \\ &= -\sum_{n=1}^{N} \sigma_{en} \left| \mathbf{W}(x) \right| \end{aligned}$$

Q: *Yikes! How are we supposed to know σ_e for **each** and **every** one of the N particles?*

A: Look closer at the equation! We don't need to all the values σ_{en}--we simply need to know the **sum** of all the values (i.e., $\sum_{n=1}^{N} \sigma_{en}$)!

To determine this sum, we just need to know the **average** value of the extinction cross-sections ($\bar{\sigma}_e$), defined as:

$$\bar{\sigma}_e \doteq \frac{1}{N}\sum_{n=1}^{N}\sigma_{en}$$

Therefore:

$$\begin{aligned}\Delta P &= -\sum_{n=1}^{N}\sigma_{en}\left|\mathbf{W}(x)\right| \\ &= -\left|\mathbf{W}(x)\right|\sum_{n=1}^{N}\sigma_{en} \\ &= -\left|\mathbf{W}(x)\right|\left(N\,\bar{\sigma}_e\right) \\ &= -N\left|\mathbf{W}(x)\right|\bar{\sigma}_e\end{aligned}$$

Recall, however, that:

$$N = n_o V = n_o A\,\Delta x$$

Therefore:

$$\begin{aligned}\Delta P &= -N\left|\mathbf{W}(x)\right|\bar{\sigma}_e \\ &= -n_o A\left|\mathbf{W}(x)\right|\bar{\sigma}_e\Delta x\end{aligned}$$

and thus:

$$\Delta\left|\mathbf{W}\right| = \frac{\Delta P}{A} = -n_o\left|\mathbf{W}(x)\right|\bar{\sigma}_e\,\Delta x$$

Finally (whew!) we can say:

$$\frac{\Delta\left|\mathbf{W}\right|}{\Delta x} = -n_o\left|\mathbf{W}(x)\right|\bar{\sigma}_e$$

And taking the limit as $\Delta x \rightarrow 0$, we have determined the following **differential equation**:

$$\frac{d\left|\mathbf{W}(x)\right|}{dx} = -\left(n_o \bar{\sigma}_e\right)\left|\mathbf{W}(x)\right|$$

This differential equation is **easily solved**:

$$\left|\mathbf{W}(x)\right| = \left|\mathbf{W}(x=0)\right| e^{-(n_o \bar{\sigma}_e)x}$$

And thus:

$$\mathbf{W}(x) = \left|\mathbf{W}(x=0)\right| e^{-(n_o \bar{\sigma}_e)x}\ \hat{\mathbf{x}}$$

The value $n_o\,\bar{\sigma}_e$ is obviously **very important** and is called the **extinction coefficient** $\boldsymbol{\kappa}_e$ of a random medium

$$\boldsymbol{\kappa}_e \doteq n_o \bar{\sigma}_e$$

Thus, the power density of a plane wave passing through a random collection of particles (with particle density n_o and average extinction cross-section $\bar{\sigma}_e$) is:

$$\mathbf{W}_i(\bar{r}) = \left|\mathbf{W}(\bar{r}=0)\right| e^{-\kappa_e\,(\hat{\mathbf{k}}_i\cdot\bar{r})}\ \hat{\mathbf{k}}_i$$
$$= \frac{\left|\mathbf{E}_0^i\right|^2}{2\eta} e^{-\kappa_e\,(\hat{\mathbf{k}}_i\cdot\bar{r})}\ \hat{\mathbf{k}}_i$$

Thus, the power density of the incident field within a collection of scatterers will diminish **exponentially** with propagation distance!

This exponential behavior will depend entirely on the **extinction coefficient** κ_e, which in turns depends on the **density** and average **extinction cross-section** of the particles.

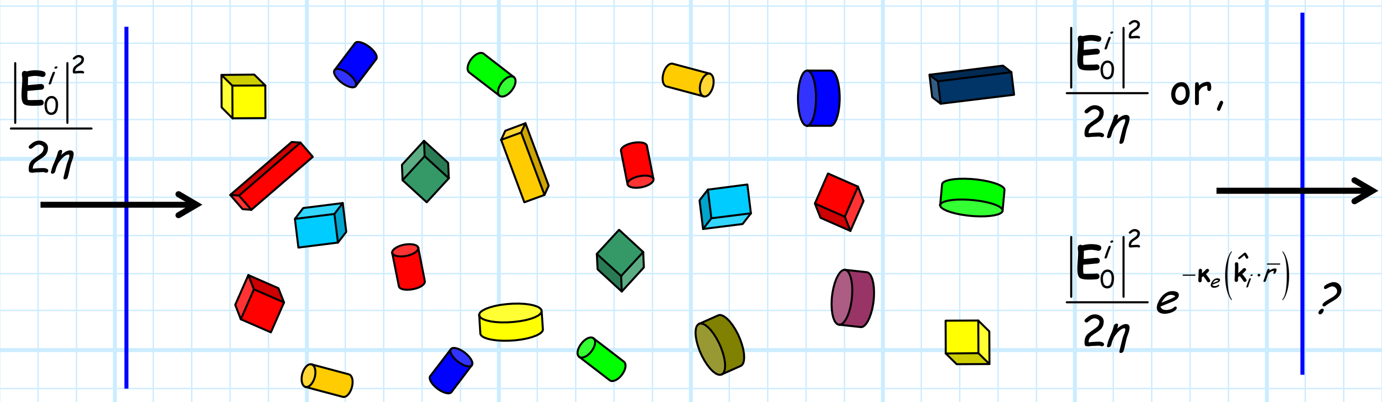
The Optical Theorem

Q: *Now I'm confused! We earlier concluded that the average field in a collection of random scatterers was simply the incident field:*

$$\langle \mathbf{E}(\bar{r}) \rangle = \mathbf{E}_i(\bar{r}) = \mathbf{E}_0^i e^{-j\bar{k}_i \cdot \bar{r}}$$

But this would result in a **constant** power density throughout the scattering volume—in other words **no extinction**!

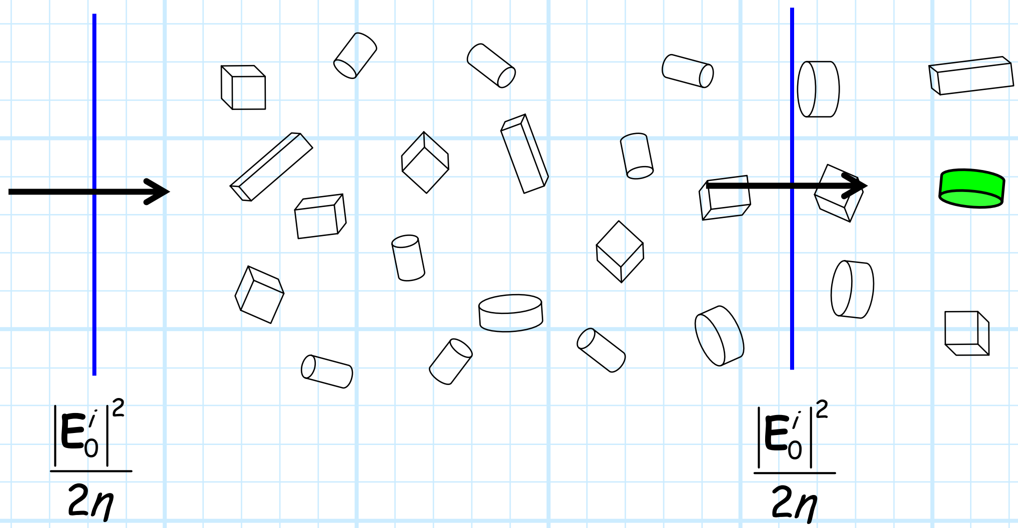
$$\left|\mathbf{W}_i(\bar{r})\right| = \frac{\left|\langle \mathbf{E}(\bar{r}) \rangle\right|^2}{2\eta} = \frac{\left|\mathbf{E}_0^i e^{-j\bar{k}_i \cdot \bar{r}}\right|^2}{2\eta} = \frac{\left|\mathbf{E}_0^i\right|^2}{2\eta} \neq \frac{\left|\mathbf{E}_0^i\right|^2}{2\eta} e^{-\kappa_e\left(\hat{\mathbf{k}}_i \cdot \bar{r}\right)} \quad ???$$



A: Recall we concluded that the coherent (i.e., average) field was equal to the incident field using the results of the **Born Approximation.**

It turns out that the Born Approximation is generally a **bad** approximation when applied to the **propagation** problem !

Recall the Born Approximation considers only **first order** scattering mechanisms. The total solution is simply a superposition of all the individual first-order solutions—and the first-order solutions effectively assume that **no other** particles are present.



But if **no** other particles are present, then extinction does **not** occur!

In other words, **extinction** is a decidedly **multiple-order** scattering effect—the incident wave encounters many particles as it propagates through the medium.

Q: *So just what **is** the coherent wave* $\langle \mathbf{E}(\bar{r}) \rangle$ *?*

A: Using an analysis similar to that which arrived at the extinction coefficient, we can determine that the coherent wave must satisfy these differential equations:

$$\frac{E_v(x)}{dx} = -j\, k_{eff}^v\, E_v(x)$$

$$\frac{E_h(x)}{dx} = -j\, k_{eff}^h\, E_h(x)$$

where k_{eff} is the complex value:

$$k_{eff}^v = k_0 + \frac{2\pi\, n_o}{k_0} \left\langle S_{vv}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)\right\rangle$$

$$k_{eff}^h = k_0 + \frac{2\pi\, n_o}{k_0} \left\langle S_{hh}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)\right\rangle$$

Note that the complex values $S_{vv}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)$ and $S_{hh}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)$ describes the **forward scattering** coefficient (i.e., $\hat{\mathbf{k}}_s = \hat{\mathbf{k}}_i$) of a particle.

The expressions above are due to a result known as the **optical theorem.** Although this result is likewise an approximation—valid only for a sparse collection of independent scatterers—it is much more accurate than the coherent wave solution using the Born approximation.

The optical theorem thus provides the solution:

$$\langle \mathbf{E}(x) \rangle = E_0^v \, e^{-j\, k_{eff}^v x} \, \hat{\mathbf{v}} + E_0^h \, e^{-j\, k_{eff}^h x} \, \hat{\mathbf{h}}$$

or more generally:

$$\langle \mathbf{E}(\bar{r}) \rangle = E_0^v \, e^{-j\, \bar{k}_{eff}^v \cdot \bar{r}} \, \hat{\mathbf{v}} + E_0^h \, e^{-j\, \bar{k}_{eff}^h \cdot \bar{r}} \, \hat{\mathbf{h}}$$

where:

$$\bar{k}_{eff}^v = k_{eff}^v \, \hat{\mathbf{k}}_i \qquad \text{and} \qquad \bar{k}_{eff}^h = k_{eff}^h \, \hat{\mathbf{k}}_i$$

Note that the power density associated with each component of the coherent wave is:

$$\mathbf{W}_v(\bar{r}) = \frac{\left|\langle E_v(\bar{r}) \rangle\right|^2}{2\eta} \hat{\mathbf{k}}_i = \frac{\left|E_0^v\right|^2}{2\eta} e^{-j\, \bar{\mathbf{k}}_i^v \cdot \bar{r}} \, e^{+j\, \bar{\mathbf{k}}_i^{v*} \cdot \bar{r}} \, \hat{\mathbf{k}}_i = \frac{\left|E_0^v\right|^2}{2\eta} e^{-2\, Im\left\{k_{eff}^v\right\} \hat{\mathbf{k}}_i \cdot \bar{r}} \, \hat{\mathbf{k}}_i$$

$$\mathbf{W}_h(\bar{r}) = \frac{\left|\langle E_h(\bar{r}) \rangle\right|^2}{2\eta} \hat{\mathbf{k}}_i = \frac{\left|E_0^h\right|^2}{2\eta} e^{-j\, \bar{\mathbf{k}}_i^h \cdot \bar{r}} \, e^{+j\, \bar{\mathbf{k}}_i^{h*} \cdot \bar{r}} \, \hat{\mathbf{k}}_i = \frac{\left|E_0^h\right|^2}{2\eta} e^{-2\, Im\left\{k_{eff}^h\right\} \hat{\mathbf{k}}_i \cdot \bar{r}} \, \hat{\mathbf{k}}_i$$

where $*$ denotes complex conjugate.

Compare this with the result in terms of the **extinction coefficient**:

$$\mathbf{W}_i(\bar{r}) = \frac{\left|\mathbf{E}_0^i\right|^2}{2\eta} e^{-\kappa_e(\hat{\mathbf{k}}_i \cdot \bar{r})}\, \hat{\mathbf{k}}_i$$

It is evident then that we can use the **optical theorem** to determine the **extinction coefficient** of a collection of scatterers:

$$\kappa_e^v = 2\,Im\left\{k_{eff}^v\right\} \qquad \text{and} \qquad \kappa_e^h = 2\,Im\left\{k_{eff}^h\right\}$$

Additionally, since $\kappa_e = n_o \langle \sigma_e \rangle$, and:

$$\begin{aligned} 2\,Im\left\{k_{eff}^v\right\} &= 2\,Im\left\{\frac{2\pi\, n_o}{k_0}\left\langle S_{vv}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)\right\rangle\right\} \\ &= n_o \frac{4\pi}{k_0} Im\left\{\left\langle S_{vv}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)\right\rangle\right\} \end{aligned}$$

we can conclude that:

$$\left\langle \sigma_e^v \right\rangle = \frac{4\pi}{k_0} Im\left\{\left\langle S_{vv}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)\right\rangle\right\}$$

$$\left\langle \sigma_e^h \right\rangle = \frac{4\pi}{k_0} Im\left\{\left\langle S_{hh}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)\right\rangle\right\}$$

Or, more specifically:

$$\sigma_e^v = \frac{4\pi}{k_0} Im\left\{ S_{vv}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)\right\}$$

$$\sigma_e^h = \frac{4\pi}{k_0} Im\left\{ S_{hh}\left(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_i\right)\right\}$$

Thus, we can use the **optical theorem** to determine the extinction coefficient of a collection of scatterers, as well as the extinction cross-section of an individual particle.

The Distorted Born Approximation

Q: *So, the Born Approximation does not account for extinction within a collection of particles. Must we completely* ***abandon*** *the Born Approximation?*

A: Not necessarily! We can abandon the Born Approximation with respect to **propagation**, but we can still use it with respect to **scattering**.

In other words, we can use the Optical Theorem to account for **extinction** when determining the **mean-field** $\langle \mathbf{E}(\bar{r}) \rangle$, but then only account for first-order scattering terms when determining the **average scattered power** $\left\langle \left| \mathbf{E}_s(\bar{r}) \right|^2 \right\rangle$.

This approach is known as the **Distorted Born Approximation,** and thus the scattered field from the *n*-th particle in the midst of a collection of N particles is specified as:

$$\begin{aligned}\mathbf{E}_s^n(\bar{r}) &= \frac{1}{r} e^{-j\,\bar{k}_s^{eff}\cdot(\bar{r}-\bar{r}_n)} \mathcal{S}_n(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i)\mathbf{E}_0^i e^{-j\bar{k}_i^{eff}\cdot\bar{r}_n} \\ &= \frac{e^{-j\,\bar{k}_s^{eff}\cdot\bar{r}}}{r} e^{-j\left(\bar{k}_i^{eff}-\bar{k}_s^{eff}\right)\cdot\bar{r}_n} \mathbf{E}_n^s\end{aligned}$$

where $\mathbf{E}_n^s \doteq \mathcal{S}_n(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i)\mathbf{E}_0^i$ and we have assumed that $k_{eff}^v = k_{eff}^h$.

Thus, the **mean** scattered field is :

$$
\begin{aligned}
\left\langle \mathbf{E}_s(\bar{r}) \right\rangle &\approx \sum_{n=1}^{N} \left\langle \mathbf{E}_s^n(\bar{r}) \right\rangle \\
&= \frac{e^{-j\,\bar{k}_s^{eff}\cdot\bar{r}}}{r} \sum_{n=1}^{N} \left\langle e^{-j\left(\bar{k}_i^{eff}-\bar{k}_s^{eff}\right)\cdot\bar{r}_n} \right\rangle \left\langle \mathbf{E}_n^s \right\rangle \\
&= 0
\end{aligned}
$$

since again we find that:

$$
\left\langle e^{-j\left(\bar{k}_i^{eff}-\bar{k}_s^{eff}\right)\cdot\bar{r}_n} \right\rangle = 0.
$$

As a result, the **average total field** (i.e., coherent field) is:

$$
\begin{aligned}
\left\langle \mathbf{E}(\bar{r}) \right\rangle &= \left\langle \mathbf{E}_i(\bar{r}) \right\rangle + \left\langle \mathbf{E}_s(\bar{r}) \right\rangle \\
&= \mathbf{E}_0^i e^{-j\,\bar{k}_i^{eff}\cdot\bar{r}} + 0 \\
&= \mathbf{E}_0^i e^{-j\,\bar{k}_i^{eff}\cdot\bar{r}}
\end{aligned}
$$

Likewise, the **average scattered power** is thus:

$$
\begin{aligned}
\left\langle \left|\mathbf{E}_s(\bar{r})\right|^2 \right\rangle &\approx \sum_{n=1}^{N}\sum_{m=1}^{N} \left\langle \mathbf{E}_s^{nH}(\bar{r})\,\mathbf{E}_s^{m}(\bar{r}) \right\rangle \\
&= \frac{e^{-j\left(\bar{k}_s^{eff}-\bar{k}_s^{eff*}\right)\cdot\bar{r}}}{r^2} \sum_{n=1}^{N}\sum_{m=1}^{N} \left\langle e^{-j\left(\bar{k}_i^{eff}-\bar{k}_s^{eff}\right)\cdot\bar{r}_n} e^{+j\left(\bar{k}_i^{eff}-\bar{k}_s^{eff}\right)^*\cdot\bar{r}_m} \right\rangle \left\langle \mathbf{E}_n^{sH}\mathbf{E}_m^{s} \right\rangle \\
&= \frac{e^{-\kappa_e \hat{k}_s\cdot\bar{r}}}{r^2} \sum_{n=1}^{N} \left\langle \left|\mathbf{E}_n^{s}\right|^2 \right\rangle \left\langle e^{-\kappa_e\left(\hat{k}_i-\hat{k}_s\right)\cdot\bar{r}_n} \right\rangle \\
&\quad + \frac{e^{-\kappa_e \hat{k}_s\cdot\bar{r}}}{r^2}\, 2\,Re\left\{ \sum_{n=1}^{N}\sum_{m\neq n}^{n-1} \left\langle e^{-j\left(\bar{k}_i^{eff}-\bar{k}_s^{eff}\right)\cdot\bar{r}_n} \right\rangle \left\langle e^{+j\left(\bar{k}_i^{eff}-\bar{k}_s^{eff}\right)^*\cdot\bar{r}_m} \right\rangle \left\langle \mathbf{E}_n^{sH}\mathbf{E}_m^{s} \right\rangle \right\}
\end{aligned}
$$

But since:

$$
\left\langle e^{-j\left(\bar{k}_i^{eff}-\bar{k}_s^{eff}\right)\cdot\bar{r}_n} \right\rangle = 0
$$

We find that:

$$
\left\langle \left|\mathbf{E}_s(\bar{r})\right|^2 \right\rangle = \frac{e^{-\kappa_e \hat{k}_s\cdot\bar{r}}}{r^2} \sum_{n=1}^{N} \left\langle \left|\mathbf{E}_n^{s}\right|^2 \right\rangle \left\langle e^{-\kappa_e\left(\hat{k}_i-\hat{k}_s\right)\cdot\bar{r}_n} \right\rangle
$$

Note that the value:

$$
\left\langle e^{-\kappa_e\left(\hat{k}_i-\hat{k}_s\right)\cdot\bar{r}_n} \right\rangle
$$

will be **real** and **positive!**

Rayleigh Scattering

Recall that **exact**, precise, solutions to scattering integral equations:

$$\mathbf{E}(\bar{r},t) = \mathbf{E}_i(\bar{r},t) + \iint \overleftrightarrow{G}_e(\bar{r},t;\bar{r}',t') \cdot \mathbf{E}(\bar{\mathbf{r}}',\mathbf{t}')\, dv'\, dt'$$

$$\mathbf{H}(\bar{\mathbf{r}},\mathbf{t}) = \mathbf{H}_i(\bar{r},t) + \iint \overleftrightarrow{G}_m(\bar{r},t;\bar{r}',t') \cdot \mathbf{H}(\bar{r}',t')\, dv'\, dt'$$

are very difficult to find—essentially **impossible** to determine!

In fact, about the **only** perfect solution we have for a **finite** scatterer is for that of a **sphere**.

This solution (i.e., for a sphere), is known as the **Mie scattering solution**. Although a sphere is a very **basic** and simple shape, its scattering solution is neither basic **nor** simple!

In fact, the solution can only be written as a **weighted superposition** of an infinite number of vector basis functions $\boldsymbol{\psi}_n(\bar{r})$:

$$\mathbf{E}_s^{sphere}(\bar{r}) = \sum_{n=0}^{\infty} a_n\, \boldsymbol{\psi}_n(\bar{r})$$

This solution is alternatively referred to as the **Mie series**.

***** The vector basis functions $\boldsymbol{\psi}_n(\bar{r})$ are very complex functions, formed from spherical **Bessel functions** and **Legendre Polynomials** (not important!).

***** The series coefficients a_n are known as **Mie coefficients**, and they depend on: 1) the **incident wave** direction and polarization (i.e., $\hat{\mathbf{k}}_i, \mathbf{E}_0^i$), 2) the **material properties** of the sphere (i.e., σ, ε, μ), and 3) the **electrical radius** of the sphere $k_0 a$.

Q: *Electrical radius? $k_0 a$? What is **that**?*

A: The value a is simply the **radius** of the sphere (e.g., a = 0.5 meters). The value $k_0 a$ is thus:

$$k_0 a = \frac{2\pi}{\lambda_0} a = 2\pi \frac{a}{\lambda_0}$$

The value $k_0 a$ effectively expresses the size (radius) of the sphere with **respect to one wavelength**.

This of course is frequently the important descriptor in electromagnetics—it's not how big something is, it's how big it is **compared to one wavelength**!

More specifically, $k_0 a$ describes the radius in terms of **electric phase**. For example, if $k_0 a = \pi$ *radians*, the radius is of the sphere is $\lambda_0/2$.

Q: *So in order to get the **exact** scattering solution, would we have to add up an **infinite** number of Mie scattering terms?*

A: Yup! To get an **exact** solution, we **must** consider an infinite number of terms. However, like the Born series, we find that at some point (i.e., some value n) the Mie coefficients will become **insignificant** (i.e., nearly zero).

In other words, the Mie series will **converge**, so that we can **truncate** the series and get a very **good** (although not perfect) scattering solution.

Q: *So, how many Mie scattering terms must we consider?*

A: It depends on **two** things: 1) the **material** properties of the sphere, and 2) the **size** $k_0 a$ of the sphere.

A sphere that is **highly scattering** (i.e., a sphere where any of the material properties $\sigma, \varepsilon,$ or μ) requires **more** Mie scattering terms than one that is less scattering.

Likewise, a **large** sphere (i.e., $k_0 a \gg 1$) requires **many more** scattering terms than a **small** sphere (i.e., $k_0 a \ll 1$).

In fact, we can apply another **asymptotic** approximation:

$$\lim_{k_0 a \to 0} \mathbf{E}_s^{sphere}(\bar{r}) = \lim_{k_0 a \to 0} \sum_{n=0}^{\infty} a_n \, \boldsymbol{\psi}_n(\bar{r}) \approx a_0 \, \boldsymbol{\psi}_0(\bar{r})$$

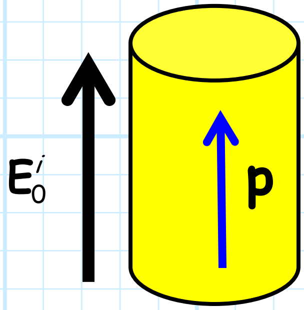
In other words, for **electrically small** spheres (i.e., $k_0 a \ll 1$), the scattered field is approximately equal to the **first** Mie scattering term.

This approximation is known as the **Rayleigh approximation**, and greatly simplifies the scattering solution—not only for electrically small spheres, but for **any** electrically small particle!

In fact, **any** electrically small particle (i.e., **all** particle dimensions are small with respect to a wavelength) is considered to be a **Rayleigh Scatterer**, with its scattering solution specified in terms of a **polarizability tensor**.

The Polarizability Tensor

Essentially, every small particle (i.e., Rayleigh Scatterer) scatterers like a simple **electric dipole**.



In other words, the incident field $\mathbf{E}_0^i$ will **polarize** the particle, creating dipole with moment **p**. The relationship between **p** and $\mathbf{E}_0^i$ is specified with the particles **polarizability tensor** $\overleftrightarrow{\mathcal{P}}$:

$$\mathbf{p} = \varepsilon_0 \overleftrightarrow{\mathcal{P}} \cdot \mathbf{E}_0^i$$

The polarizability tensor of a particle is **completely** dependent on particle properties such as size, shape, and material.

Conversely, it is completely **independent** of incident or scattering direction, or polarization, or even frequency ω_0!

The scattering matrix for an electric dipole with moment **p** is:

$$\mathcal{S} = -\frac{k_0^2}{4\pi\varepsilon_0}\left(\hat{\mathbf{k}}_s \times \hat{\mathbf{k}}_s \times \mathbf{p}\right)$$

$$= -\frac{k_0^2}{4\pi}\left(\hat{\mathbf{k}}_s \times \hat{\mathbf{k}}_s \times \overleftrightarrow{\mathcal{P}} \cdot \mathbf{E}_0^i\right)$$

The elements of the scattering matrix can be even more **simply** stated as:

$$\mathcal{S}_{vv} = \frac{k_0^2}{4\pi}\left(\hat{\mathbf{v}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{v}}_i\right) \qquad \mathcal{S}_{hv} = \frac{k_0^2}{4\pi}\left(\hat{\mathbf{h}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{v}}_i\right)$$

$$\mathcal{S}_{vh} = \frac{k_0^2}{4\pi}\left(\hat{\mathbf{v}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{h}}_i\right) \qquad \mathcal{S}_{hh} = \frac{k_0^2}{4\pi}\left(\hat{\mathbf{h}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{h}}_i\right)$$

Note that the scattering matrix elements **are** dependent on incident and scattering directions $\hat{\mathbf{k}}_i$ and $\hat{\mathbf{k}}_s$, but **only** because they affect the directions of polarization vectors $\hat{\mathbf{h}}_i$, $\hat{\mathbf{v}}_s$, etc.

We now can conclude **many** things about Rayleigh scatters, based on our previous analysis and discussions.

For example, the **scattering cross-section** of a Rayleigh particle is:

$$\sigma_{vv}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) = 4\pi \left| S_{vv}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) \right|^2 = k_0^2 \left| \hat{\mathbf{v}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{v}}_i \right|^2$$

$$\sigma_{vh}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) = 4\pi \left| S_{vh}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) \right|^2 = k_0^2 \left| \hat{\mathbf{v}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{h}}_i \right|^2$$

$$\sigma_{hv}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) = 4\pi \left| S_{hv}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) \right|^2 = k_0^2 \left| \hat{\mathbf{h}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{v}}_i \right|^2$$

$$\sigma_{hh}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) = 4\pi \left| S_{hh}(\hat{\mathbf{k}}_s,\hat{\mathbf{k}}_i) \right|^2 = k_0^2 \left| \hat{\mathbf{h}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{h}}_i \right|^2$$

While the **extinction** cross-section is approximately:

$$\sigma_e^v = \frac{4\pi}{k_0} Im\left\{ S_{vv}\left(\hat{\mathbf{k}}_i,\hat{\mathbf{k}}_i\right)\right\} = k_0\, Im\left\{ \hat{\mathbf{v}}_i \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{v}}_i \right\}$$

$$\sigma_e^h = \frac{4\pi}{k_0} Im\left\{ S_{hh}\left(\hat{\mathbf{k}}_i,\hat{\mathbf{k}}_i\right)\right\} = k_0\, Im\left\{ \hat{\mathbf{h}}_i \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{h}}_i \right\}$$

And thus the **effective propagation constant** in a random collection of Rayleigh Scatterers is:

$$k_{eff}^v = k_0 + \frac{2\pi\, n_o}{k_0} \left\langle S_{vv}\left(\hat{\mathbf{k}}_i,\hat{\mathbf{k}}_i\right)\right\rangle = k_0 + \frac{n_o k_0}{2}\, \hat{\mathbf{v}}_i \cdot \left\langle \overleftrightarrow{\mathcal{P}} \right\rangle \cdot \hat{\mathbf{v}}_i$$

$$k_{eff}^h = k_0 + \frac{2\pi\, n_o}{k_0} \left\langle S_{hh}\left(\hat{\mathbf{k}}_i,\hat{\mathbf{k}}_i\right)\right\rangle = k_0 + \frac{n_o k_0}{2}\, \hat{\mathbf{h}}_i \cdot \left\langle \overleftrightarrow{\mathcal{P}} \right\rangle \cdot \hat{\mathbf{h}}_i$$

Q: *OK, I see that the polarization tensor is **all** we need to describe Rayleigh scattering, but what **is** this polarization tensor?*

A: For the simplest Rayleigh Scatterer—a dielectric **sphere**—we find that the polarizability tensor is:

$$\overleftrightarrow{\mathcal{P}} = v_o \, 3 \, \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \, \overleftrightarrow{\mathcal{I}}$$

where v_o is the **volume** of the sphere ($v_o = 4\pi a^3/3$), ε_r is the **relative dielectric** of the sphere, and $\overleftrightarrow{\mathcal{I}}$ is the **identity** tensor.

The value P, defined as:

$$P \doteq 3 \, \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

is the **normalized** polarizability of a sphere (i.e., the polarizability per unit volume).

Therefore, we can conclude that:

$$\sigma_{vv}(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i) = k_0^2 \left| \hat{\mathbf{v}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{v}}_i \right|^2 = k_0^2 \, v_o \left| P \right|^2 \left| \hat{\mathbf{v}}_s \cdot \mathbf{v}_i \right|^2$$

$$\sigma_{vh}(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i) = k_0^2 \left| \hat{\mathbf{v}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{h}}_i \right|^2 = k_0^2 \, v_o \left| P \right|^2 \left| \hat{\mathbf{v}}_s \cdot \hat{\mathbf{h}}_i \right|^2$$

$$\sigma_{hv}(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i) = k_0^2 \left| \hat{\mathbf{h}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{v}}_i \right|^2 = k_0^2 \, v_o \left| P \right|^2 \left| \hat{\mathbf{h}}_s \cdot \mathbf{v}_i \right|^2$$

$$\sigma_{hh}(\hat{\mathbf{k}}_s, \hat{\mathbf{k}}_i) = k_0^2 \left| \hat{\mathbf{h}}_s \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{h}}_i \right|^2 = k_0^2 \, v_o \left| P \right|^2 \left| \hat{\mathbf{h}}_s \cdot \hat{\mathbf{h}}_i \right|^2$$

and that:

$$\sigma_e^v = k_0\, Im\left\{\hat{\mathbf{v}}_i \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{v}}_i\right\} = k_0\, v_o\, Im\{P\} = (k_0 a)^3 \frac{\Lambda_0^2}{3\pi} Im\{P\}$$

$$\sigma_e^h = k_0\, Im\left\{\hat{\mathbf{h}}_i \cdot \overleftrightarrow{\mathcal{P}} \cdot \hat{\mathbf{h}}_i\right\} = k_0\, v_o\, Im\{P\} = (k_0 a)^3 \frac{\Lambda_0^2}{3\pi} Im\{P\}$$

Note that $\sigma_e^v = \sigma_e^v$, and that the extinction coefficient increases **proportionally** with $(k_0 a)^3$.

We find that the propagation constant of a collection of **spherical** scatterers is therefore:

$$k_{eff}^v = k_0 + \frac{n_o k_0}{2}\langle v_o P\rangle = k_0\left(1 + \frac{f_v}{2}\langle P\rangle\right)$$

$$k_{eff}^h = k_0 + \frac{n_o k_0}{2}\langle v_o P\rangle = k_0\left(1 + \frac{f_v}{2}\langle P\rangle\right)$$

where f_v is the **fractional volume**, defined as:

$$f_v = n_o \langle v_o \rangle$$

Note that this value is approximately the **fraction** of the total volume that is **filled** with spheres. Typically, a scattering medium is considered to be sparse if $f_v \leq 0.1$ (i.e., less than 10 % filled).

Q: *OK, so we now know the scattering solution for* ***spherical*** *Rayleigh scatters, but what about non-spherical particles?*

A: We find that the polarizability tensor for particles that are **roughly** spherical (e.g., an ice crystal) is typically very **close** to that of a sphere. As a result, we often use these spherical Rayleigh scattering solutions to **approximate** the scattering and extinction of non-spherical Rayleigh scatterers!

Mixing Formula

The propagation constant for a wave propagating in a **homogeneous** dielectric medium ε is:

$$\mathbf{k} = \frac{\omega}{v_p}\hat{\mathbf{k}}_i = \omega\sqrt{\varepsilon\mu_0}\,\hat{\mathbf{k}}_i$$

Thus, for a random medium with **effective** propagation constant $\mathbf{k}_{eff}$, we can define an **effective permittivity** ε_{eff}:

$$\mathbf{k}_{eff} = \omega\sqrt{\varepsilon_{eff}\mu_0}\,\hat{\mathbf{k}}_i$$

For a random collection of **spherical Rayleigh scatterers**, with volume fraction f_v, it can be shown that this **effective permittivity** is approximately:

$$\varepsilon_{eff} = \varepsilon\left[\frac{1+2f_v(\varepsilon_s-\varepsilon)/(\varepsilon_s+2\varepsilon)}{1-f_v(\varepsilon_s-\varepsilon)/(\varepsilon_s+2\varepsilon)}\right]$$

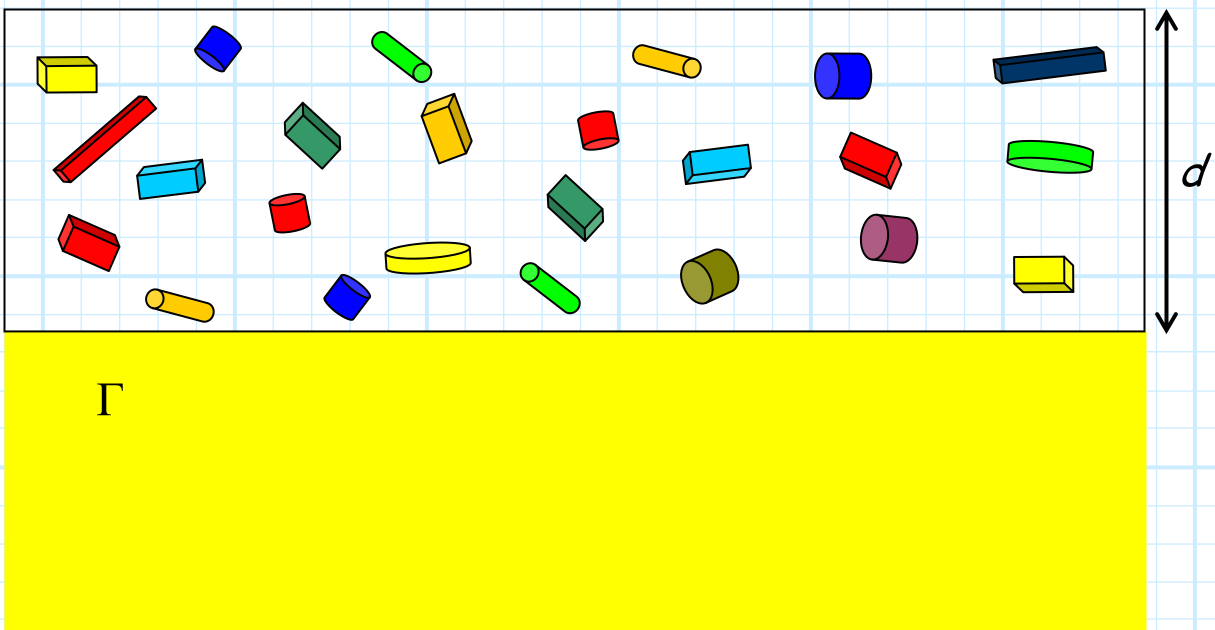
where ε_s is the dielectric constant of the **spherical scatterers**, and ε is the dielectric of the **background material** (i.e., the material that the spherical particles) are embedded in).

The previous equation is an example of a **dielectric mixing formula**. This particular formula is valid when the Rayleigh scatterers are sparse (i.e., $f_v < 0.1$) and the Rayleigh particles are very small $ka \ll 1$.

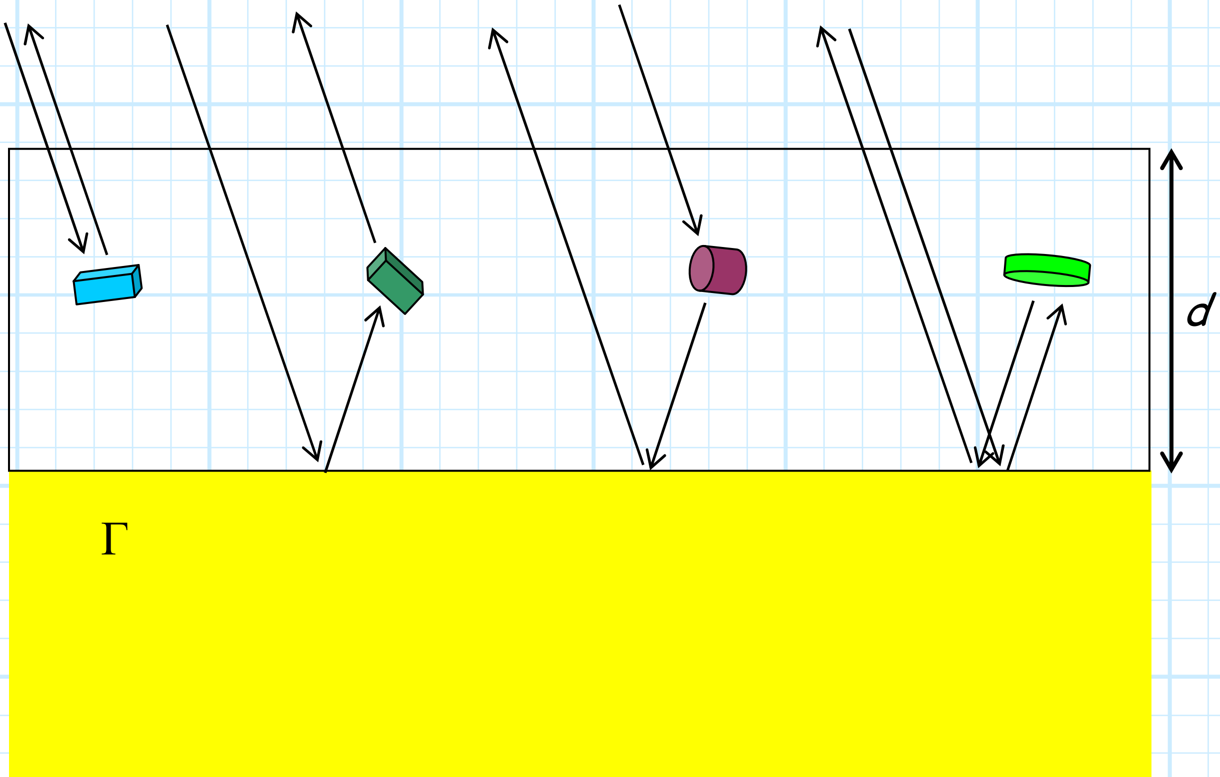
There are other mixing formula solutions, which are valid for more **densely** packed and/or **larger** scatterers.

Volume Scattering from a Layer of Scatterers

Consider now the scattering from a **layer** of Rayleigh Scatterers above a dielectric half space.



There are actually **four** first-order scattering mechanisms associated with the total scattering from this layer.



We can use the **optical theorem** or **mixing formula** to determine the effective propagation constant of the **mean** field in the scattering layer.

We can then use the **Distorted Born approximation** to determine the **average scattered power** from the random scatterers, carefully considering **each** of the **four** first-order scattering mechanisms.

Typically, we seek to find the **backscattering coefficient** σ_0 of the random media, which is the backscattering cross-section of **one square meter** of the layer.