Course Notes 10 – The Discrete Fourier Transform

10.0 Introduction

In this chapter we will derive and study the Discrete Fourier Transform. The DFT is a rather curious mathematical entity, in that it does not exist in the matrix of Fourier transformations described below:

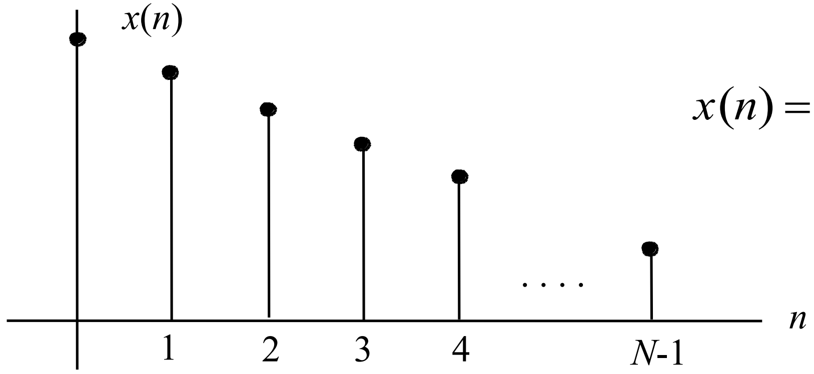
temporal characteristics

	discrete	continuous
periodic	discrete & periodic Discrete Fourier Series (DFS)	continuous & periodic Fourier Series
non-periodic	discrete & non-periodic DTFT	continuous & non-periodic Fourier Transform

For the DFT we would like to Fourier transform a **discrete and non-periodic time-domain** signal into a **discrete and periodic (on 2π) frequency-domain signal**.

10.1 Representation of Periodic Sequences – The DFS

Let's begin by considering $x(n)$, a finite sequence such as



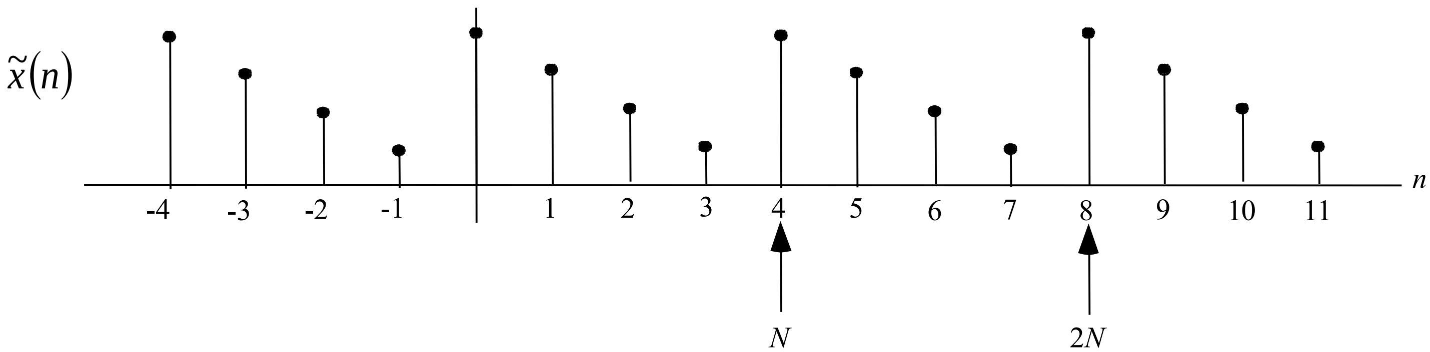
$$x(n) = 0 \quad \text{for } n < 0 \text{ and } n \geq N$$

and we know that $\mathrm{DTFT}\{x(n)\} = X(\omega)$ is a continuous function of ω

Next, let's define another sequence, which we will call $\tilde{x}(n)$, to be periodic with period N, so that it satisfies the property:

$$\tilde{x}(n) = x(n \pm kN) \quad \forall\, n \text{ (where } k \text{ is an integer)}$$

for example: with $N = 4$



Note that $\tilde{x}(n)$ does <u>not</u> have a Fourier Transform (the sequence is <u>not</u> absolutely summable), or a z-transform (the sequence does <u>not</u> converge). However, it <u>does</u> have a Fourier Series representation:

$$\tilde{x}(n) = \sum_{k=-\infty}^{\infty} c_k e^{jn\omega_k}$$

There are some important observations regarding this representation:

1. If $\tilde{x}(n)$ is periodic in N, then $e^{jn\omega_k}$ is also periodic in N. That is:

$$e^{jn\omega_k} = e^{j(n+N)\omega_k} = e^{jn\omega_k} e^{jN\omega_k}$$

require this = 1 to satisfy periodicity (i.e., $N\omega_k = 2\pi k$)

Therefore $\omega_k = \frac{2\pi k}{N}$, with k an integer

2. There are only N distinct exponentials (i.e., 0, 1, ..., N–1) of the form:

$$e_k(n) = e^{j\frac{2\pi nk}{N}} \qquad \text{for} \;\; k = 0, 1, \ldots, N-1$$

such that,

$$e_0(n) = e^{j\frac{2\pi n(0)}{N}} \Rightarrow \quad e_N(n) = e^{j\frac{2\pi n(N)}{N}} = 1$$

$$e_1(n) = e^{j\frac{2\pi n(1)}{N}} \Rightarrow \quad e_{N+1}(n) = e^{j\frac{2\pi n(N+1)}{N}} = e^{j\frac{2\pi n}{N}}$$

$$e_2(n) = e^{j\frac{2\pi n(2)}{N}} \Rightarrow \quad e_{N+2}(n) = e^{j\frac{2\pi n(N+2)}{N}} = e^{j\frac{4\pi n}{N}}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$e_{N-1}(n) = e^{j\frac{2\pi n(N-1)}{N}} \Rightarrow \quad e_{2N-1}(n) = e^{j\frac{2\pi n(2N-1)}{N}} = e^{-j\frac{2\pi n}{N}}$$

first period

second period

Therefore,

$$\tilde{x}(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi kn}{N}}$$

so the discrete Fourier series (DFS) needs only N complex exponential terms, rather than the infinite number required in the continuous case.

We can recognize the dual relation between the time and frequency domains by allowing:

$$c_k \Rightarrow \frac{1}{N}\tilde{X}(k)$$

Therefore, the DFS is

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi kn}{N}}$$

Next, lets derive an inversion formula - that is, obtain $\tilde{X}(k)$ given $\tilde{x}(n)$

Use the following results:

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi kn}{N}} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{Otherwise} \end{cases}$$

Also,
$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & a=1 \\ \frac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$
from geometric sum formula

Perform the following:

$$\sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi nr}{N}} = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi kn}{N}} \right] e^{-j\frac{2\pi nr}{N}}$$

$$= \sum_{k=0}^{N-1} \tilde{X}(k) \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi n(k-r)}{N}} \right]$$

Consider 2 cases:

1. $k = r \longrightarrow \sum_{n=0}^{N-1} e^{j\frac{2\pi n(k-r)}{N}} = N$

2. $k \neq r \longrightarrow \sum_{n=0}^{N-1} e^{j\frac{2\pi n(k-r)}{N}} = \dfrac{1-e^{j\frac{2\pi(k-r)}{N}\cdot N}}{1-e^{j\frac{2\pi(k-r)}{N}}}$

Note that $k - r =$ integer in all cases so that in the numerator

$$1 - e^{j2\pi(\text{integer})} = 0$$

So these cases simply to the expression

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi n(k-r)}{N}} = \begin{cases} N, & k = r \\ 0, & k \neq r \end{cases}$$

Therefore,

$$\widetilde{X}(k) = \sum_{n=0}^{N-1} \widetilde{x}(n) e^{-j\frac{2\pi nk}{N}}$$

Fourier series coefficients

for convenience we shall define:

$$W_N = e^{-j\frac{2\pi}{N}}$$

So that the Fourier Series pair can be written as:

$$\boxed{\begin{aligned} \tilde{x}(n) &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)W_N^{-nk}, \qquad \forall\, n \\ \tilde{X}(k) &= \sum_{n=0}^{N-1}\tilde{x}(n)W_N^{nk}, \qquad \forall\, k \end{aligned}}$$

Synthesis equation

Analysis equation

Note: both $\tilde{x}(n)$ and $\tilde{X}(k)$ are **periodic** with period N.

10.2 Properties of the Discrete Fourier Series

Most DFS properties are analogous to z-transform properties. Some important properties are illustrated below:

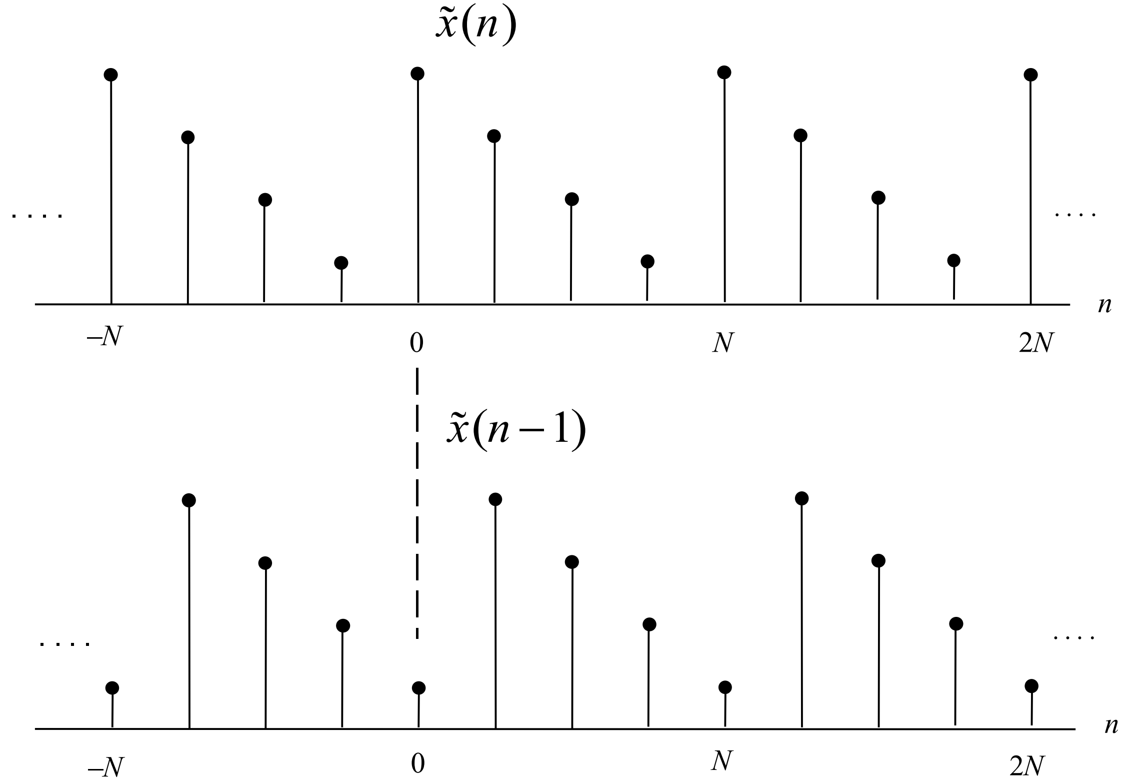
1. Shift of a Sequence

Shift in "time"

$$\tilde{x}(n-m) \Leftrightarrow W_N^{km} \tilde{X}(k)$$

Shift in frequency

$$\tilde{X}(k-l) \Leftrightarrow W_N^{-nl} \tilde{x}(n)$$



Any shift greater than a period (i.e., $M > N$) cannot be distinguished from a shorter shift

2. Symmetry Properties:

for $\tilde{x}(n)$ real, $\tilde{X}(k)$ is complex in general

$$\tilde{X}(k) = \tilde{X}_R(k) + j\tilde{X}_I(k)$$

Even real part:

$$\tilde{X}_R(k) = \tilde{X}_R(-k)$$

$$= \tilde{X}_R(N-k)$$

Odd imaginary part:

$$\tilde{X}_I(k) = -\tilde{X}_I(-k)$$

$$= -\tilde{X}_I(N-k)$$

So that:

$$\tilde{X}(k) = \left|\tilde{X}(k)\right| e^{j\arg\left\{\tilde{X}(k)\right\}}$$

$$\left|\tilde{X}(k)\right| \Rightarrow even$$

$$\arg\left\{\tilde{X}(k)\right\} \Rightarrow odd$$

3. Periodic Convolution:

Given that:

$$\tilde{x}_1(n) \Rightarrow \tilde{X}_1(k) \qquad \text{Period } N$$
$$\tilde{x}_2(n) \Rightarrow \tilde{X}_2(k) \qquad \text{Period } N$$

Then we would like to know what sequence corresponds to:

$$\tilde{X}_3(k) = \tilde{X}_1(k) \cdot \tilde{X}_2(k) \qquad \text{for } k = 0,1,\ldots,N-1$$

where: $$\tilde{x}_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}_3(k) W_N^{-kn}$$

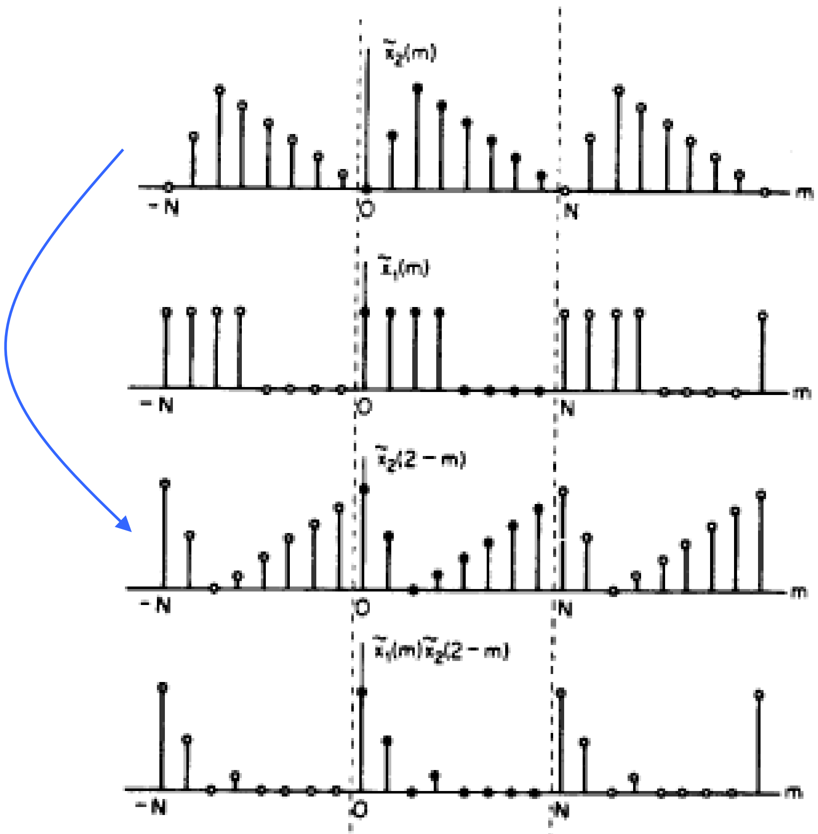
We should expect some sort of convolution since multiplication in frequency corresponds to convolution in time:

$$\tilde{x}_3(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1}\tilde{x}_1(m)W_N^{km} \cdot \sum_{r=0}^{N-1}\tilde{x}_2(r)W_N^{kr}\right]W_N^{-kn}$$

$$\tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m) \sum_{r=0}^{N-1} \tilde{x}_2(r) \left[\frac{1}{N} \sum_{k=0}^{N-1} W_N^{-k(n-m-r)} \right]$$

using $W_N = e^{-j\frac{2\pi}{N}}$

$$= \begin{cases} 1, & r = (n-m) + \ell N \\ 0, & \text{otherwise} \end{cases}$$



$$\tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m) \tilde{x}_2(n-m)$$

this convolution accounts for periodicity

summing these yields $\longrightarrow$ $\tilde{x}_3(2)$

$\tilde{X}(k)$ can be interpreted as samples on the unit circle of the z-transform of one period of $\tilde{x}(n)$

that is, let $x(n)$ represent one period of $\tilde{x}(n)$. Then it has a z-transform:

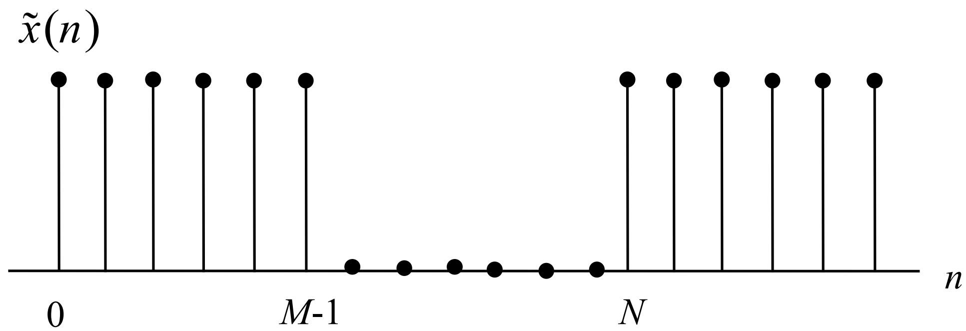
$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

that is, $x(n)$ is a finite sequence of length N.

Therefore, the z-transform of $x(n)$ and the DFS of $\tilde{x}(n)$ are related as follows:

$$\tilde{X}(k) = X(z)\Big|_{z = e^{j\frac{2\pi k}{N}} = W_N^{-k}}$$

Example:

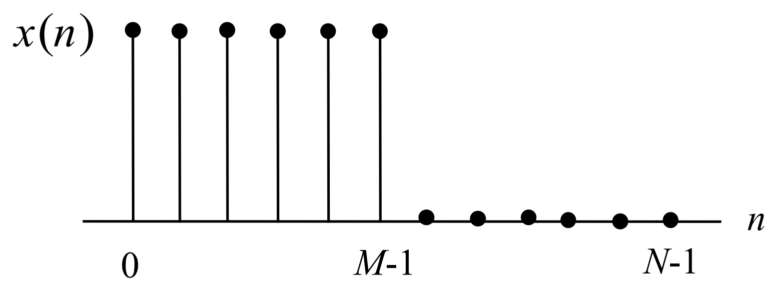


$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{M-1} e^{-j\frac{2\pi kn}{N}} = \frac{1 - e^{-j\frac{2\pi kM}{N}}}{1 - e^{-j\frac{2\pi k}{N}}}$$

after some algebra:

$$\tilde{X}(k) = e^{-j\frac{\pi k}{N}(M-1)} \cdot \frac{\sin\left(\frac{\pi k}{N} \cdot M\right)}{\sin\left(\frac{\pi k}{N}\right)}$$

Note that this is equivalent to obtaining sample values of the z-transform (on the unit circle) of the following sequence:

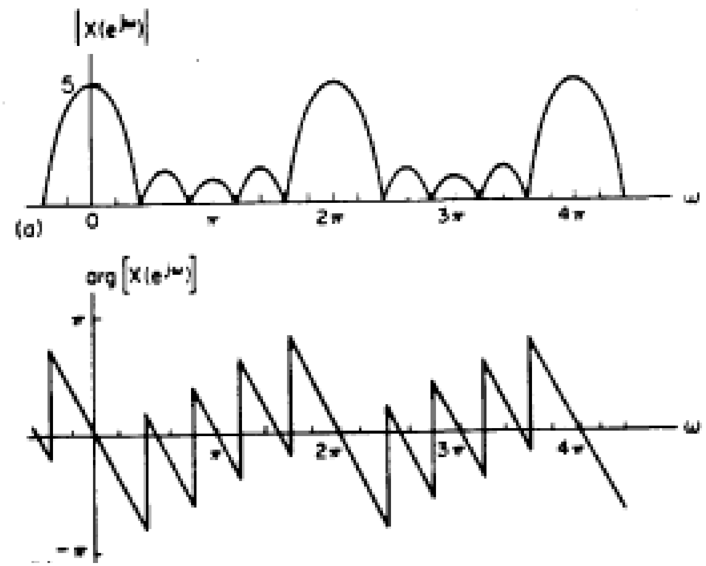


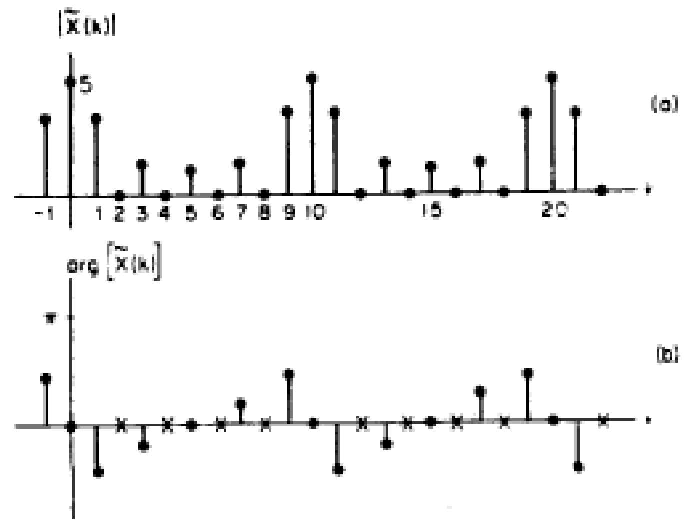
Evaluate X(z) on the (continuous) unit circle yields the DTFT

Now we are discretizing the DTFT in frequency (i.e. samples on the unit circle)

that is:

$$X(z)\Big|_{z=e^{j\omega}} = X(\omega) \Rightarrow X(\omega)\Big|_{\omega=\frac{2\pi k}{N}} = \left[e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot \frac{\sin\left(\frac{M\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right]_{\omega=\frac{2\pi k}{N}} = \tilde{X}(k)$$





10.3 Sampling the Fourier Transform

In this section we discuss the transition from the Discrete-Fourier Series (DFS) to the Discrete Fourier Transform (DFT).

To accomplish this we need to consider the relation between a non-periodic sequence:

$$x(n) \Leftrightarrow X(z)$$

and a periodic sequence

$$\tilde{x}(n) \Leftrightarrow \tilde{X}(k)$$

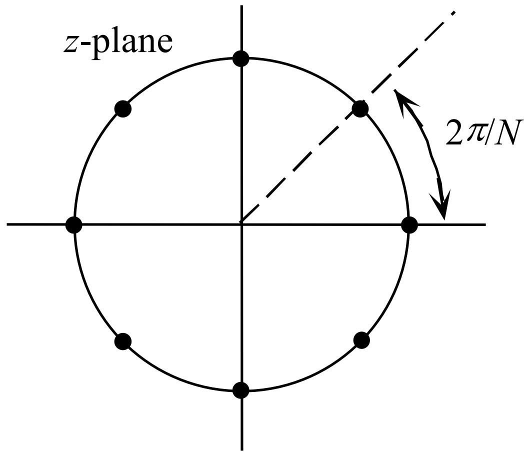
where $\tilde{X}(k)$ corresponds to samples of $X(z)$ on the unit circle.

We know that the finite sequence $x(n)$ has a z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The ROC includes the unit circle (this condition is always met for finite sequences)

Now evaluate this z-transform around the unit circle (at N equally spaced points).



In this case:

$$z = e^{j\frac{2\pi k}{8}}, \quad \text{with } k = 0,1,2,\ldots,7$$

where we have allowed:

$$\omega_k = \frac{2\pi k}{8}$$

By evaluating $X(z)$ on the unit circle, <u>we obtain a periodic **sequence** in frequency</u>

$$X(z)\Big|_{z=W_N^{-k}} = \sum_{n=-\infty}^{\infty} x(n) W_N^{kn} = \tilde{X}(k) \quad \text{where,} \quad W_N = e^{-j\frac{2\pi}{N}}$$

But we already know the DFS for a periodic sequence to be:

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-nk}$$

Therefore, let's investigate the relation between $x(n)$ and $\tilde{x}(n)$

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X(z)\Big|_{z=W_N^{-k}} \right] W_N^{-kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x(m) W_N^{km} \right] W_N^{-kn}$$

$$X(z)\Big|_{z=W_N^{-k}} = \sum_{m=-\infty}^{\infty} x(m) z^{-m}\Big|_{z=W_N^{-k}}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \left[\frac{1}{N} \sum_{k=0}^{N-1} W_N^{-k(n-m)} \right]$$

as before, this is = 1 when $m = n$,
(actually, when $m = n + rN$),
and zero otherwise

so that, $$\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n + rN)$$

that is, $$\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n + rN) = x(n \text{ modulo } N) = x((n))_N$$

where we have adopted the notation:

$$(n \text{ modulo } N) \Rightarrow ((n))_N$$

for example:

$$\begin{array}{ll} ((0))_2 = 0 & ((N-1))_N = N-1 \\ ((1))_2 = 1 & ((N))_N = 0 \\ ((2))_2 = 0 & ((N+1))_N = 1 \\ ((3))_2 = 1 & \end{array}$$

since $\tilde{x}(n) = x((n))_N$, then what is $x(n)$ with respect to $\tilde{x}(n)$?

We know that:

$$x(n) = \begin{cases} \tilde{x}(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

So a convenient notation in relating these sequences is the <u>rectangular sequence</u>.

$$R_N(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Recall the rectangular window used for FIR filter design (truncated the infinite ideal filter)

so that: $x(n) = \tilde{x}(n) R_N(n)$

Note:

1. A periodic sequence is formed by overlapping successive repetitions of the non-periodic sequence.
2. As long as $x(n)$ has length < N, then each period of $\tilde{x}(n)$ will be a replica of $x(n)$.
3. $x(n)$ of length N(or less) can be represented exactly by N samples of its z-transform on the unit circle.

In summary:

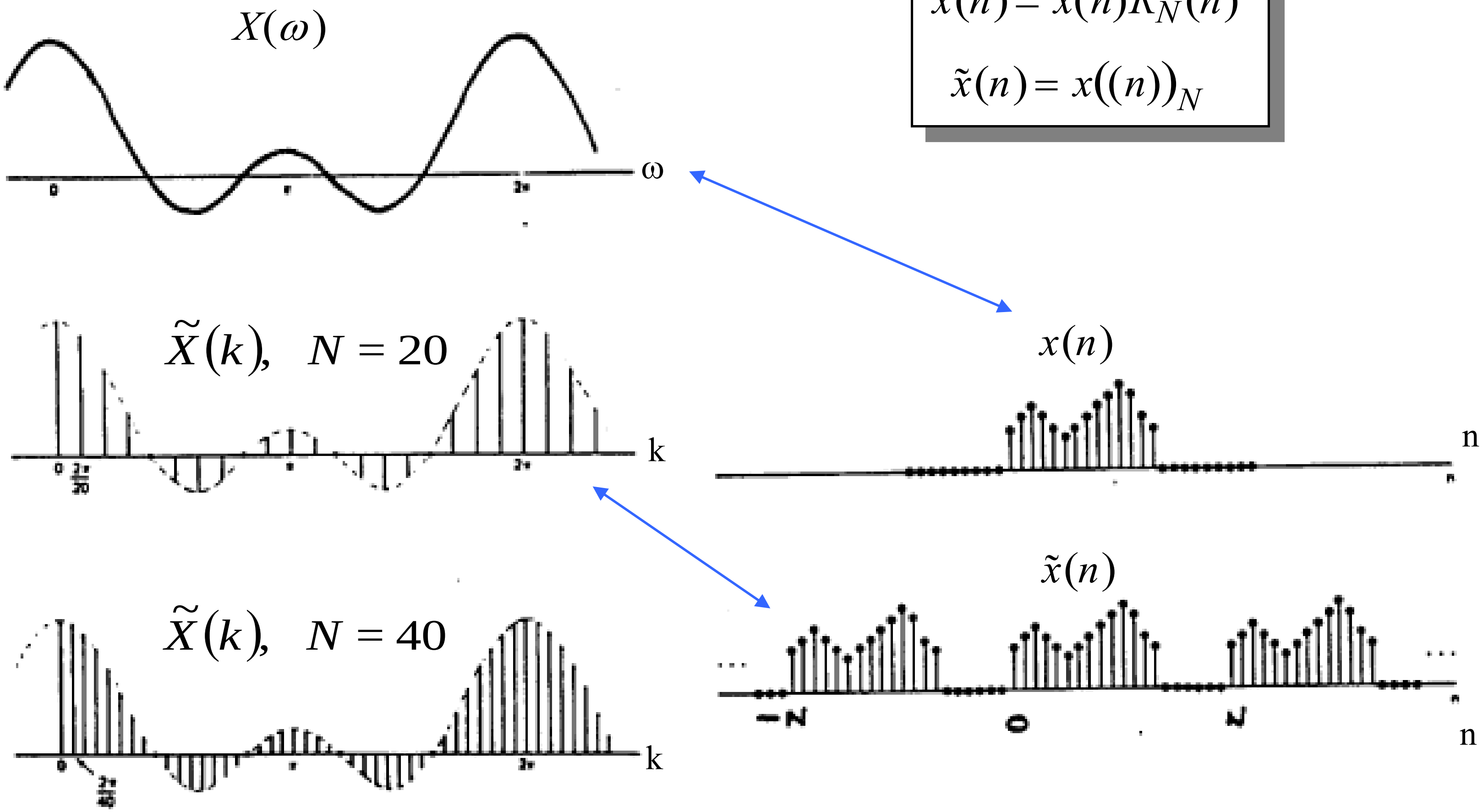
$$\boxed{\begin{aligned} \tilde{x}(n) &= x((n))_N \\ x(n) &= \tilde{x}(n) R_N(n) \end{aligned}}$$

These concepts are illustrated by an example on the next page.

These points are illustrated below:

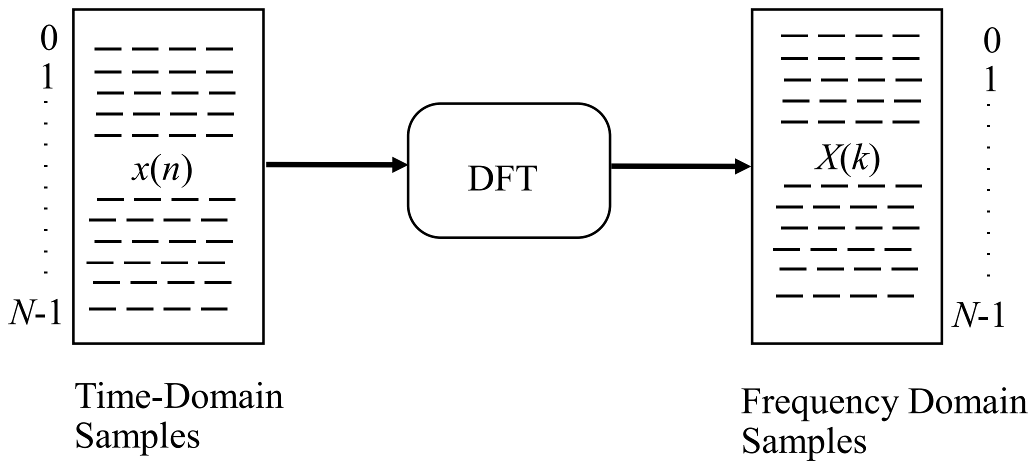
$$x(n) = \tilde{x}(n) R_N(n)$$

$$\tilde{x}(n) = x((n))_N$$



10.4 Fourier Representation of Finite-Duration Sequences: The Discrete Fourier Transform (DFT)

We can now define the Discrete Fourier Transform (DFT) as an operation which takes N samples of a discrete-time signal (sequence) and produces N samples of its Fourier transform



Where does this operation fit in with our other frequency analysis tools?

	discrete	continuous
periodic	discrete & periodic <u>Discrete Fourier Series (DFS)</u>	continuous & periodic <u>Fourier Series</u>
non-periodic	discrete & non-periodic <u>DTFT</u>	continuous & non-periodic <u>Fourier Transform</u>

Where is the DFT?

A Fourier representation of finite duration sequences can have two equivalent interpretations:

1. $x(n),\ X(k)$ are <u>one period</u> of $\tilde{x}(n),\ \tilde{X}(k)$
2. $x(n),\ X(k)$ are related by <u>samples of the z-transform on the unit circle</u>. That is:

since $Z\{x(n)\} = X(z)$ and $X(z)\Big|_{z=e^{j\omega}} = X(\omega) = \mathrm{DTFT}\{x(n)\}$

then $X(\omega)\Big|_{\omega=\frac{2\pi k}{N}} = X(k) = DFT\{x(n)\}$

A corresponding relation exists between the frequency domain counterparts $X(k)$ and $\tilde{X}(k)$

We want the Fourier coefficients of the finite sequence $x(n)$ to be a finite sequence $X(k)$ corresponding to <u>one period</u> of $\tilde{X}(k)$

From the DFS:

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn}, \ \forall k \qquad \tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-kn}, \ \forall n$$

$$X((k))_N = \sum_{n=0}^{N-1} x((n))_N W_N^{kn} \qquad x((n))_N = \frac{1}{N} \sum_{k=0}^{N-1} X((k))_N W_N^{-kn}$$

all are modulo N

Which is equivalently written as follows:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad 0 \le k \le N\text{-}1$$

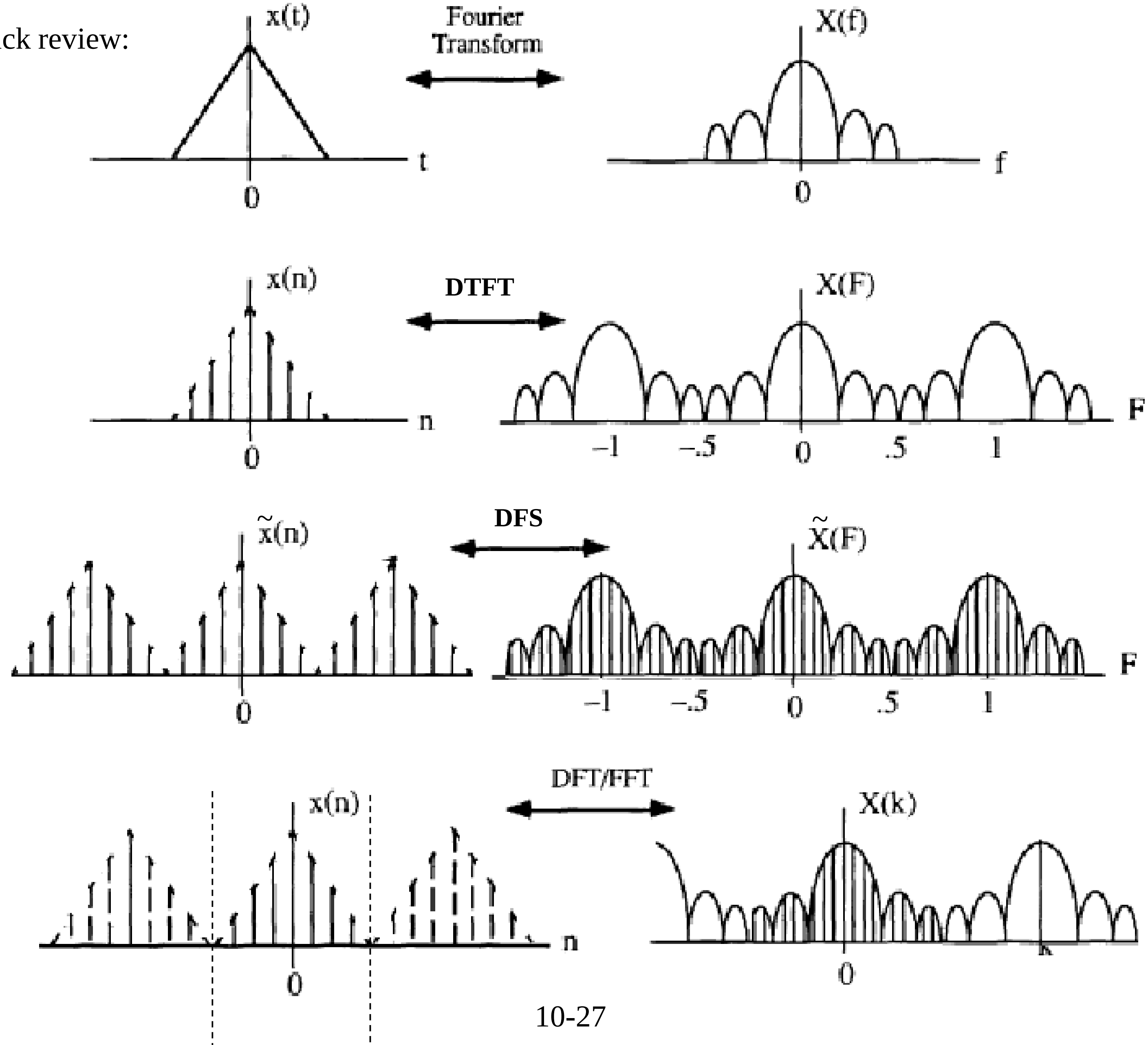
$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad 0 \le n \le N\text{-}1$$

This is the Discrete Fourier Transform (DFT) pair

$$x(n) \Leftrightarrow X(k)$$

Always remember when using the DFT that a <u>finite length sequence is represented as one period of a periodic sequence</u>.

A quick review:



10.5 Properties of the DFT

1. Linearity

if $$x_3(n) = ax_1(n) + bx_2(n)$$

then the DFT of $x_3(n)$ is

$$X_3(k) = aX_1(k) + bX_2(k)$$

> if $x_1(n)$ has length N_1
> and $x_2(n)$ has length N_2
> then $x_3(n)$ has length $max[N_1, N_2]$

Suppose that $N_1 > N_2$, then when computing the DFT of $x_1(n)$ and $x_2(n)$:

$$X_1(k) = \sum_{n=0}^{N_1-1} x_1(n) W_{N_1}^{kn}, \qquad 0 \le k \le N_1 - 1$$

$$X_2(k) = \sum_{n=0}^{N_1-1} x_2(n) W_{N_1}^{kn}, \qquad 0 \le k \le N_1 - 1$$

This implies that the shorter sequences must be augmented by zeros.

This is also required in convolution operations.

2. Periodicity

If $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(k+N) = X(k) \quad \text{for all } k$$

3. Circular Symmetry of a Sequence

An N-point sequence is called *circularly even* if it is symmetric about the point zero on the circle. This implies:

$$x(N-n) = x(n) \qquad 1 \le n \le N-1$$

An N-point sequence is called *circularly odd* if it is antisymmetric about the point zero on the circle.

$$x(N-n) = -x(n) \qquad 1 \le n \le N-1$$

The *time reversal* of an N-point sequence is attained by reversing its samples about the point zero on the circle:

$$x((-n))_N = x(N-n) \qquad 0 \le n \le N\text{-}1$$

4. <u>Circular Shift of a Sequence</u>

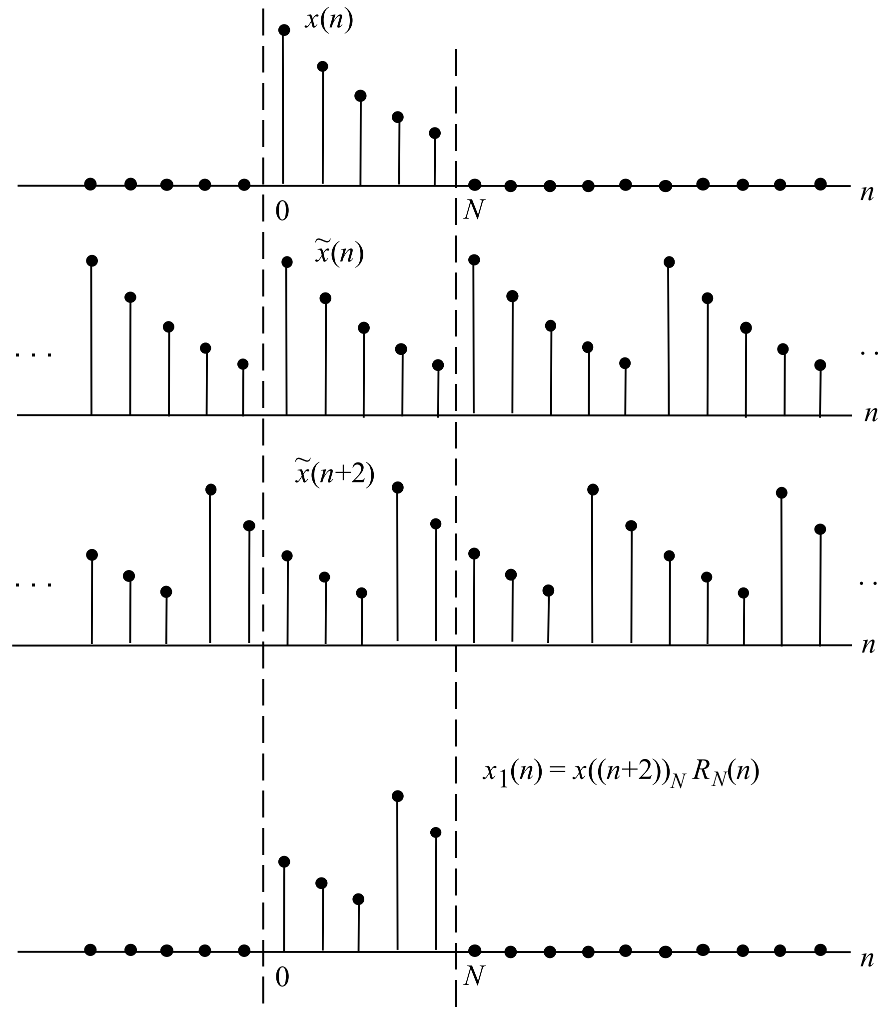
A (linear) delay shift of the periodic sequence is analogous to the rotation of a circle.

That is, if:

$$\tilde{x}_1(n) = \tilde{x}(n+m) = x((n+m))_N$$

then,

$$x_1(n) = x((n+m))_N R_N(n)$$



We can show that the relation between the DFT of $x_1(n)$ and the DFT of $x(n)$ is

$$X_1(k) = W_N^{-km} X(k)$$

circular time shift $\longrightarrow$ $x((n+m))_N R_N(n) \Leftrightarrow W_N^{-km} X(k)$

circular frequency shift $\longrightarrow$ $X((k+\ell))_N R_N(k) \Leftrightarrow W_N^{n\ell} x(n)$

5. <u>Symmetry Properties of the DFT</u>

Suppose we have a sequence $x(n)$, which is finite and,

$$\mathrm{DFT}\{x(n)\} = X(k)$$

We have stated that $X(k)$ is complex, in general. therefore

$$X(k) = \underbrace{X_r(k)}_{\mathrm{Re}[X(k)]} + j\underbrace{X_i(k)}_{\mathrm{Im}[X(k)]}$$

Since $$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}, \qquad 0 \le k \le N-1$$

$$= \underbrace{\sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{N}\right)}_{X_r(k)} - j \underbrace{\sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi kn}{N}\right)}_{X_i(k)}$$

For $x(n)$ purely real, note that:

$$X_r(-k) = X_r(k)$$

$$X_i(-k) = -X_i(k)$$

When $x(n)$ is purely real, $X_r(k)$ is even and $X_i(k)$ is odd.

Similarly, when $x(n)$ is purely imaginary, $X_r(k)$ is odd and $X_i(k)$ is even.

If $x(n)$ is real and even, then

$$x_e(n) \Leftrightarrow X_r(k)$$

If $x(n)$ is real and odd, then

$$x_o(n) \Leftrightarrow jX_i(k)$$

All the symmetry properties of the DFT can be deduced from the following:

$$x(n) = x_r^e(n) + x_r^o(n) + jx_i^e(n) + jx_i^o(n)$$

$$X(k) = X_r^e(k) + X_r^o(k) + jX_i^e(k) + jX_i^o(k)$$

6. Circular Convolution

Convolution of two finite sequences using the DFT is a Circular Convolution, which is actually derived from periodic convolution.

Consider two finite sequences, $x_1(n)$ and $x_2(n)$ of duration N, such that,

$$x_1(n) \Leftrightarrow X_1(k)$$

$$x_2(n) \Leftrightarrow X_2(k)$$

Determine $x_3(n)$ where $X_3(k)$ is the product of the DFTs of $x_1(n)$ and $x_2(n)$,

$$X_3(k) = X_1(k) \cdot X_2(k)$$

As usual, we can allow $x_3(n)$ to correspond to one period of $\tilde{x}_3(n)$. Also, recall the following expression for periodic convolution that we developed when studying the DFS:

$$\tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m)\tilde{x}_2(n-m)$$

Therefore, since $$x(n) = \tilde{x}(n) R_N(n)$$

then: $$x_3(n) = \left[\sum_{m=0}^{N-1} \tilde{x}_1(m) \tilde{x}_2(n-m) \right] \cdot R_N(n)$$

modulo N accounts for periodic nature

and since: $$\tilde{x}(n) = x((n))_N$$

$$x_3(n) = \left[\sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N \right] \cdot R_N(n)$$

This is an <u>N-point circular convolution</u>, which is represented as:

$$x_3(n) = x_1(n) \; \textcircled{\scriptsize N} \; x_2(n)$$

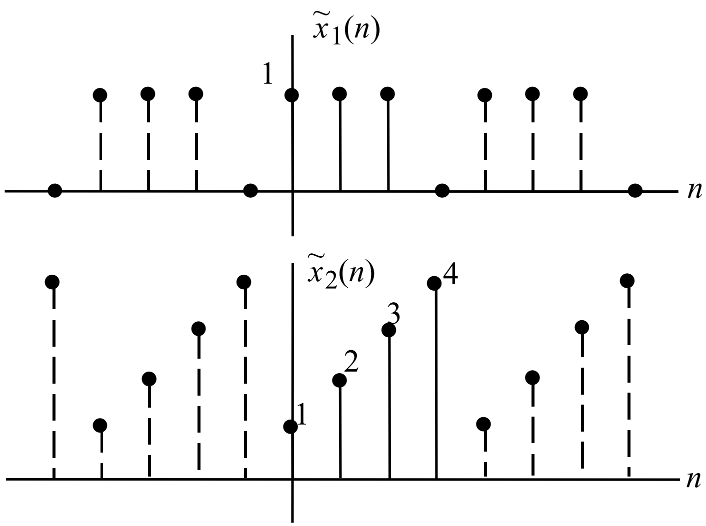
In conclusion, multiplication of DFTs corresponds to circular convolution in the time domain. The dual also exists whereby the multiplication of two finite sequences is the circular convolution of their respective DFTs.

As an example, consider the following processes:

$$\tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m)\tilde{x}_2(n-m)$$

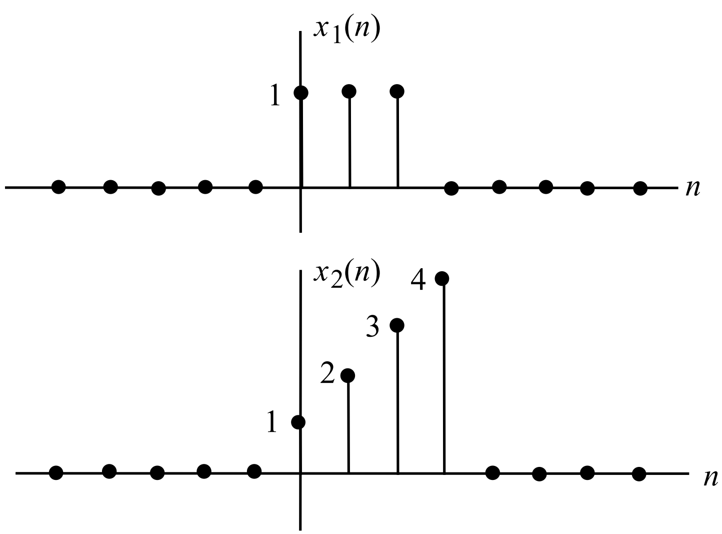
$$x_3(n) = \left[\sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N\right] \cdot R_N(n)$$

Periodic Convolution



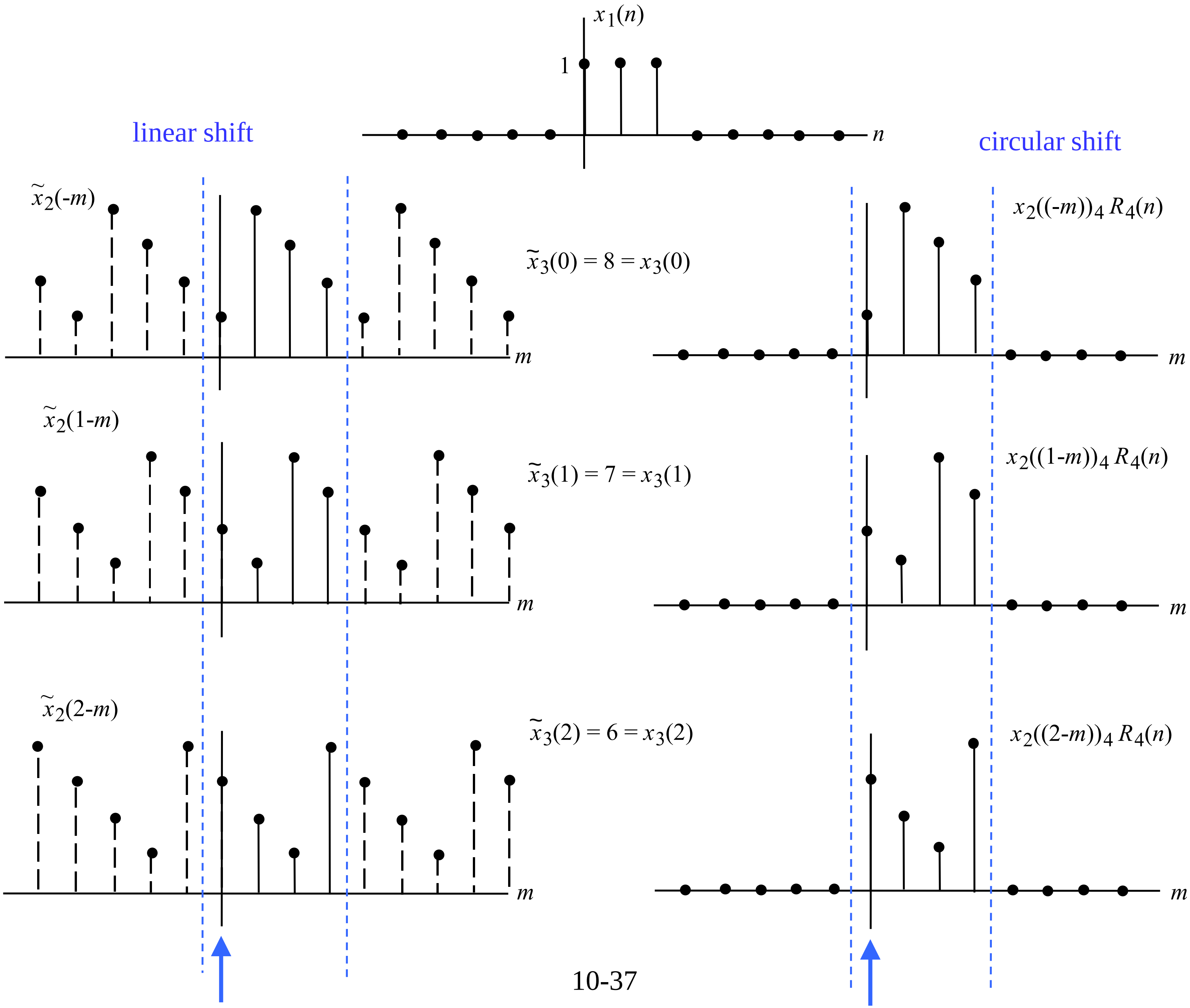
$$\tilde{x}_3(n) = \tilde{x}_1(n) * \tilde{x}_2(n)$$

Circular Convolution

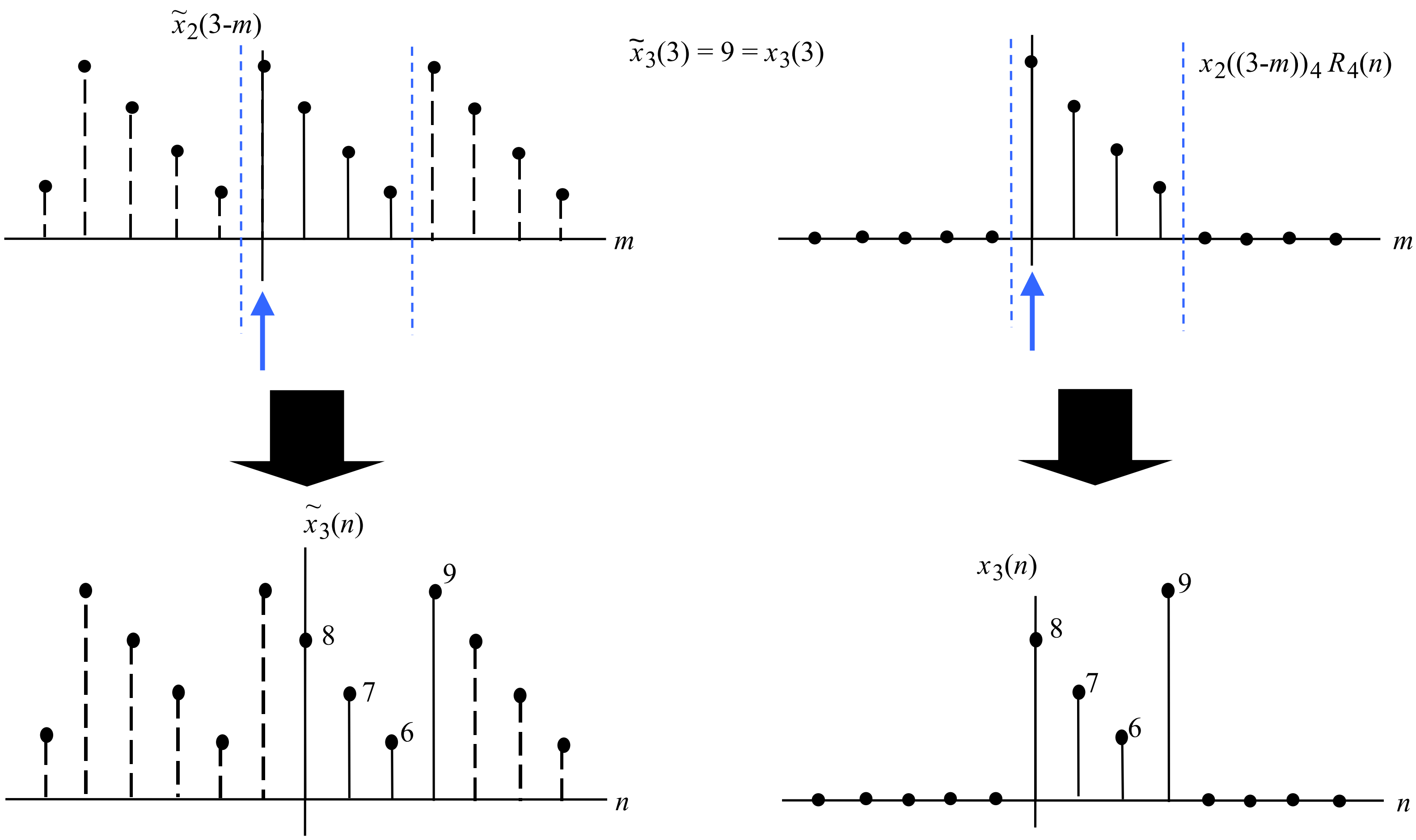


$$x_3(n) = x_1(n) \; (N) \; x_2(n)$$

with $N = 4$



$x_1(n)$
1
n
linear shift
circular shift
$\tilde{x}_2(-m)$
m
$\tilde{x}_3(0) = 8 = x_3(0)$
$x_2((-m))_4\, R_4(n)$
$\tilde{x}_2(1-m)$
$\tilde{x}_3(1) = 7 = x_3(1)$
$x_2((1-m))_4\, R_4(n)$
$\tilde{x}_2(2-m)$
$\tilde{x}_3(2) = 6 = x_3(2)$
$x_2((2-m))_4\, R_4(n)$



We have established that multiplication of DFTs corresponds to circular convolution of the sequences. <u>However, most useful applications require **linear convolution**</u>.

10.6 Linear Convolution

How can we ensure that circular convolution has the effect of a linear convolution?

Linear convolution of two sequences, each of length N is:

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) x_2(n-m)$$

Note that $x_3(n)$ will be of length $2N-1$ (i.e., will have $2N-1$ possible nonzero points).

To obtain a linear convolution using the DFT, the DFT of each sequence must be computed on the basis of $2N$-1 points.

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) W_{2N-1}^{nk}, \qquad 0 \le k \le 2N-2$$

where $W_{2N-1} = e^{-j\frac{2\pi}{2N-1}}$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) W_{2N-1}^{nk}, \qquad 0 \le k \le 2N-2$$

so that:

$$x_3(n) = \frac{1}{2N-1}\left[\sum_{k=0}^{2N-2} X_1(k) X_2(k) W_{2N-1}^{-nk}\right] \cdot R_{2N-1}(n)$$

which is a linear convolution

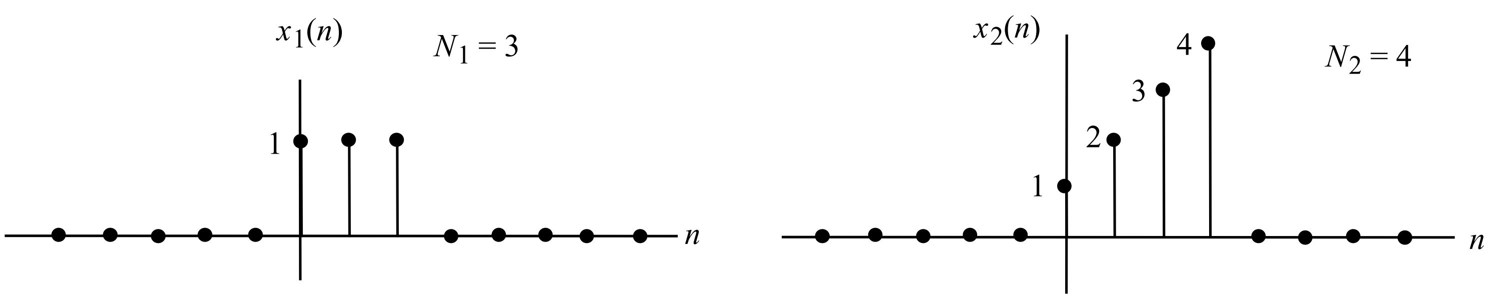
When convolving two sequences of unequal length, for example

$$x_1(n) \Rightarrow N_1$$

$$x_2(n) \Rightarrow N_2$$

then: $$x_1(n) * x_2(n) \Rightarrow N_1 + N_2 - 1$$

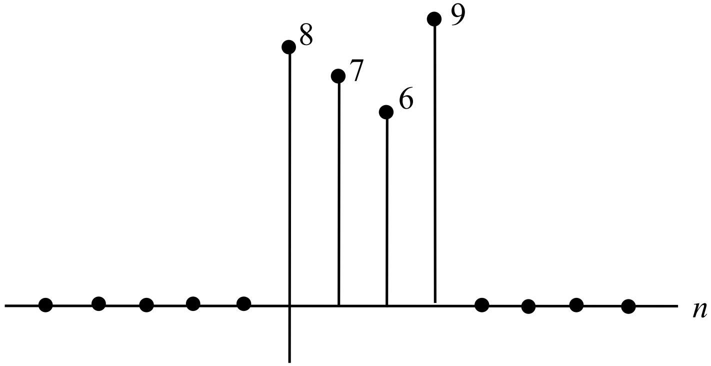
so take the DFT based on this length. Here's an example:



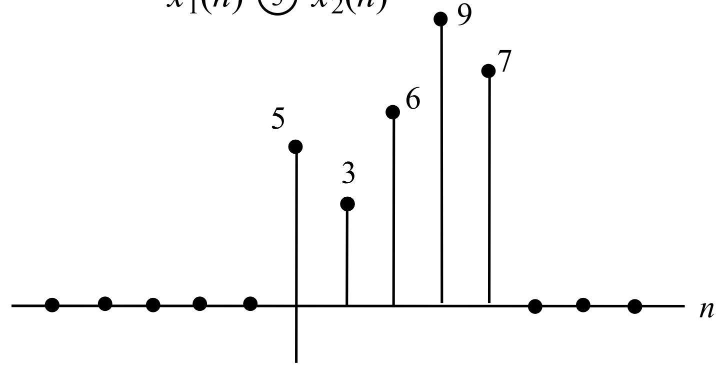
We perform a circular convolution with increasing values of N in which $N \geq N_1 + N_2 - 1 = 6$ is needed to obtain the linear convolution result.

Verify that the following results are obtained for $N = 4, 5, 6, 7$

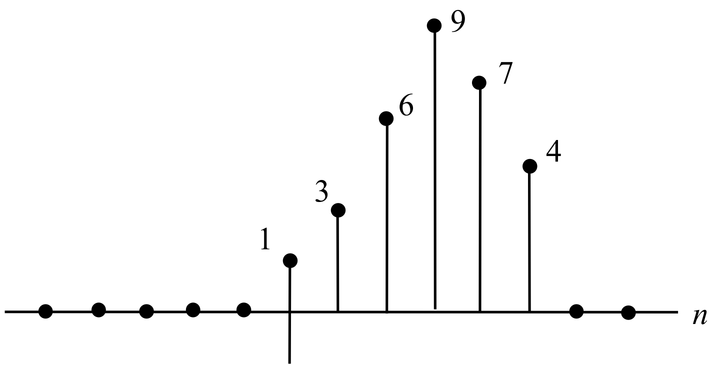
$x_1(n)$ ④ $x_2(n)$



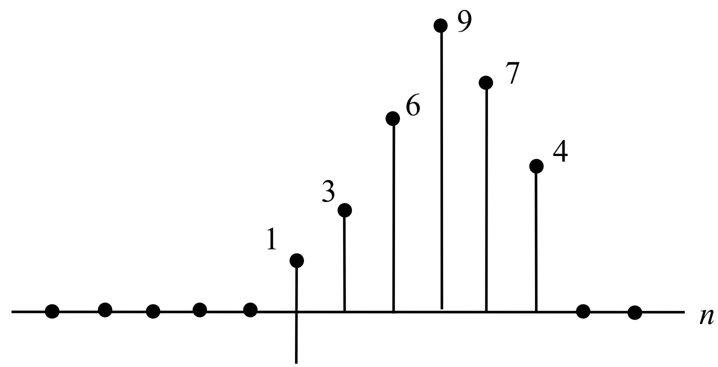
$x_1(n)$ ⑤ $x_2(n)$



$x_1(n)$ ⑥ $x_2(n)$



$x_1(n)$ ⑦ $x_2(n)$



Observations:

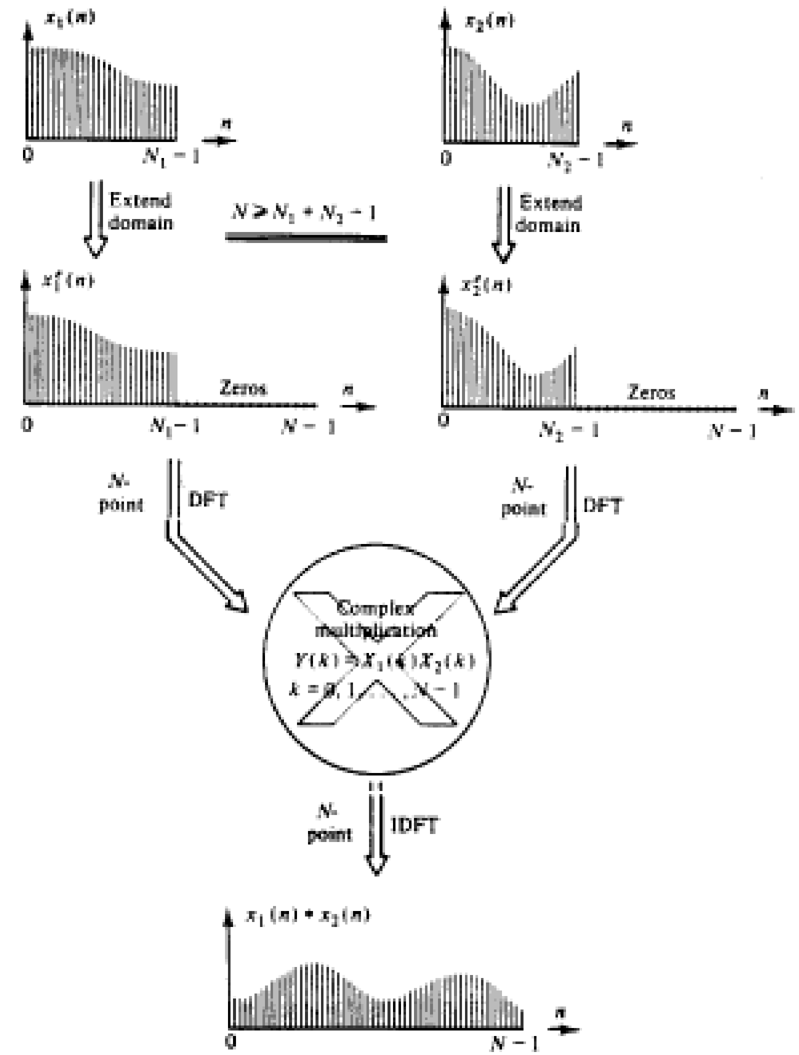
1. $$x_1(n) * x_2(n) = x_1(n) \; ⓝ \; x_2(n), \qquad \text{for } N \geq N_1 + N_2 - 1$$

2. In general, if $x_3(n) = x_1(n) * x_2(n)$

$$x_1(n) \; ⓝ \; x_2(n) = \left[\sum_{r=-\infty}^{\infty} \tilde{x}_3(n + rN) \right] \cdot R_N(n)$$

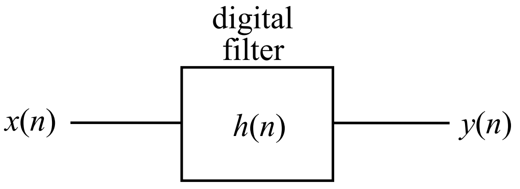
3. In summary, performing linear convolution using the DFT requires the following steps:

   - Choose $N \geq N_1 + N_2 - 1$
   - Make $x_1(n)$ and $x_2(n)$ be N points long by appending zeros as necessary
   - Compute the N-point DFTs $X_1(k)$ and $X_2(k)$, for $k = 0, 1, \ldots, N - 1$
   - Multiply $X_3(k) = X_1(k)\, X_2(k)$, for $k = 0, 1, \ldots, N - 1$
   - Compute N-point inverse DFT (IDFT) $X_3(k)$ to obtain $x_3(n)$.


$x_1(n)$
n
0
$N_1 - 1$
$x_2(n)$
n
0
$N_2 - 1$
Extend domain
$N \geq N_1 + N_2 - 1$
Extend domain
$x_1^e(n)$
Zeros
n
0
$N_1 - 1$
$N - 1$
$x_2^e(n)$
Zeros
n
0
$N_2 - 1$
$N - 1$
N-point
DFT
N-point
DFT
Complex multiplication
$Y(k) = X_1(k)X_2(k)$
$k = 0, 1, \ldots, N - 1$
N-point
IDFT
$x_1(n) * x_2(n)$
n
0
$N - 1$

Linear Filtering with the DFT

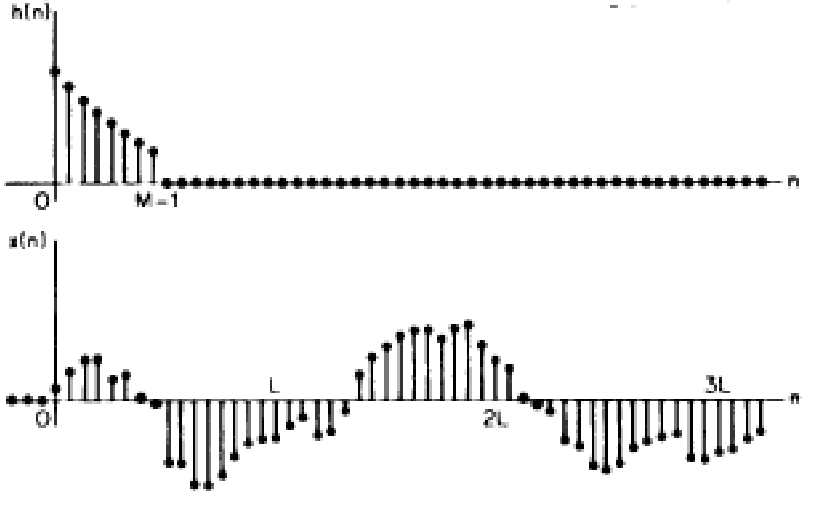
Consider the following case:



where $x(n)$ is an extremely long sequence (e.g. streaming signal) and $h(n)$ is a finite length sequence

The previous procedure will not be convenient to use due to the length of $x(n)$. However, we can use principles discovered in the previous procedure to implement a linear convolution operation. This operation is known as <u>Block Filtering</u>.

Suppose we require the convolution of the following two sequences:



There are two techniques we can use to ensure that the convolution of $h(n)$ and $x(n)$ is linear:

- Overlap-add technique
- Overlap-save technique

For both techniques the filter has length M and the (long) input sequence is segmented into blocks of length L where we assume $L >> M$. The DFTs (and subsequent IDFTs) will be performed on blocks of length $N = L + M - 1$.

Overlap-Add Technique

Break $x(n)$ into L-length sections, append $M - 1$ zeros to each section, then compute the N-point DFT as

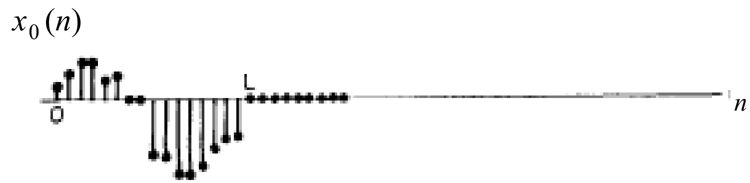
$$x_0(n) = \{x(0),\ x(1),\ \cdots,\ x(L-1),\ \overbrace{0,\ 0,\ \cdots,\ 0}^{M-1}\} \quad \Leftrightarrow \quad X_0(k) \qquad k = 0, 1, \ldots, N-1$$

$$x_1(n) = \{x(L),\ x(L+1),\ \cdots,\ x(2L-1),\ \overbrace{0,\ 0,\ \cdots,\ 0}^{M-1}\} \quad \Leftrightarrow \quad X_1(k) \qquad k = 0, 1, \ldots, N-1$$

$$\vdots \qquad\qquad\qquad \vdots$$

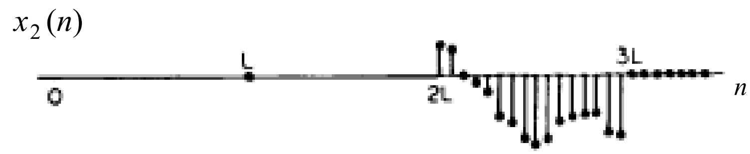
The filter impulse response is increased in length by $L - 1$ zeros and an N-point DFT is computed as

$$h(n) = \{h(0),\ h(1),\ \cdots,\ h(M-1),\ \overbrace{0,\ 0,\ \cdots,\ 0}^{L-1}\} \quad \Leftrightarrow \quad H(k) \qquad k = 0, 1, \ldots, N-1$$



$$\xrightarrow{\text{DFT}} \quad X_0(k) \qquad k = 0,1,\ldots,N-1$$

$$\xrightarrow{\text{DFT}} \quad X_1(k) \qquad k = 0,1,\ldots,N-1$$



$$\xrightarrow{\text{DFT}} \quad X_2(k) \qquad k = 0,1,\ldots,N-1$$

Next, each set of the *N*-point DFTs is multiplied with the *N*-point DFT of the filter to obtain

$$Y_\ell(k) = H(k)\,X_\ell(k) \qquad k = 0,1,\ldots,N-1 \text{ and } \ell = 0,1,2,\ \ldots$$

The IDFT of each $Y_\ell(k)$ therefore yields $y_\ell(n)$ which are then recombined as

$$y(n) = y_0(n) + y_1(n-L) + y_2(n-2L) + \cdots$$

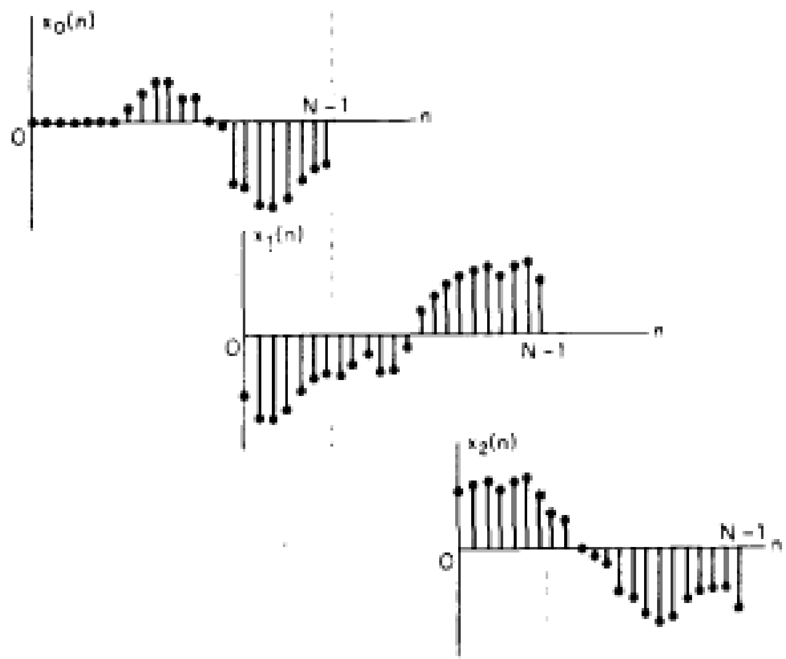
Overlap-Save Technique

In this technique, the input data is segmented into blocks of length N in which contiguous blocks overlap $M-1$ data points. Thus the input sequences are

$$x_0(n)=\{\underbrace{0,\ 0,\ \cdots,\ 0}_{M\text{–}1\text{ points}},\ x(0),\ x(1),\ \cdots,\ x(L-1)\}$$

$$x_1(n)=\{\underbrace{x(L-M+1),\ \cdots,\ x(L-1)}_{M\text{–}1\text{ data points from } x_0(n)},\ \underbrace{x(L),\ \cdots,\ x(2L-1)}_{L\text{ new data points}}\}$$

$$x_2(n)=\{\underbrace{x(2L-M+1),\ \cdots,\ x(2L-1)}_{M\text{–}1\text{ data points from } x_1(n)},\ \underbrace{x(2L),\ \cdots,\ x(3L-1)}_{L\text{ new data points}}\}$$ and so forth.



$$x_0(n) \Leftrightarrow X_0(k) \qquad k=0,1,\ldots,N-1$$

$$x_1(n) \Leftrightarrow X_1(k) \qquad k=0,1,\ldots,N-1$$

$$x_2(n) \Leftrightarrow X_2(k) \qquad k=0,1,\ldots,N-1$$

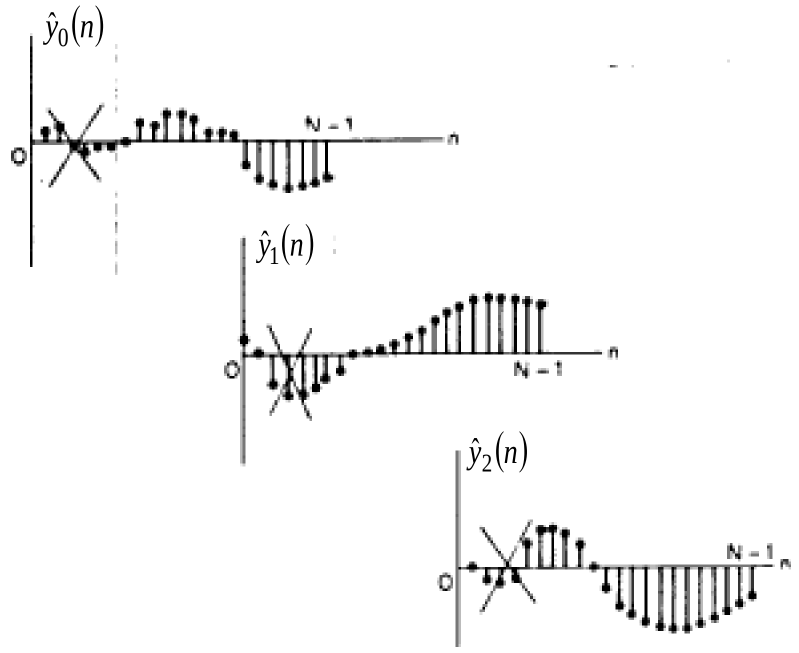
The N-point DFTs of the input sequences are multiplied with the N-point DFT of the filter to obtain

$$\hat{Y}_\ell(k) = H(k)\, X_\ell(k) \qquad k = 0, 1, \ldots, N-1 \text{ and } \ell = 0, 1, 2, \ldots$$

The IDFT of each $\hat{Y}_\ell(k)$ therefore yields $\hat{y}_\ell(n)$.

The first $M - 1$ terms of each output sequence are discarded as they are corrupted by aliasing resulting in

$$y_k(n - M + 1) = \begin{cases} \hat{y}_k(n), & M - 1 \le n \le N - 1 \\ 0, & \text{otherwise} \end{cases}$$



Hence, the resulting output which is the same as for linear convolution is

$$y(n) = y_0(n) + y_1(n - L) + y_2(n - 2L) + \cdots$$