Course Notes 12 – Fourier Analysis of Signals Using the DFT

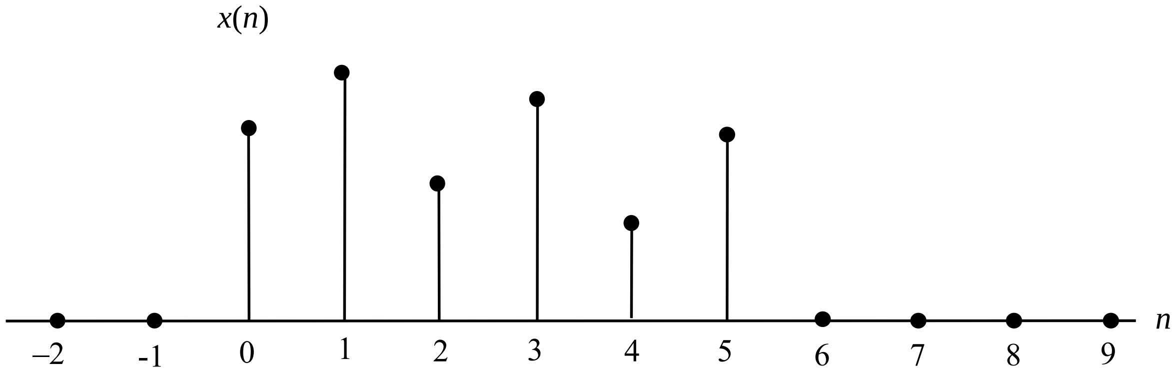
12.0 Fourier Analysis of Signals Using the DFT

The spectral computation of three types of signals will be considered:

1. Short sequences
2. Long sequences
3. Continuous-time signals

Short Sequences

The description "short" is based on the criterion that the number of data points is less than or equal to the memory size or the available computation length of the FFT. For example, consider the following sequence:



The Fourier Transform (DTFT) of this sequence is:

$$X(\omega) = \sum_{n=0}^{5} x(n) e^{-j\omega n}$$

Which we described as the z-transform evaluated along the unit circle.

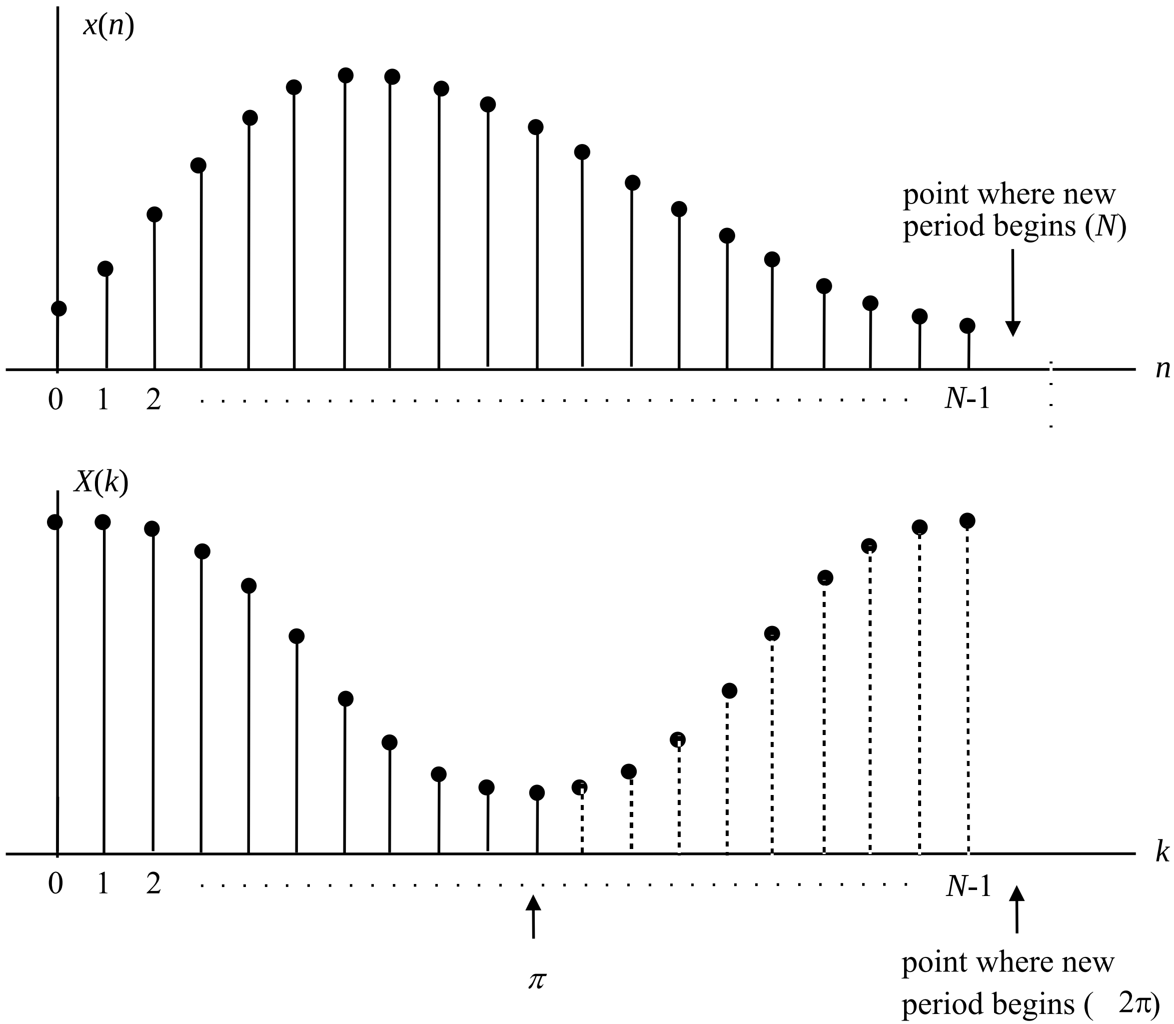
and the DFT is

$$X(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{5} x(n) e^{-j\frac{2\pi kn}{N}}$$

$$X(k) = \sum_{n=0}^{5} x(n) W_6^{kn}$$

This DFT is exact. No approximations are involved in the computation and the results are entirely consistent with the DFS.

Now, suppose we take the DFT of a short N-point sequence, such as shown on the next page

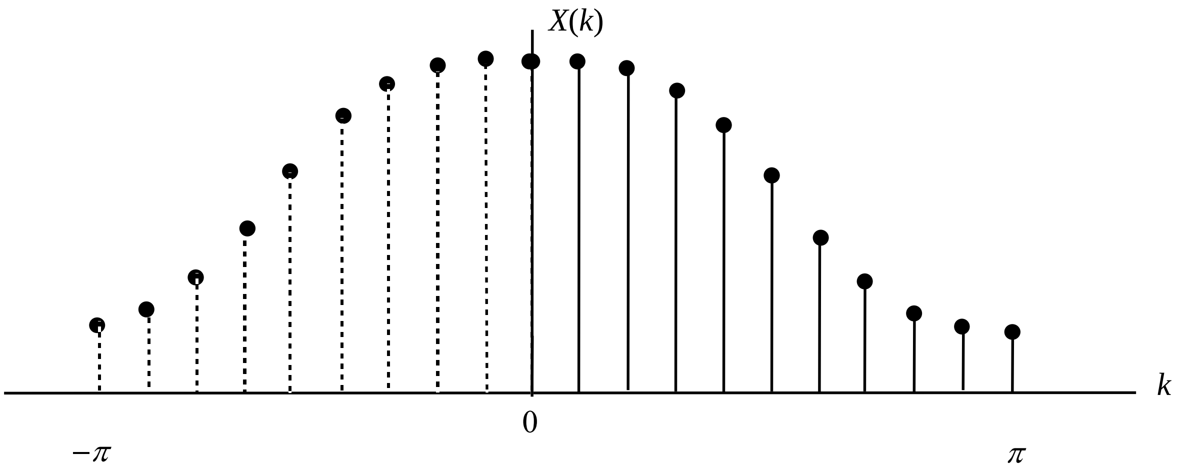


Note that the N-points of the DFT are spaced by

$$\Delta\omega = \frac{2\pi}{N}$$

We can improve the appearance of the result of this DFT in a couple of simple ways.

1. A more convenient representation can be obtained by using the principle of periodicity to show the DFT from $-\pi < \omega < \pi$.



2. The apparent spectral resolution can be improved by zero-padding the sequence $x(n)$ with zeros. Note that the resolution is actually unaffected; zero-padding merely alters the number and spacing of the DFT frequency samples.

To see this, we can create a new sequence $g(n)$:

$$g(n) = \begin{cases} x(n), & n = 0, 1, \ldots, N-1 \\ 0, & n = N, N+1, \ldots, M-1 \end{cases}$$

It is easy to show that the <u>DTFT</u> of $x(n)$ and $g(n)$ are identical. That is:

Index change by ignoring the zero-padding

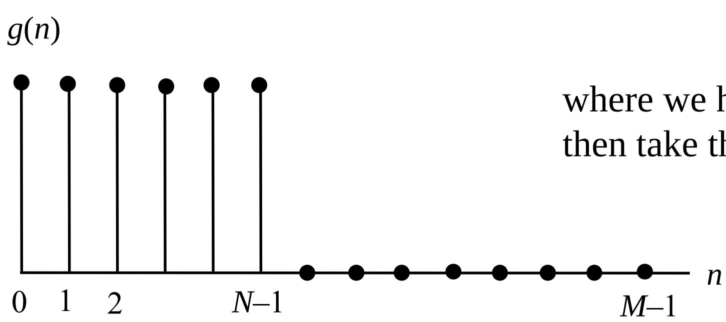
$$G(\omega) = X(\omega)$$

$$G(\omega) = \sum_{n=0}^{M-1} g(n) e^{-j\omega n} = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} = X(\omega)$$

So the DFT of $g(n)$ yields a discrete spectrum $G(k)$ whose components are spaced by;

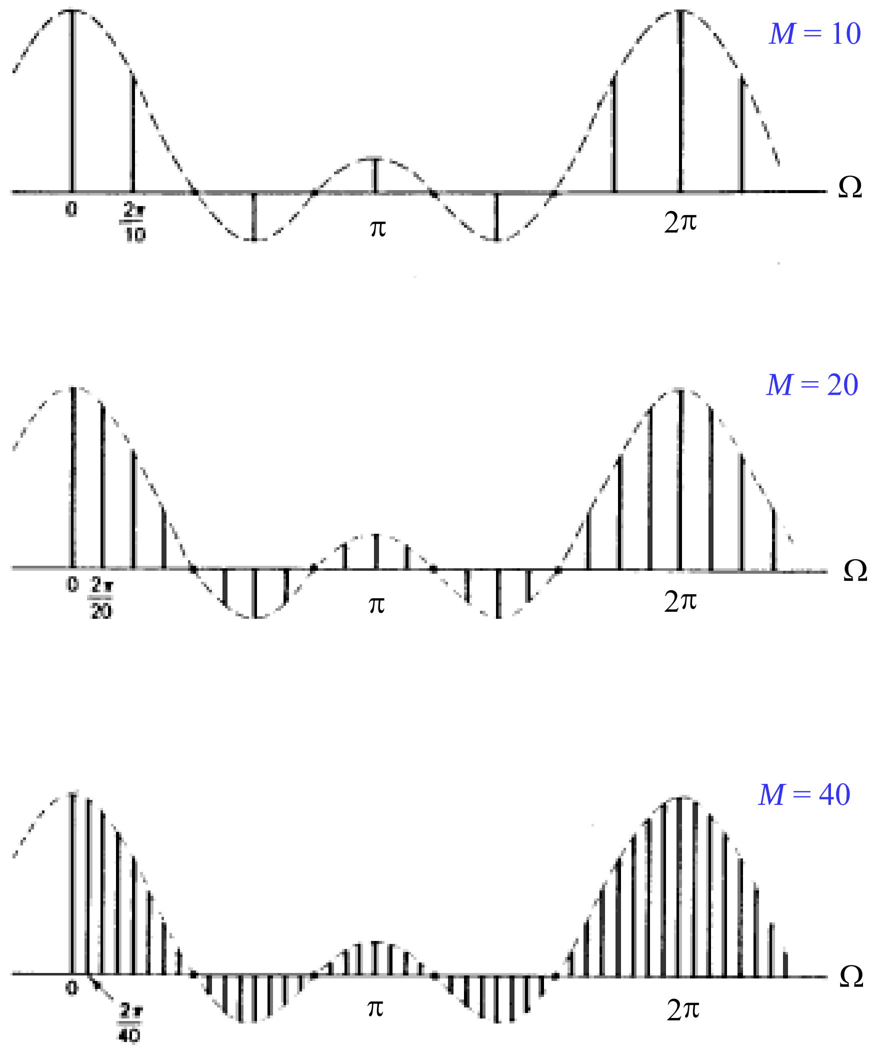
$$\Delta\omega = \frac{2\pi}{M}$$

For example, if the sequence to be transformed is:



where we have zero-padded with M–N zeros, then take the M-point DFT as follows:

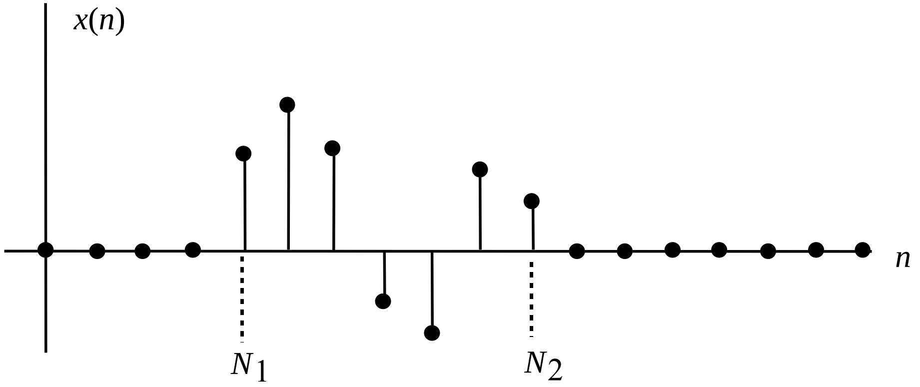
The following spectra are obtained for various values of *M*:



In theory, if we let $M \rightarrow \infty$ the DFT discrete spectrum would become the continuous DTFT spectrum

In addition to improving the presentation of the DFT, another reason for zero-padding is that many FFT algorithms require the number of data points to be a power of 2.

Consider the following sequence



That is, $x(n)$ is nonzero only over the interval $n = N_1, N_1+1, ..., N_2-1, N_2$

For purposes of computing the spectrum, the effective length of this sequence is N_2-N_1+1.

How do we compute the spectrum of this sequence? Consider two techniques:

1. <u>Shift the sequence to the origin and pad with zeros</u>. That is:

$$g(n) = \begin{cases} x(n+N_1), & n = 0, 1, \ldots, N_2 - N_1 \\ 0, & n = N_2 - N_1 + 1, \ldots, M-1 \end{cases}$$

pad with ($M - N_2 + N_1 - 1$) zeros

And we can show that:

Phase shift due to delay shift

$$\mathrm{DFT}\{g(n)\} = G(k) = e^{j\frac{2\pi k}{M}\cdot N_1} X(\omega)\Big|_{\omega=\frac{2\pi k}{M}}$$

where:

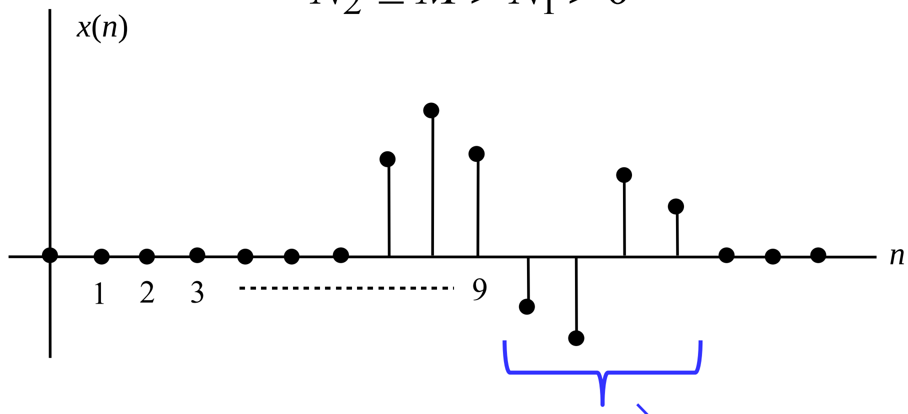
$$|G(\omega)| = |X(\omega)|$$

$$Arg[G(\omega)] = Arg[X(\omega)] + \omega N_1$$

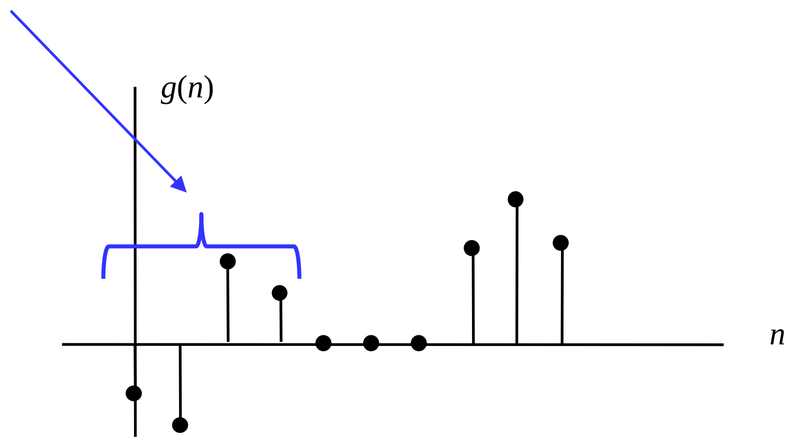
Phase

2. <u>Periodically extend the sequence and pad with zeros</u>. For example, when

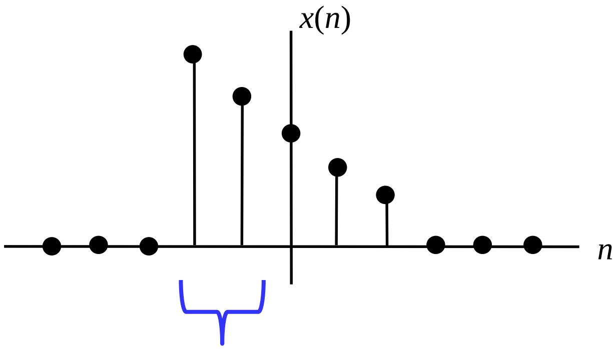
$$N_2 \geq M > N_1 > 0$$ (with M = 10, the DFT length)



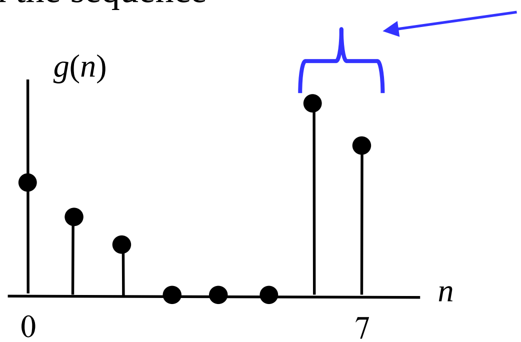
$$g(n) = \begin{cases} x(n+M), & n = 0, 1, \ldots, N_2 - M \\ 0, & n = N_2 - M + 1, \ldots, N_1 - 1 \\ x(n), & n = N_1, \ldots, M\text{ -1} \end{cases}$$



Similarly for non-causal sequences with $N_1 < 0$ and $N_2 > 0$ (with $M = 8$)



Then form the sequence



$$g(n) = \begin{cases} x(n), & n = 0, 1, \ldots, N_2 \\ 0, & n = N_2 + 1, \ldots, M + N_1 - 1 \\ x(n - M), & n = M + N_1, \ldots, M - 1 \end{cases}$$

In both cases we can show that

$$G(k) = X(\omega)\big|_{\omega = \frac{2\pi k}{M}}, \qquad k = 0,\ 1,\ \ldots,\ M - 1$$

if computed with M samples

$$G(k) = X(k)$$

So the DFT of a finite sequence is the same as the DFT of its periodic extension.

Long Sequences

Whether a sequence is considered "long" is based on the criterion that the number of data points is greater than the memory size (or larger than the desired FFT length). Often this consideration is based on processing speed and allowable latency.

For an infinite sequence:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

In order to apply the DFT to long sequences, the sequence must be truncated into a short sequence. Let $g(n)$ be the truncated sequence, so that:

$$g(n) = \begin{cases} h(n), & n = 0, \ \pm 1, \ \pm 2, \ \ldots, \ \pm M \\ 0, & |n| > M \end{cases}$$

which is a finite sequence of length $2M+1$. The spectrum of $g(n)$ can be computed with the DTFT as:

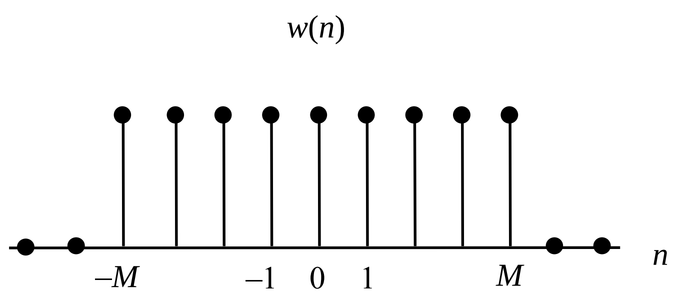
$$G(\omega) = \sum_{n=-M}^{M} h(n) e^{-j\omega n}$$

Since $g(n)$ is a finite sequence, the DFT can be used to compute the sampled version of $G(\omega)$. The problem, however, is that we want to compute $H(\omega)$; therefore, the relation between $H(\omega)$ and $G(\omega)$ must be established.

To accomplish this, let's examine the effects of truncation. Define

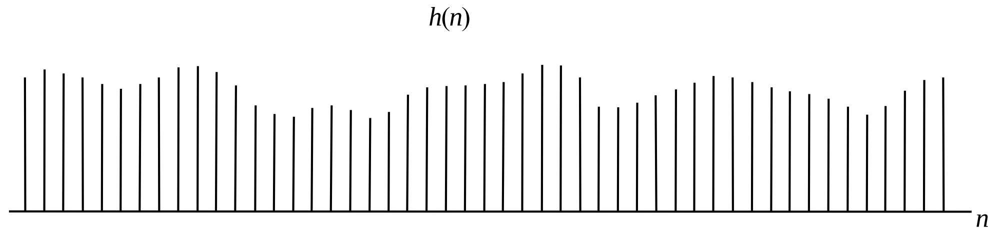
$$w(n) = \begin{cases} 1, & n = 0, \ \pm 1, \ \pm 2, \ \ldots, \ \pm M \\ 0, & |n| > M \end{cases}$$

as a rectangular window of length $2M+1$



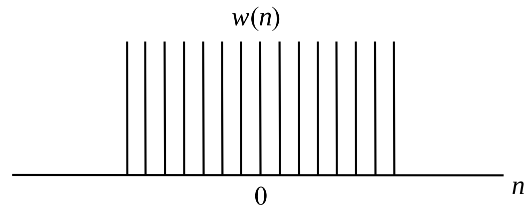
so that $\qquad g(n) = w(n) \cdot h(n) \qquad$ for all n

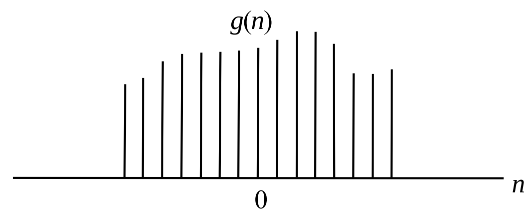
for example



windowing is a
multiplication process
in the time domain:

$$g(n) = w(n) \cdot h(n)$$





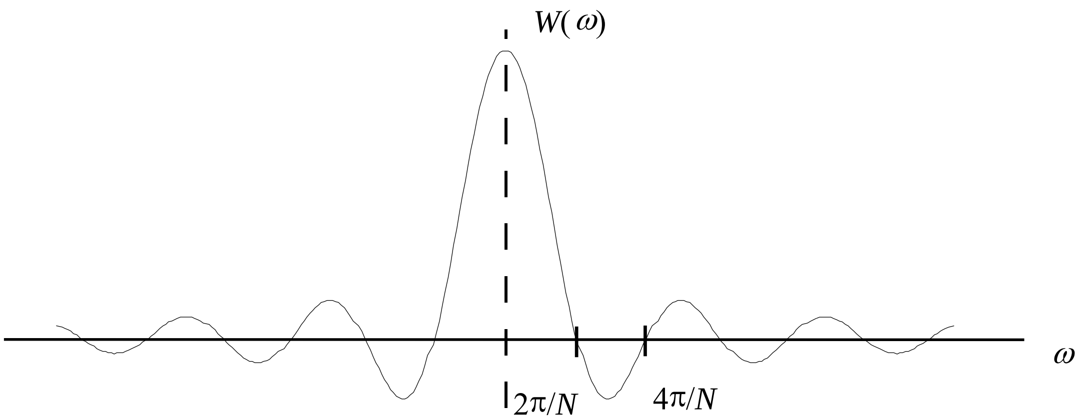
Therefore, in the frequency domain we have:

$$G(\omega) = W(\omega) * H(\omega)$$

and the spectrum of the truncated sequence is obtained by the convolution of the original spectrum with the spectrum of the rectangular window.

The spectrum of the window function is:

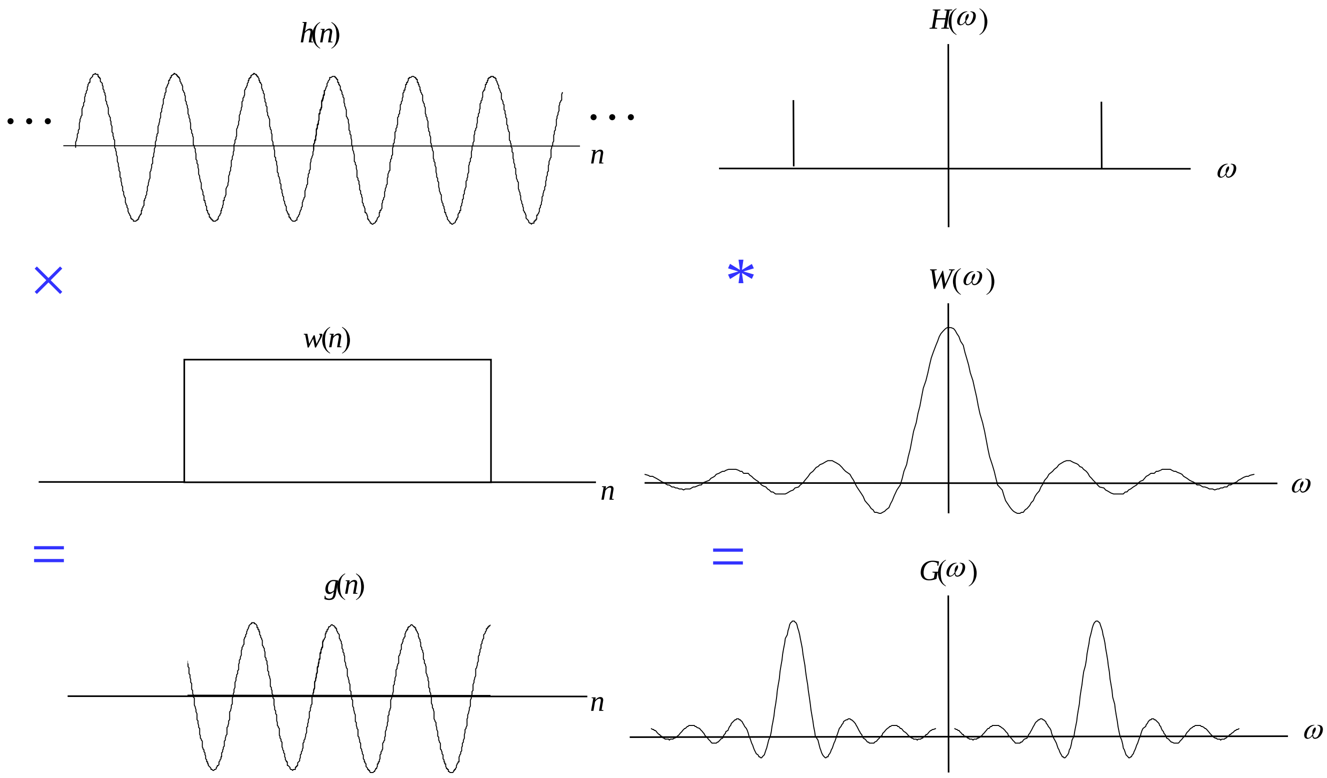
$$F\{w(n)\} = W(\omega) = \frac{\sin\left(\omega N/2\right)}{\sin\left(\omega/2\right)} \qquad \text{(where } N = 2M+1 \text{ in this case)}$$



As N increases, the mainlobe becomes narrower. In the limit, $W(\omega)$ approaches an impulse as $w(n)$ becomes infinitely long. That is:

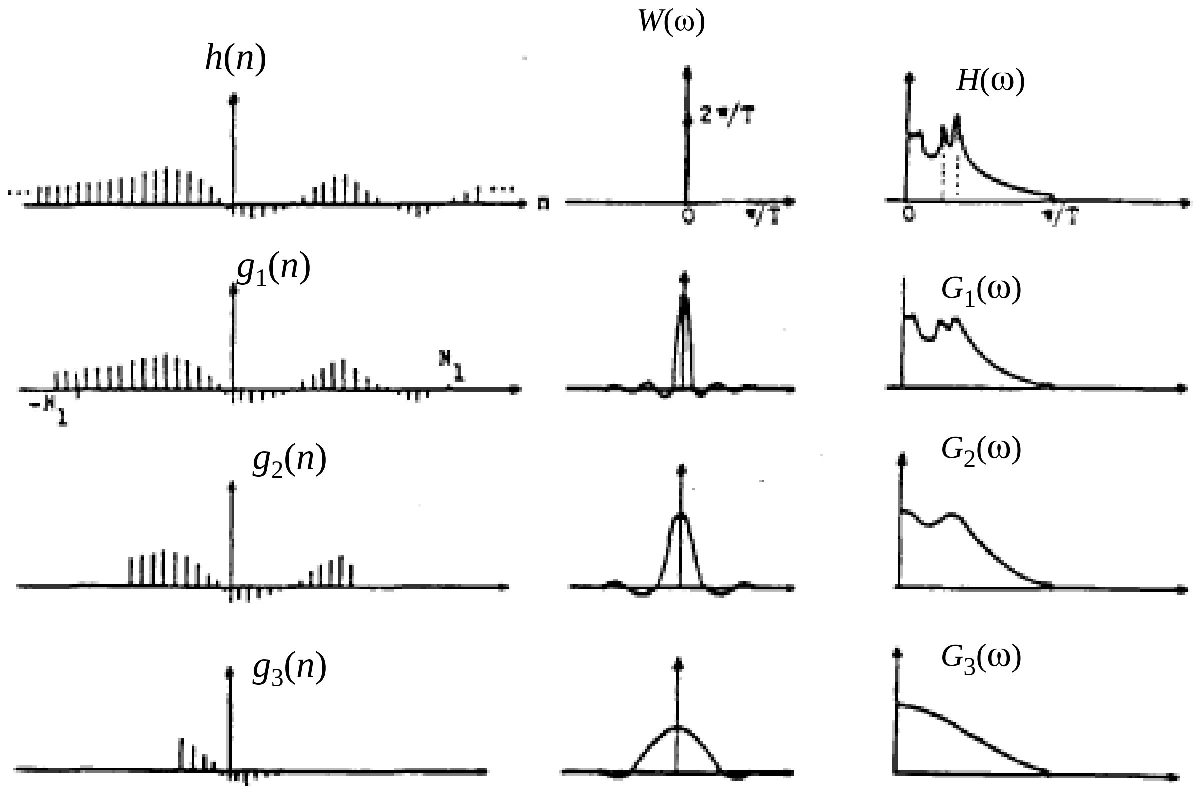
$$\lim_{N\to\infty} W(\omega) = \delta(\omega)$$

We can observe the phenomenon called "leakage" via the example below:



Note that the window smears the spectrum of the original signal in a manner determined by the window. Obviously, the length of the window determines the amount of spectral smearing.

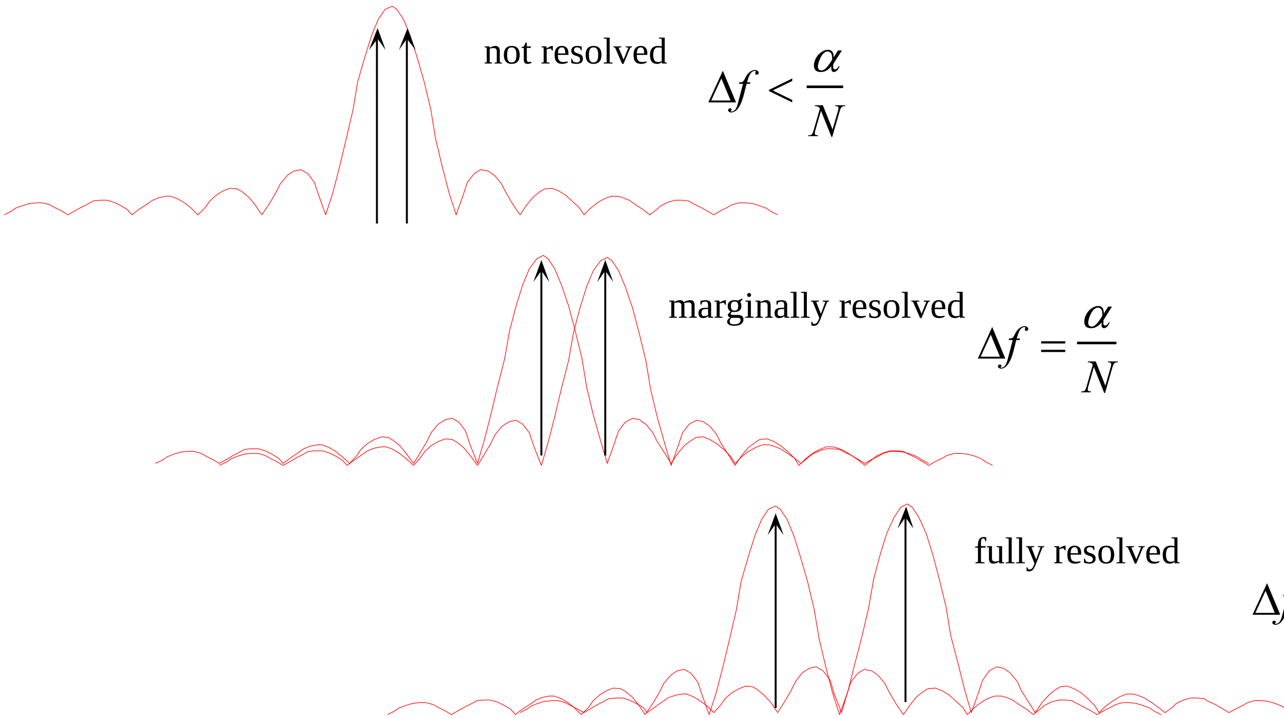
Now consider the further complications caused by leakage with the example below:



Since it appears that our ability to resolve fine spectral details is obscured by windows we can state that <u>**spectral resolution** is degraded by the windowing process</u>.

The same thing occurs for **spatial resolution** if we decrease the number of $\lambda/2$ spaced elements in an antenna array

Consider how we can “resolve” close frequency components:



$$\Delta f < \frac{\alpha}{N}$$

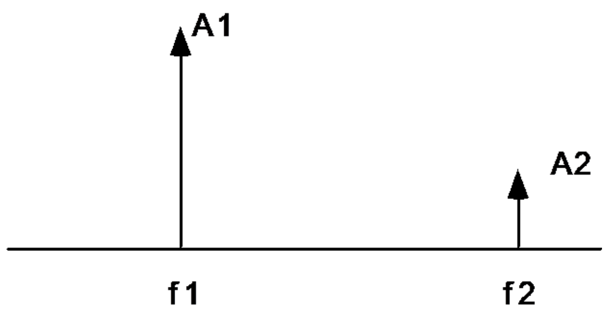
$$\Delta f = \frac{\alpha}{N}$$

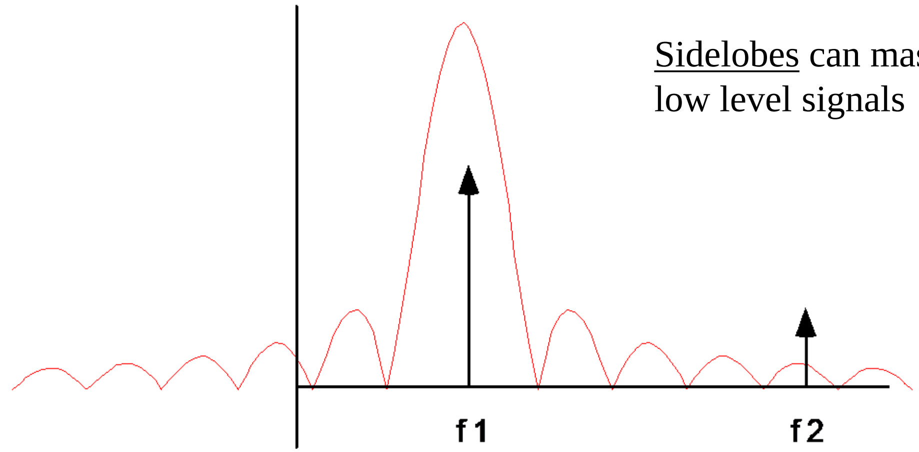
$$\Delta f > \frac{\alpha}{N}$$

α/N is the spectral width (bandwidth) of the window, where α depends on the particular window

Note: **resolution** is explicitly defined as $1/N$, but the resolving “separability” of two signals is a bit more complicated

Another important parameter related to windows is **dynamic range**. Dynamic range is primarily influenced by the type of window used.





Sidelobes can mask low level signals

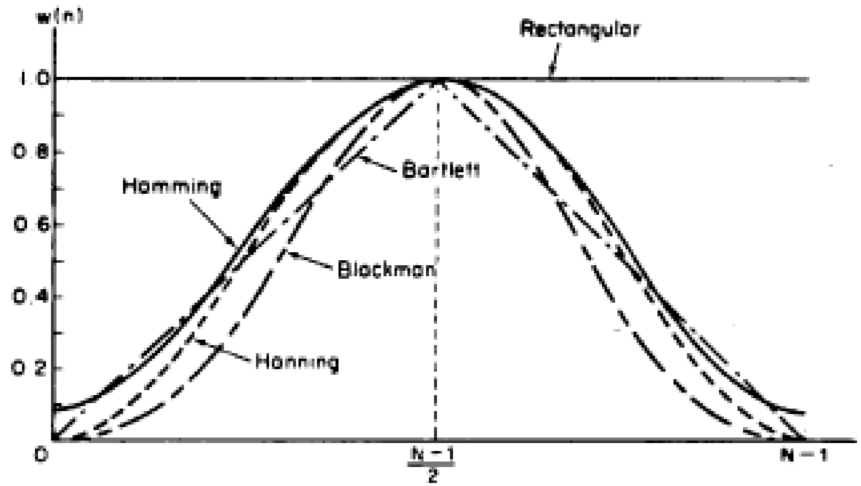
The rectangular window is only one of many possible window types used in Fourier analysis. Others are shown on the next page.

In summary, several choices are required for effective signal analysis:

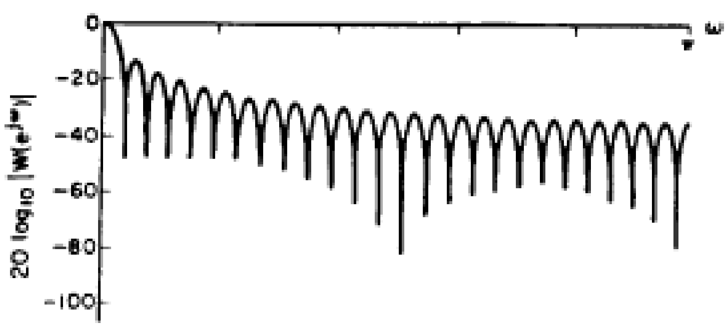
- Select window type to provide required dynamic range
- Select window length (if possible) to give required frequency separability
- Select FFT/DFT length (zero-padding) to provide a useful frequency grid

… and even more, but that requires getting into 844 material

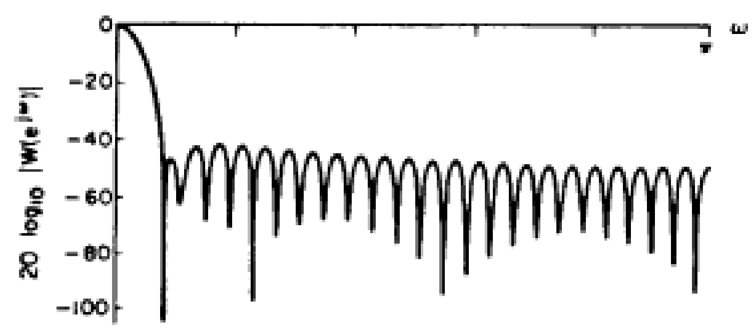
Here are some commonly used window types and their spectra:



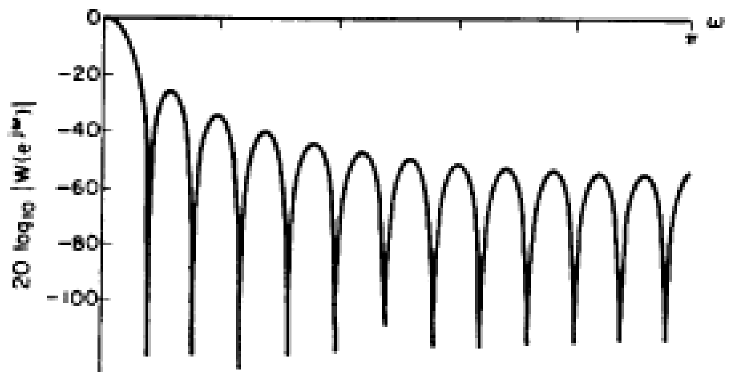
Rectangular



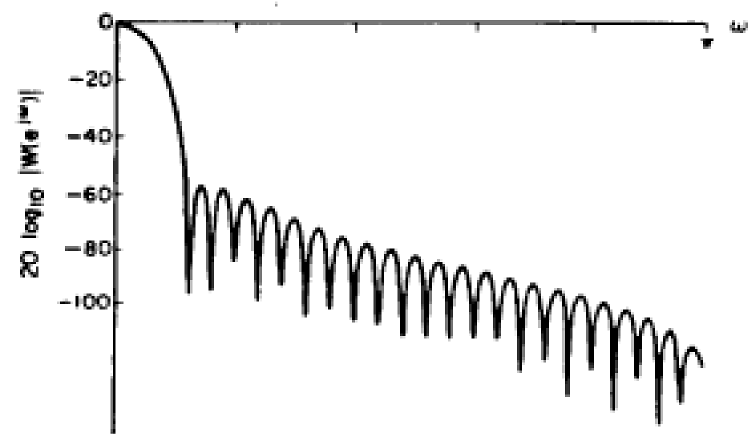
Hamming



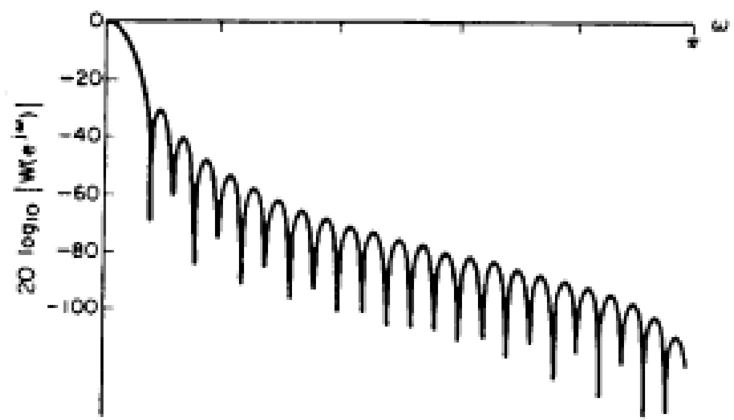
Bartlett:



Blackman:

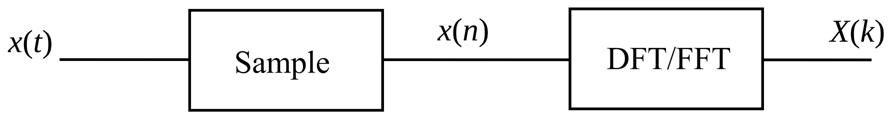


Hanning:



Continuous Time Signals

Given the following process:



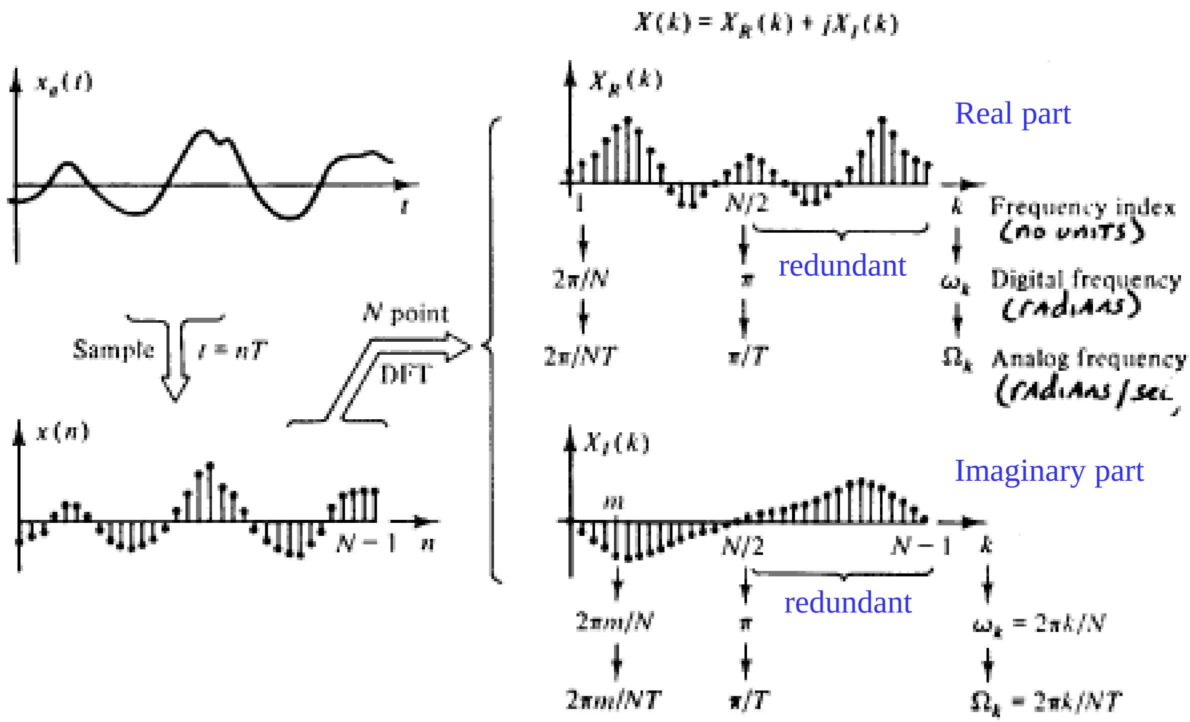
Recall that if $x(n)$ is real (as is sometimes the case), the following properties apply:

$X_i(k)$ is odd $\qquad |X(k)|$ is even

$X_r(k)$ is even $\qquad \angle X(k)$ is odd

$$\text{where} \quad X(k) = X_r(k) + jX_i(k)$$

When the analog signal $x(t)$ is sampled and the Fourier transform of $x(n)$ is taken (via the FFT or DFT), the axis in the transformed domain has several valid interpretations, as shown on the next page.



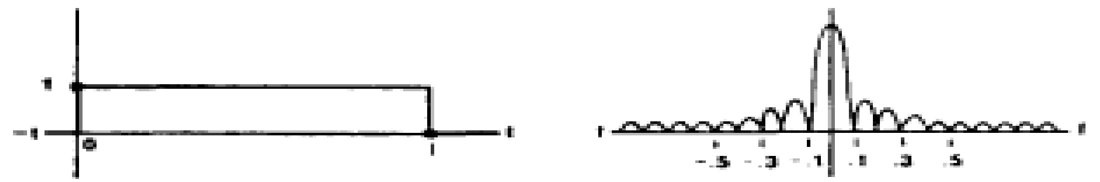
Note that all frequency components greater than *N*/2 are redundant. Also note that the frequency grid is:

$$\Delta\omega = \frac{2\pi}{N} \quad \text{radians}$$

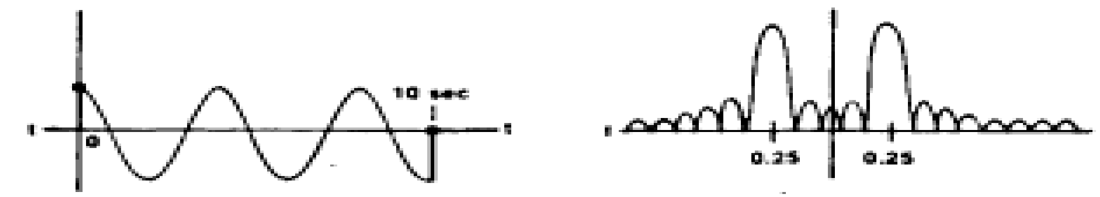
$$\Delta\Omega = \frac{2\pi}{NT} \quad \text{radians/sec}$$

TIME DOMAIN	FREQUENCY DOMAIN

Cosine signal

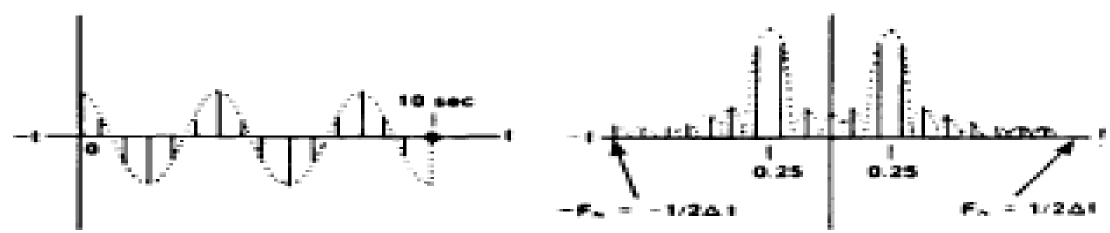


Rectangular window



Rectangular windowed cosine

Impulse train for sampling



Windowed and sampled cosine

12.1 DFT Analysis of Sinusoidal Signals

A further look at the effects of windowing

We will use a sinusoidal signal and windowing techniques to demonstrate that:

- Windowing smears or broadens the impulses in the theoretical Fourier representation and therefore the exact frequency is less well-defined (leakage)

- Windowing reduces the ability to resolve sinusoidal signals that are closely spaced in frequency

For example, let the input signal be:

$$x_c(t) = A_0 \cos(\Omega_0 t + \theta_0) + A_1 \cos(\Omega_1 t + \theta_1) \qquad -\infty < t < \infty$$

Hence, the sampled signal is

$$x(n) = A_0 \cos(\omega_0 n + \theta_0) + A_1 \cos(\omega_1 n + \theta_1) \qquad -\infty < n < \infty$$

where $\omega_0 = \Omega_0 T$ and $\omega_1 = \Omega_1 T$

The <u>windowed</u> signal is

$$v(n) = A_0\, w(n)\, \cos(\omega_0 n + \theta_0) + A_1\, w(n)\, \cos(\omega_1 n + \theta_1) \qquad -\infty < n < \infty$$

Now determine the DTFT of $v(n)$. Let $\theta_0 = \theta_1 = 0$ for convenience.

$$v(n) = A_0\, w(n) \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \right) + A_1\, w(n) \left(\frac{e^{j\omega_1 n} + e^{-j\omega_1 n}}{2} \right)$$

$$= \frac{A_0}{2} w(n)\, e^{j\omega_0 n} + \frac{A_0}{2} w(n)\, e^{-j\omega_0 n} + \frac{A_1}{2} w(n)\, e^{j\omega_1 n} + \frac{A_1}{2} w(n)\, e^{-j\omega_1 n}$$

So that the DTFT is

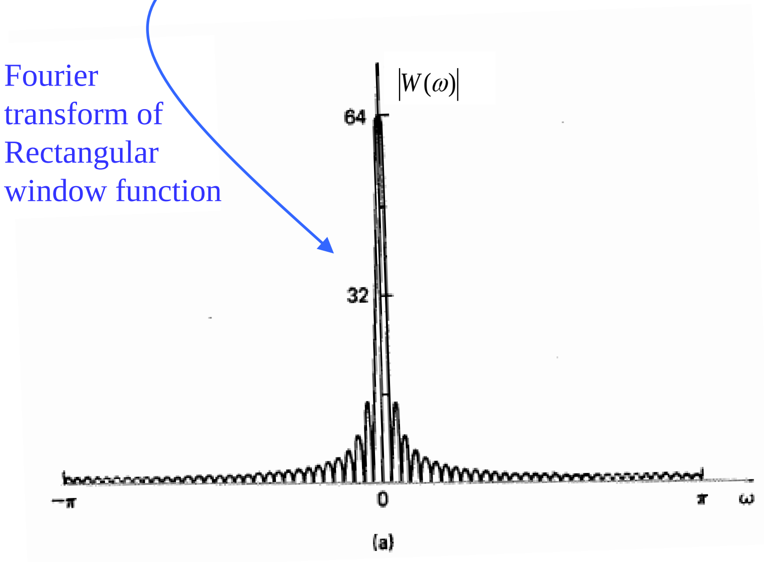
$$V(\omega) = \frac{A_0}{2} W(\omega - \omega_0) + \frac{A_0}{2} W(\omega + \omega_0) + \frac{A_1}{2} W(\omega - \omega_1) + \frac{A_1}{2} W(\omega + \omega_1)$$

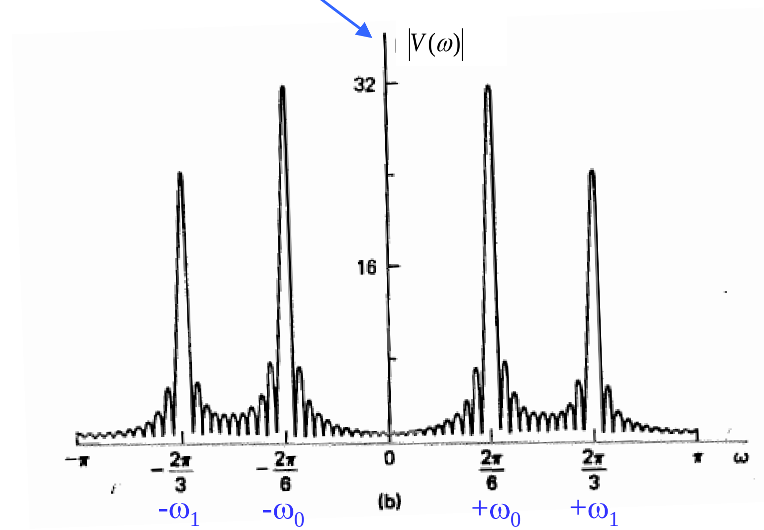
Example: $x(n)=\cos\left(\frac{2\pi n}{6}\right)+0.75\cos\left(\frac{2\pi n}{3}\right) \qquad -\infty \le n \le \infty$

$$w(n)=\begin{cases}1, & 0\le n\le 63\\ 0, & \text{otherwise}\end{cases}$$

$v(n)=x(n)\,w(n)$

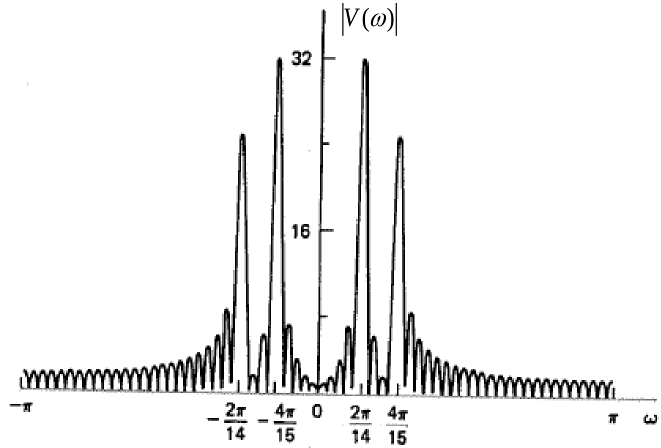
Fourier transform of Rectangular window function



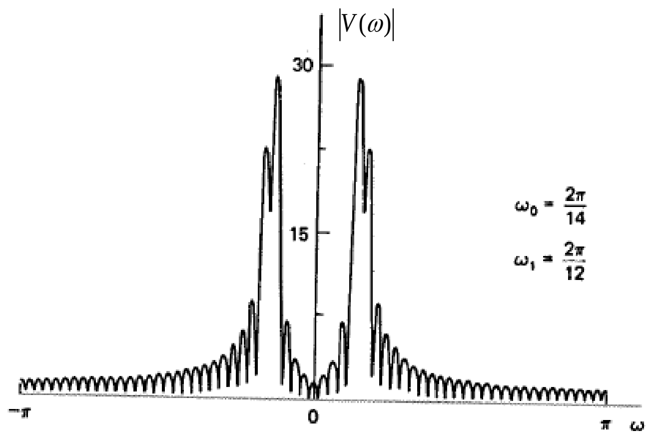


Note: $V(\omega)$ appears as the Fourier transform of the window function, weighted and replicated at $\pm\,\omega_0$ and $\pm\,\omega_1$

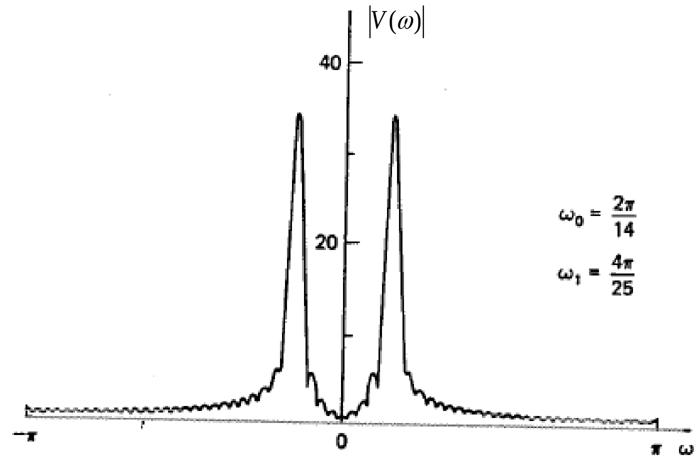
Now let the sinusoids move closer together. Note the <u>resolution is dictated by window length *N*</u>.



(c)



(d)



(e)

In summary, the two primary effects on a spectrum as a result of applying a window to the signal are:

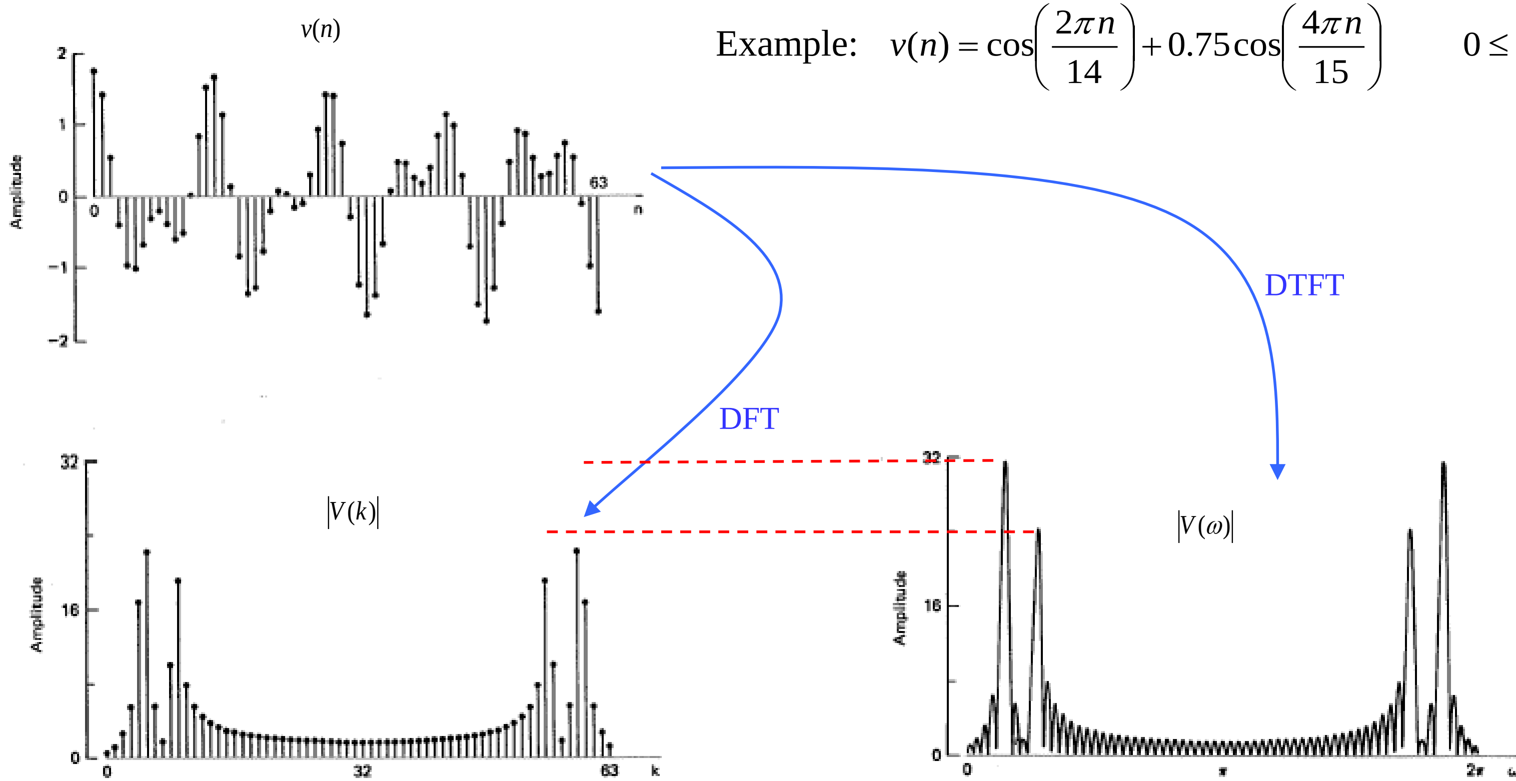
- degraded resolution (influenced by window length)
- leakage (influenced by relative amplitudes of mainlobe and sidelobes)

__The effect of spectral sampling__

Spectral sampling, as imposed by the DFT, can sometimes produce misleading results.

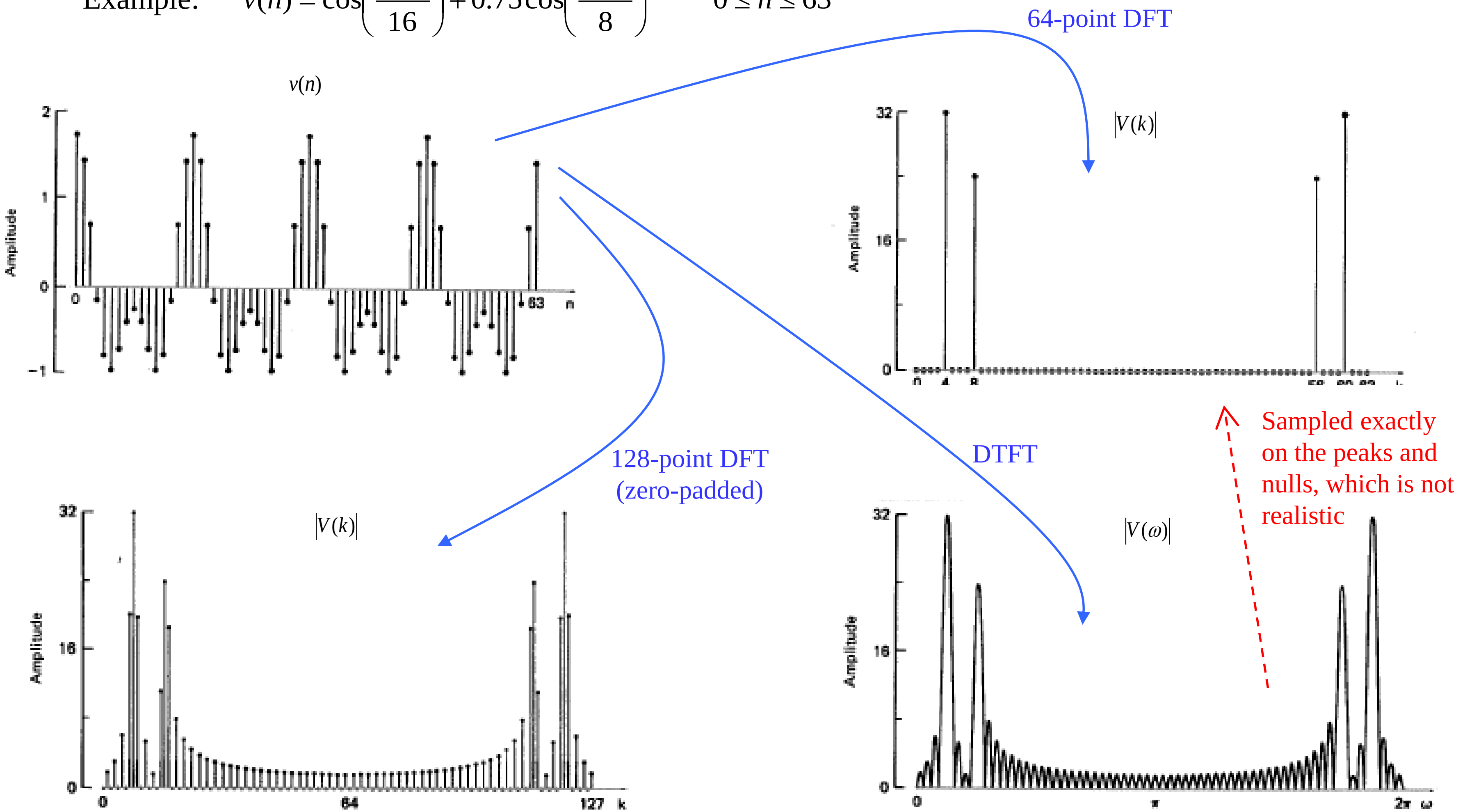
Notice in the example below that the locations of the peaks in the DFT values do not necessarily coincide with the exact frequency locations of the peaks in the DTFT since the __spectral peaks can lie between spectral samples__. Also, the relative amplitudes of peaks in the DFT will not necessarily reflect the relative amplitudes of the true spectral peaks.

Example: $v(n) = \cos\left(\frac{2\pi n}{14}\right) + 0.75\cos\left(\frac{4\pi n}{15}\right) \qquad 0 \le n \le 63$



In this other case, the frequency of the sinusoids coincide exactly with the DFT samples.

Example: $$v(n) = \cos\left(\frac{2\pi n}{16}\right) + 0.75\cos\left(\frac{2\pi n}{8}\right) \qquad 0 \le n \le 63$$



12.2 Summary

- For sequences shorter than the DFT/FFT length, the periodic assumption of the DFT can be used to "wrap around" the sequence if needed
- For sequences in which we must consider only a finite segment, windowing effects occur
    1. Mainlobe width of the window's frequency response limits resolution
    2. Sidelobes of the window limit dynamic range
    3. The separability of closely-spaced signals, or those with very different amplitudes, can be hindered
- An N-point real valued signal corresponds to N unique frequency values ($N/2$ real and $N/2$ imaginary). Note: if the N-point signal is complex, then there is no longer redundancy in the frequency domain.
- The discrete frequency representation provided by a DFT may miss important details of the continuous DTFT.
    - Zero padding can improve the visibility of the DFT (approach closer to DTFT response), though it cannot improve upon the resolution imposed by window length.
    - Note: there are "super-resolution" techniques that exist (MUSIC, ESPRIT, RISR, etc.), but they require more sophisticated signal processing

That's one I developed