<u>**Conventional Beamforming (for uniform linear arrays)**</u>

Consider an antenna array comprised of N identical antenna elements that are in a linear configuration (see Figure 1 below) with the elements indexed from 0 to $N-1$. The elements have equal spacing d.

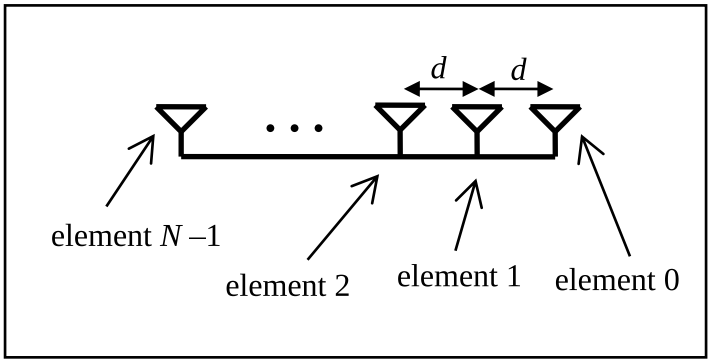

Figure 1. Uniform Linear Array Antenna

As shown in Figure 2, we will establish the geometry such that the *spatial angle* ϕ for a received plane wave is defined as 0° for the direction normal to the array. This is also known as the array boresight. Spatial receive directions for $\phi = -90°$ and $\phi = +90°$ are also shown. Note that the direction of element numbering and the definition of angles can vary depending on context (e.g. electromagnetics community puts 0° at one end). Just remember to be consistent.

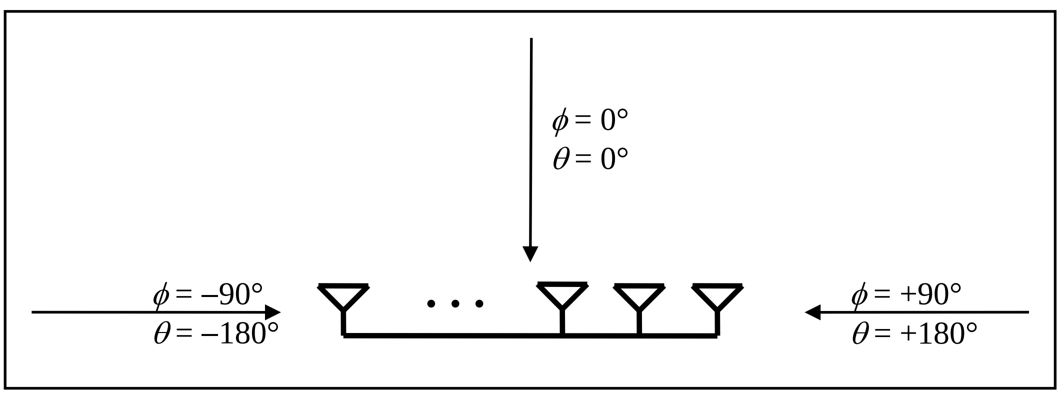

Figure 2. Receive Signal Geometry for Uniform Linear Array
(half-wavelength element spacing)

We shall assume that, for an incident signal, the difference of arrival time between the elements at the <u>ends</u> of the array (the two farthest elements) is much less than the reciprocal of the signal's bandwidth. This condition is the *narrowband array assumption*. By making this assumption, we can treat the signal received at any antenna element as simply a phase shift of the signal received on any other element (<u>at the same time instant</u>). Therefore, we can define an *electrical angle* θ that characterizes this phase shift between adjacent antenna elements. For a uniform linear array (which implies equal element spacing), the relationship between spatial angle ϕ and electrical angle θ is thus

$$\theta = \frac{2\pi d}{\lambda} \sin \phi \tag{1}$$

where λ is the wavelength of the incident signal. Figure 2 depicts the relationship between spatial angle ϕ and electrical angle θ when $d = \lambda / 2$ (known as half-wavelength spacing). Note that a spatial equivalent to Nyquist sampling exists such that $d \le \lambda / 2$ yields no aliasing. When there is spatial aliasing, we will see *grating lobes* (extra mainlobes) in different, yet predictable, spatial directions.

In Fig. 3 we see a single plane-wave being received by the array (the arrows are parallel and point in the direction of propagation for the plane-wave). The dashed blue lines perpendicular to the direction of propagation are "lines of constant phase", meaning that along that *phase front* is the same temporal point in the signal. Therefore, at the fixed points in space where the antenna elements lie, the signal will sweep past. However, because the signal is coming in at an angle different from boresight (at 0°), a given phase front will encounter the antenna elements at different times, with a fixed amount of delay between the arrival at each element.

Specifically, the plane wave is incident from the particular spatial angle ϕ_o, which corresponds to electrical angle θ_o. Relative to array element 0 (the *reference element*), each subsequent antenna element encounters a phase shift depending on how far away it is from the reference element due to the time delay involved with traveling the additional distance.

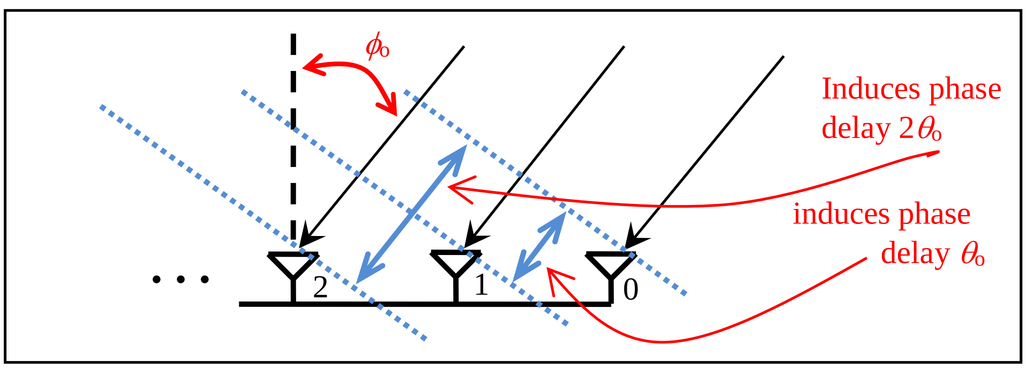

Figure 3. Phase delays introduced by path length differences

In general, for a received signal corresponding to some electrical angle θ, the received (and time sampled) signal across the array at discrete-time index n can be expressed as the $N \times 1$ vector

$$\mathbf{x}(n) = \left[x(n) \quad x(n)e^{-j\theta} \quad x(n)e^{-j2\theta} \quad \cdots \quad x(n)e^{-j(N-1)\theta} \right]^T \tag{2}$$

for $(\bullet)^T$ the vector transpose operation. Noting that the $x(n)$ appears in every term of (2) due to our previous narrowband array assumption, (2) can be re-expressed as

$$\mathbf{x}(n) = x(n)\,\mathbf{s}_\theta = x(n)\left[1 \;\; e^{-j\theta} \quad e^{-j2\theta} \;\; \cdots \;\; e^{-j(N-1)\theta}\right]^T, \tag{3}$$

where $\mathbf{s}_\theta = [1 \;\; e^{-j\theta} \quad e^{-j2\theta} \;\; \cdots \;\; e^{-j(N-1)\theta}]^T$ is the ***spatial steering vector***.

Just like the filtering of a time-domain signal, we can also filter spatial signals. Defining a length-*N* spatial filter as the $N \times 1$ vector

$$\mathbf{w} = \left[w_0 \;\; w_1 \;\; w_2 \;\; \cdots \;\; w_{N-1}\right]^T, \tag{4}$$

the output of the spatial filter at discrete-time index *n* is

$$\begin{aligned} y(n) &= \mathbf{w}^H \mathbf{x}(n) \\ &= \sum_{k=0}^{N-1} w_k^* \left[x(n)\, e^{-jk\theta} \right] \\ &= x(n) \sum_{k=0}^{N-1} w_k^* e^{-jk\theta} \quad \text{\{spatial Fourier transform!\}} \\ &= x(n)\, \mathbf{w}^H \mathbf{s}_\theta \end{aligned} \tag{5}$$

where $(\bullet)^H$ is the <u>complex-conjugate transpose</u> (or Hermitian) operation. We also see that the third line in (5) is just a DTFT, albeit using electrical angle θ instead of digital frequency ω. There are numerous approaches to designing a spatial filter $\mathbf{w}$, but a simple approach can be borrowed directly from FIR time-domain filters.

<u>How to steer a beam</u>

It is often the desired to "steer" the receive beam such that it maximally collects a desired incident signal. If the *direction-of-arrival* (DOA) of a signal is known, then it's electrical angle θ can be easily determined using (1). Hence, the filter that <u>maximizes the receive signal-to-noise ratio</u> (also known as the spatial *matched filter*) is simply

$$\mathbf{w} = \mathbf{s}_\theta \tag{6}$$

and applied in the same manner as (5). However, this filter produces *spatial sidelobes* that follow a sinc-like shape as a function of θ, which tend to be rather high (the first and largest sidelobe is only 13 dB down from the mainlobe). High spatial sidelobes mean that interference from other spatial angles can more easily corrupt the desired received signal.

How to construct a (modestly) better beamformer

Alternatively, if we express one of the window functions (*e.g.* Hamming, Hanning, Blackman, etc) as a vector denoted as **t**, then the filter can be modified as

$$\mathbf{w} = \mathbf{t} \odot \mathbf{s}_\theta \tag{7}$$

where $\odot$ is the *Hadamard product* (term-by-term multiplication of the elements in the respective vectors). Thus the spatial filter in (7) will yield lower spatial sidelobes, though this comes at the cost of a wider *mainbeam* (and thus degraded *spatial resolution*) and some SNR loss. In the array processing nomenclature, the window **t** is called a *taper*.

For a given spatial filter $\mathbf{w}$, the *beampattern* can be determined by plotting $|\,\mathbf{w}^H\,\mathbf{s}_\theta\,|$ for a discretized grid on $-\pi \le \theta \le \pi$. Figures 4 and 5 provide a depiction of the beampattern (in absolute and dB, respectively) for a spatial filter employing a rectangular taper (i.e. no taper) when the mainbeam is pointed at spatial angle $\phi = 0°$ ($\theta = 0°$) and $N = 10$.

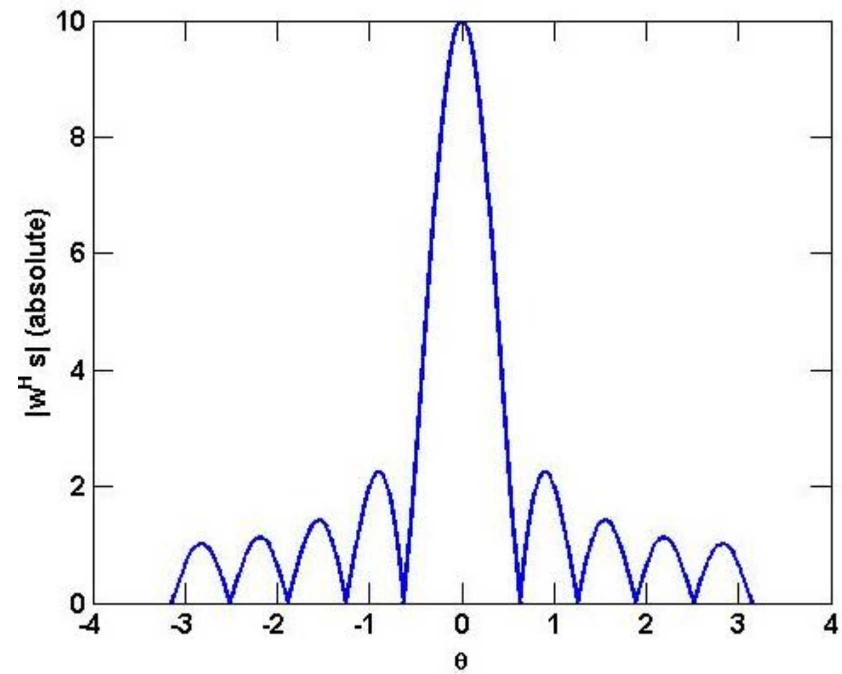

Figure 4. Absolute beampattern for rectangular taper steered to $\phi = 0°$ ($\theta = 0°$) with $N = 10$ elements (angle plotted in radians)

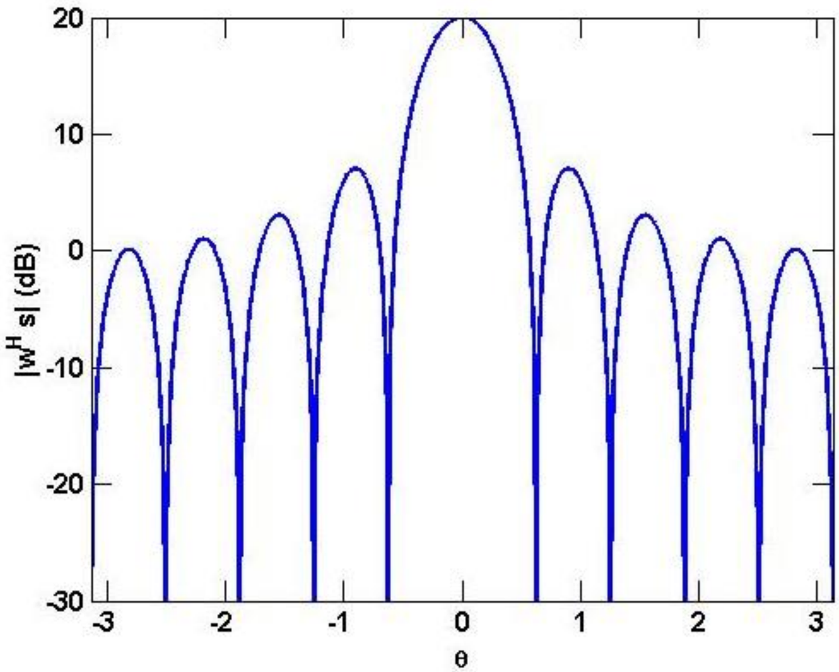

Figure 5. Beampattern (in dB, which is $20\log_{10}|\mathbf{w}^H\mathbf{s}_\theta|$) for rectangular taper steered to $\phi = 0°$ ($\theta = 0°$) with $N = 10$ elements (angle plotted in radians)