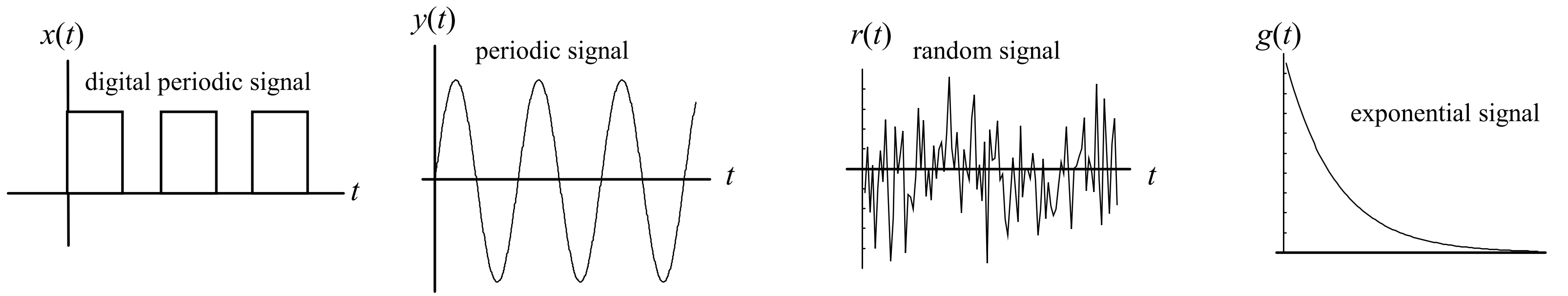
Course Notes 2 – Discrete Time Signals and Systems

2.0 Introduction

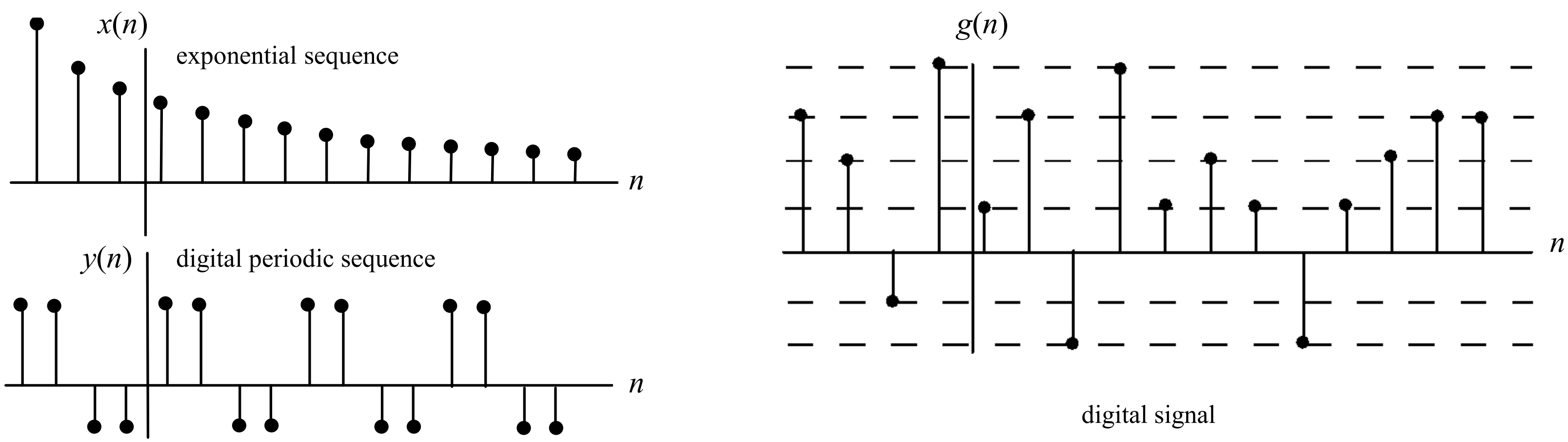
Classification of Signals

We can classify signals according to their independent variable (usually time) and their dependent variable (usually amplitude) as either continuous or discrete.

Continuous-Time Signals:



Discrete-Time Signals:

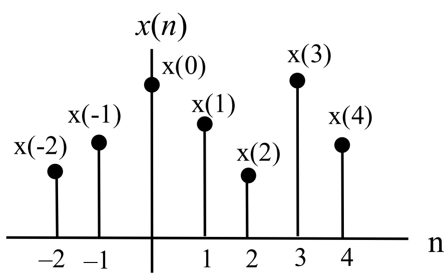


Signal Representations

1. Math (analytical) model:

$$x(n) = a^n \cos \omega_0 n$$

2. Graphical model:



3. Indexed list of numbers:

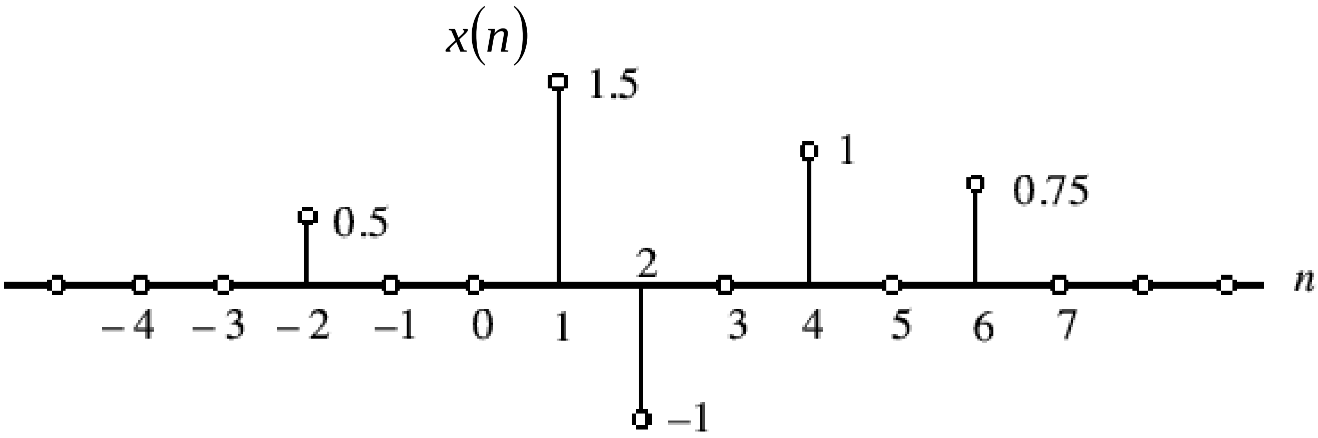
n	$x(n)$
0	2
1	3
2	9
3	0
4	1
5	2
⋮	⋮
N	2

4. Superposition of delayed impulses:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) = \sum_{k=-2}^{4} x(k)\delta(n-k)$$

$$= x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + \ . \ . \ . \ + x(4)\delta(n-4)$$

<u>Example</u>: An arbitrary sequence can be represented in the time-domain as a weighted sum of some basic sequence and its delayed (advanced) versions



$$x(n) = 0.5\delta(n+2) + 1.5\delta(n-1) - \delta(n-2) + \delta(n-4) + 0.75\delta(n-6)$$

Another representation of sequences, which is used in some textbooks is a numerical alternative to the graphical representation using set notation

$$x(n) = \{3, -1, \underset{\uparrow}{-2}, 5, 0, 4, -1\}$$

A finite 7-point sequence

Arrow indicates $n = 0$

$$x(n) = \{\underset{\uparrow}{0}, 2, 5, 4\}$$

A finite 4-point sequence

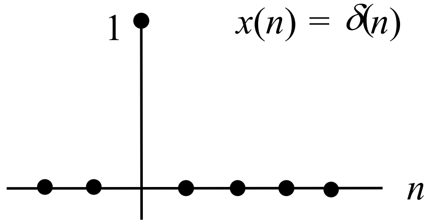
$$x(n) = \{\ldots, 0, \underset{\uparrow}{0}, 1, 2, 9, 7, 4, 0, \ldots\}$$

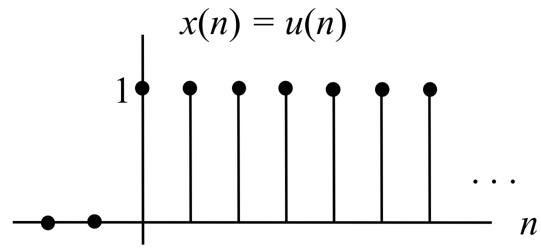
infinite sequence (two sided)

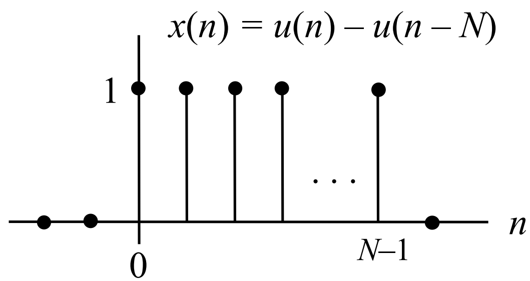
$$x(n) = \{\underset{\uparrow}{5}, 6, 8, 2, 9, 7, 3, 0, 0, \ldots\}$$

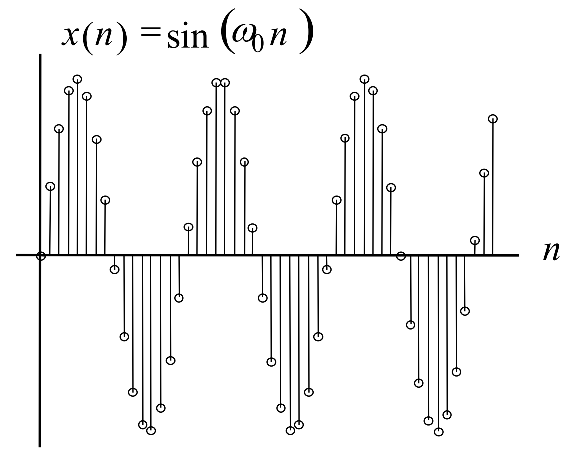
infinite sequence (one-sided)

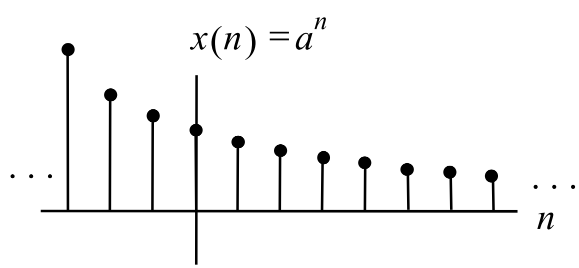
2.1 Discrete-Time Signals: Sequences











The sinusoidal sequence has special significance, so we will take a closer look at it

Other interesting classification of sequences based on time-domain behavior are:

- **Finite-length sequence**. (The length of a finite-length sequence can be arbitrarily increased by zero-padding, *i.e.*, by appending it with zeros).

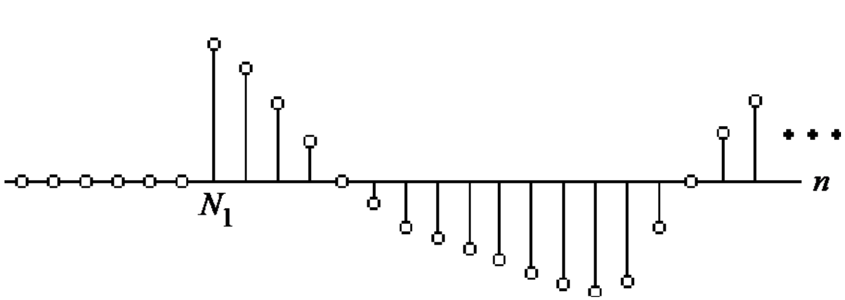
- A **right-sided sequence** $x(n)$ has zero-valued samples for $n < N_1$

- If $N_1 \geq 0,$ a right-sided sequence is called a **causal sequence**

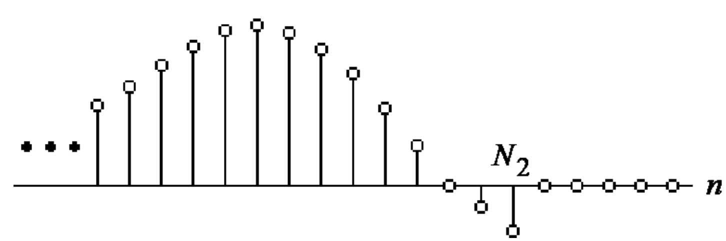
- A **left-sided sequence** $x(n)$ has zero-valued samples for $n > N_2$

- If $N_2 \leq 0,$ a left-sided sequence is called a **anti-causal sequence**

right-sided sequence

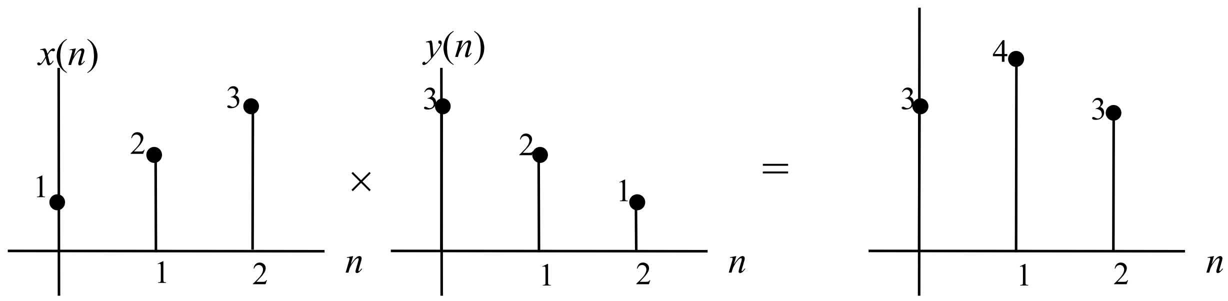


left-sided sequence

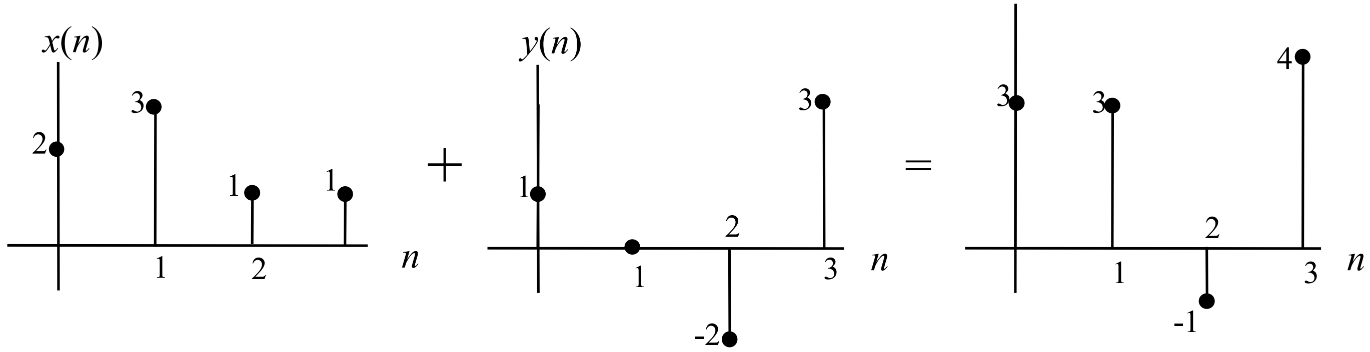


Operations on Sequences (also think of as vectors)

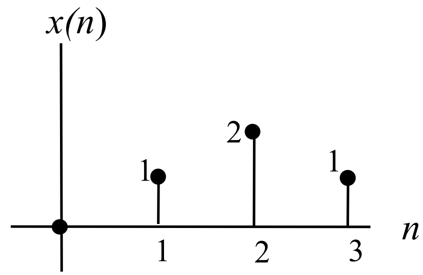
1. Multiplication: $y(n)\,x(n)$
   (requires both sequences to be same length. Zero filling is implied)

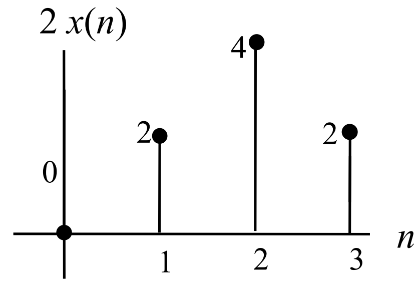


2. Addition: $y(n) + x(n)$
   (requires both sequences to be same length. Zero filling may be necessary)

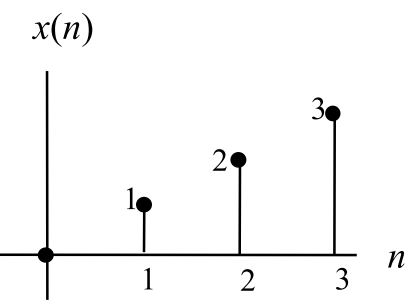


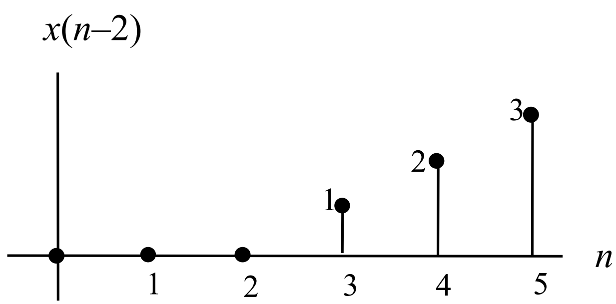
3. <u>Scalar Multiplication</u>: $a\,x(n)$

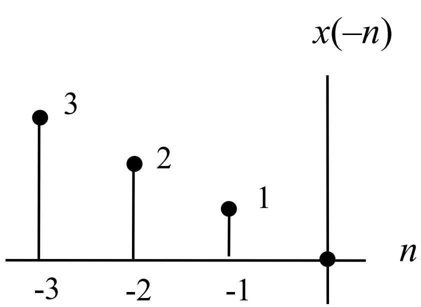


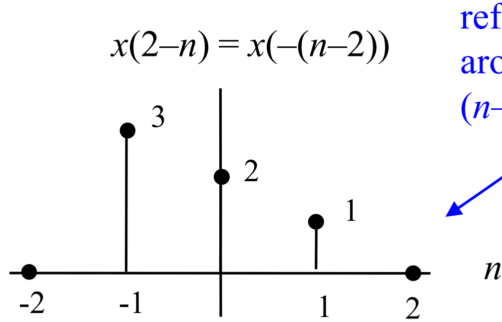


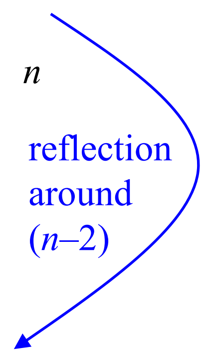
4. <u>Translation and Reflection</u>







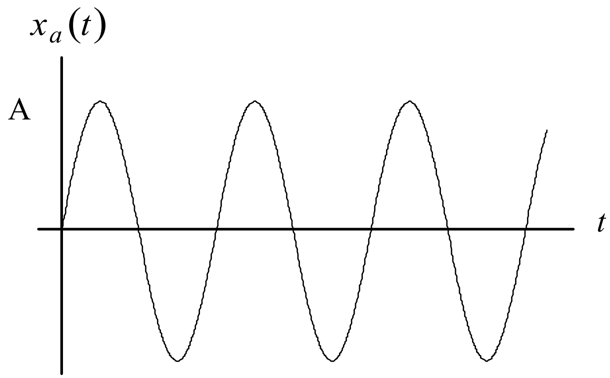




n	$x(n)$	n-2	$x(n$-2)	-n	x(-n)	2-n	x(2-n)
-3	0	-5	0	3	3	5	0
-2	0	-4	0	2	2	4	0
-1	0	-3	0	1	1	3	3
0	0	-2	0	0	0	2	2
1	1	-1	0	-1	0	1	1
2	2	0	0	-2	0	0	0
3	3	1	1	-3	0	-1	0
4	0	2	2	-4	0	-2	0
5	0	3	3	-5	0	-3	0

The Concept of "Digital" Frequency

Begin by considering the analog sinusoid:

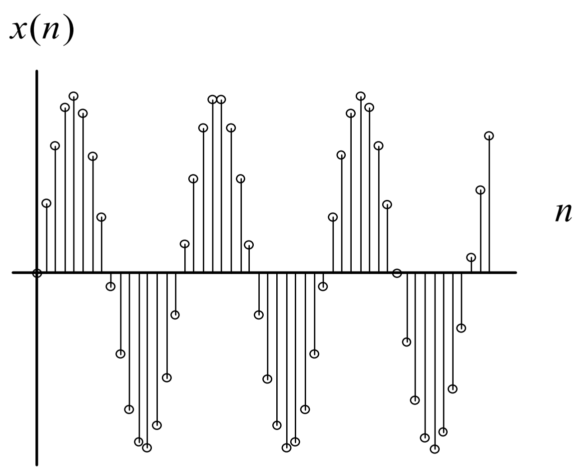


$$x_a(t) = A\cos(\Omega_o t), \qquad -\infty \le t \le \infty$$

where **Ω** is the continuous frequency variable (radians/sec)

$\Omega_o = 2\pi F_o$ (F_o is the frequency in Hz or cycles/sec)

Now compare with the discrete sinusoid:



$$x(n) = A\cos(\omega_o n), \qquad -\infty \le n \le \infty$$

where ω is the discrete frequency variable (radians/sample)

$\omega_o = 2\pi f_o$ (f_o is the frequency in cycles/sample)

How are Ω and ω related?

The continuous and discrete frequency variables are related through the sampling process as follows:

$$x(n) = x(t)\Big|_{t=nT} = A\cos(\Omega_o t)\Big|_{t=nT} = A\cos(\Omega_o nT) = A\cos(\omega_o n)$$

sample $x(t)$ every T-seconds

Note that if: $\cos(\omega_o n) = \cos(\Omega_o nT)$

then

$$\omega = \Omega T$$

This expression relates the essential concepts of the continuous and discrete frequency domains (we can likewise write $f = FT$ by dividing both sides by 2π)

Since T is the interval between samples of the continuous-time signal, then $1/T$ is the sampling frequency. That is,

$F_s = \frac{1}{T}$ and therefore, $f = \frac{F}{F_s}$ is often called the "normalized frequency"

Sampling frequency (or rate) in samples/second

There is an important relationship between signals and sequences which we will rely upon throughout this course. This relationship is described below:

$$x(t)\Big|_{t=nT} = x(nT) \Rightarrow x(n)$$

analog signal
(continuous-time)

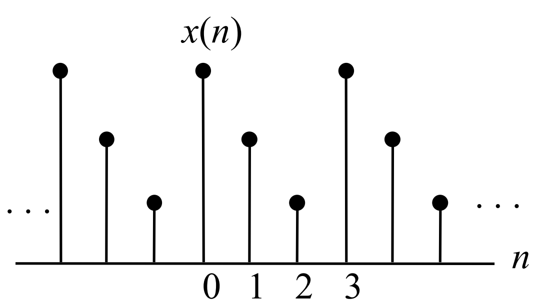
time series
(still continuous-time)

sequence
(discrete-time)

Periodic Discrete-Time Signals and Sinusoids

It is an interesting property of discrete sinusoids that they need not be periodic. To see why this is true, let's define the concept of periodicity in discrete-time signals.

Periodic sequences have the property, $x(n \pm kN) = x(n)$



In this case $N = 3$

In order for sinusoidal sequences to be periodic, it is necessary for the following condition to apply:

$$\cos\left(\omega_o\left[n+N\right]\right)=\cos\left(\omega_o n\right)$$

using a trigonometric identity:

$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$

$$\cos\left(\omega_o n\right)\cos\left(\omega_o N\right)-\sin\left(\omega_o n\right)\sin\left(\omega_o N\right)=\cos\left(\omega_o n\right)$$

$\cos(\omega_o N) \rightarrow 1$ $\quad$ $\sin(\omega_o N) \rightarrow 0$

The periodicity requirement will be met as long as,

$$\omega_o N = 2\pi k \qquad (k, N \text{ integers})$$

then, $$\omega_o = \frac{2\pi k}{N}$$

or, $$f_o = \frac{k}{N}$$

the "fundamental frequency" (f_o)

# of full sinusoid cycles to perfectly repeat (k)

period (in samples), # of samples to perfectly repeat (N)

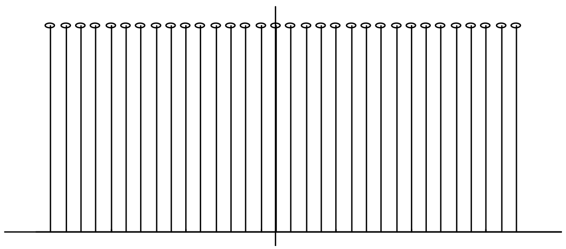
<u>Significance</u>: The sinusoidal sequence is periodic if f_o is a rational number (ratio of integers).

The concept of the sinusoidal sequence can be illustrated by taking the sequence

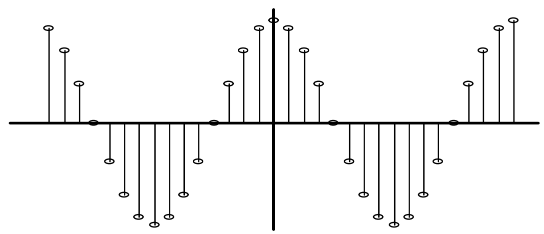
$$x(n) = \cos(\omega_0 n)$$

and allowing ω_o to increase from $\omega_o = 0$ to $\omega_o = 2\pi$. We can observe that increasing ω_o does not necessarily result in a higher rate of oscillation.

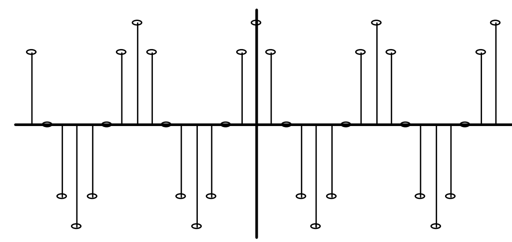
a. $x(n) = \cos(0 \cdot n)$



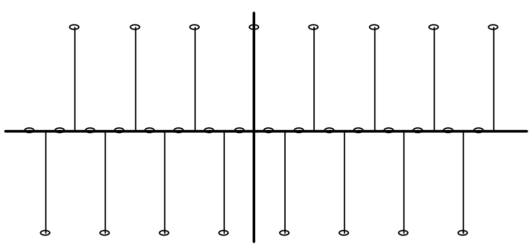
b. $x(n) = \cos\left(2\pi \cdot \frac{n}{16}\right)$



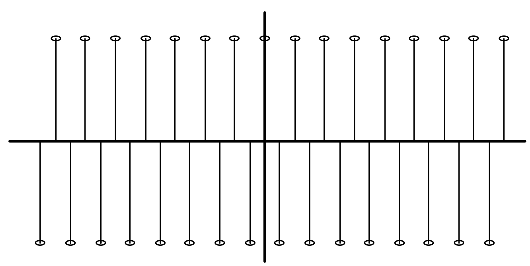
c. $x(n) = \cos\left(2\pi \cdot \frac{n}{8}\right)$



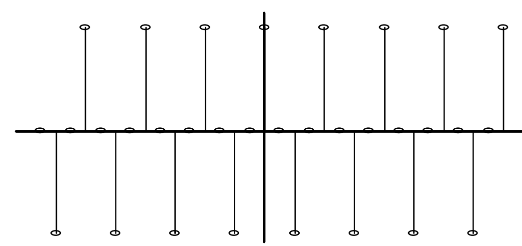
d. $x(n) = \cos\left(2\pi \cdot \frac{n}{4}\right)$



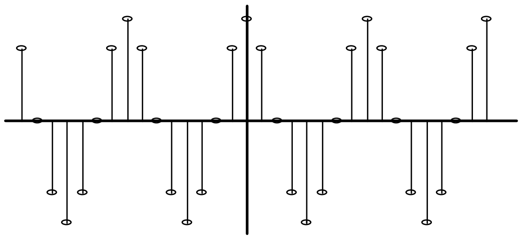
e. $x(n) = \cos\left(2\pi \cdot \frac{n}{2}\right)$



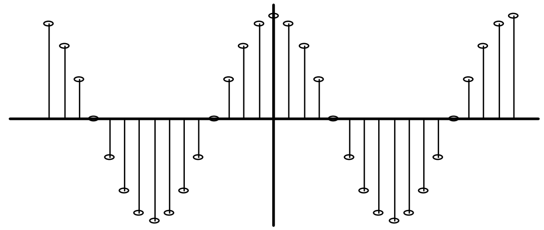
f. $x(n) = \cos\left(2\pi \cdot \frac{3}{4} n\right)$



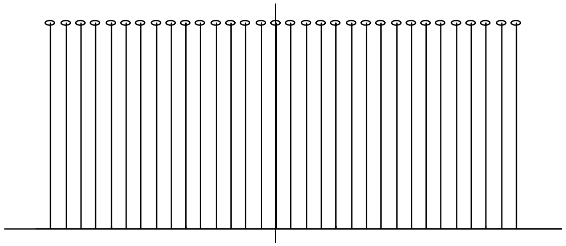
g. $x(n) = \cos\left(2\pi \cdot \frac{7}{8} n\right)$

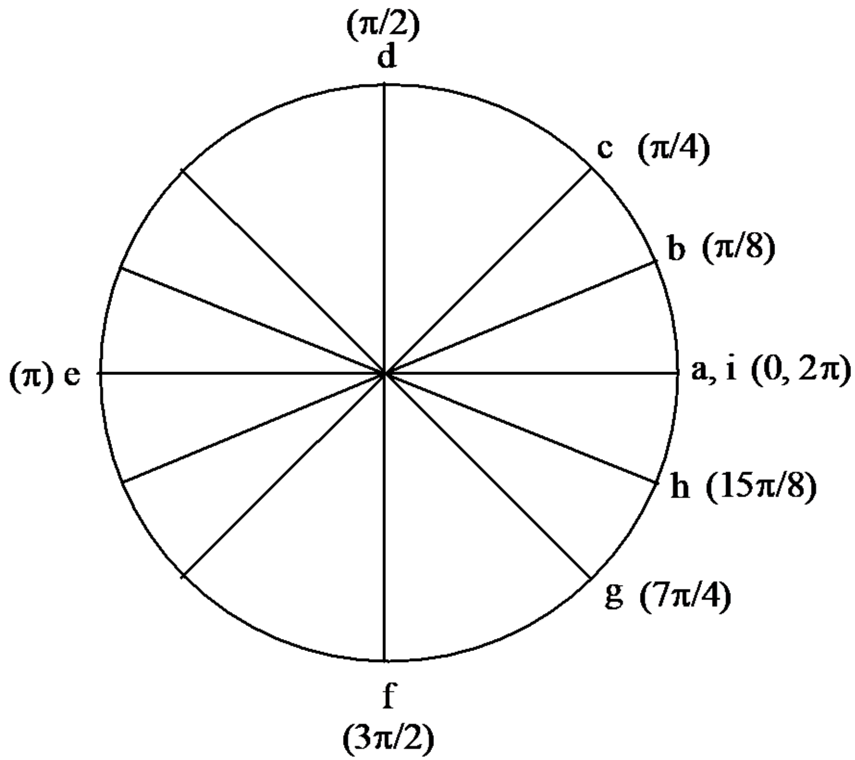


h. $x(n) = \cos\left(2\pi \cdot \frac{15}{16} n\right)$



i. $x(n) = \cos(2\pi \cdot n)$





Digital Frequency Concepts

- A frequency ω_o in the neighborhood of $\omega = 2\pi k$ is indistinguishable from a frequency in the neighborhood of $\omega = 0$

- A frequency ω_o in the neighborhood of $\omega = \pi(2k + 1)$ is indistinguishable from a frequency in the neighborhood of $\omega = \pi$.

- Frequencies in the neighborhood of $\omega = 2\pi k$ are called **low frequencies**.

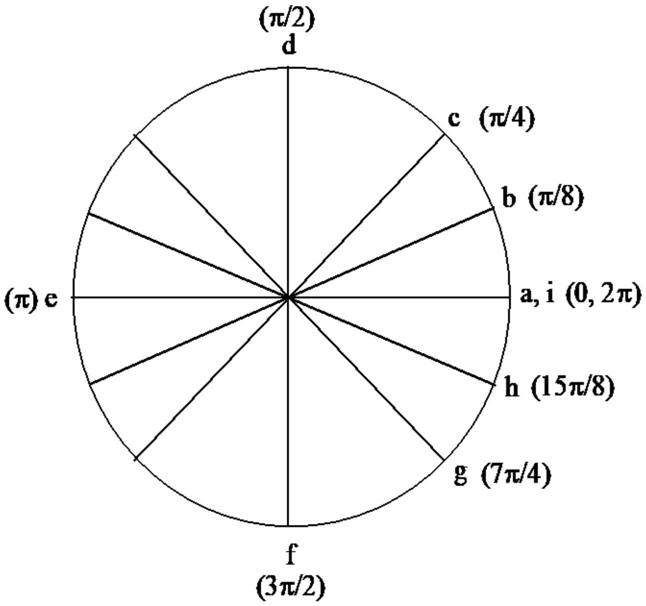
- Frequencies in the neighborhood of $\omega = \pi(2k + 1)$ are called **high frequencies**.

- The following is a low frequency signal:

$$v_1[n] = \cos(0.1\pi n) = \cos(1.9\pi n)$$

- The following is a high frequency signal:

$$v_2[n] = \cos(0.8\pi n) = \cos(1.2\pi n)$$



Complex Exponential Sequences are also sinusoids (from Euler's identity)

$$e^{j\omega_o n} = \cos(\omega_o n) + j\sin(\omega_o n)$$

This is a complex sinusoidal sequence. When are complex exponential sequences periodic?

$$e^{j\omega_o n} \stackrel{?}{=} e^{j\omega_o (n+N)}$$

$$= e^{j\omega_o n} \cdot e^{j\omega_o N}$$

need this equal to 1 for equality (*i.e.*., periodicity) to hold.

Since $e^{j2\pi k} = 1$ for k an integer,

then we need: $\omega_o N = 2\pi \cdot k$ (same as before)

so periodicity occurs when $\frac{k}{N}$ is rational

Some Useful Identities

$e^{j\pi/2} = j$

$e^{j2\pi n} = 1$ for all n

$e^{j\pi n} = -1$ for n odd

$e^{j(\omega_o + 2\pi)n} = e^{j\omega_o n}$ for all n

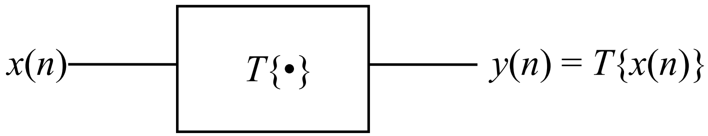
$e^{j(\omega_o + \pi)n} = -e^{j\omega_o n}$ for n odd

The following table summarizes the important differences between continuous and discrete complex exponentials.

• Distinct signals for distinct values of Ω_o.	• Identical signals for exponentials at frequencies separated by 2π
• Periodic for any choice of Ω_o.	• Periodic only if $$\omega_o = \frac{2\pi k}{N}$$ (k and N are integers)
• Fundamental frequency = $\Omega_o = 2\pi F_o$	• Fundamental frequency of the sequence is: $$\frac{\omega_o}{k} = \frac{2\pi}{N} \quad \text{or} \quad \frac{f_o}{k} = \frac{1}{N}$$ (assuming k and N have no common factors)
• Fundamental Period $$T_o = \frac{2\pi}{\Omega_o}$$ or since $\Omega_o = 2\pi F_o$, then $$T_o = \frac{1}{F_o}$$	• Fundamental period of the sequence is: $$N = \frac{2\pi k}{\omega_o}$$ or since $\omega_o = 2\pi f_o$ $$N = \frac{k}{f_o}$$

2.2 Discrete-Time Systems

A discrete-time system transforms a sequence $x(n)$ into $y(n)$ in a manner defined by $T\{\bullet\}$



$T\{\bullet\}$ defines a rule or mapping. Some examples:

- A list

$$
\begin{aligned}
y(1) &= T\{x(1)\} \\
y(2) &= T\{x(2)\} \\
y(3) &= T\{x(3)\} \\
&\vdots \\
y(N) &= T\{x(N)\}
\end{aligned}
$$

- A mathematical rule

$$y(n) = x^2(n) \qquad \text{square law device}$$

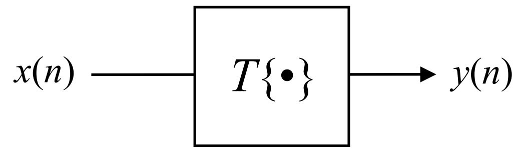
- A difference equation

$$y(n) = \frac{1}{2}\left[x(n) + x(n-1)\right] \qquad \text{digital filter}$$

by placing restrictions on $T\{\bullet\}$ we can form different classes of discrete-time systems.

Classification of DT Systems

1. Static vs. Dynamic Systems



- Static System

$y(n)$ is dependent upon $x(n)$ only and not upon $y(n-k)$ or $x(n-k)$ where k is any integer other than zero. Also called a “memoryless system”.

for example: $y(n) = a\,x(n) + b\,x^3(n)$

- Dynamic System

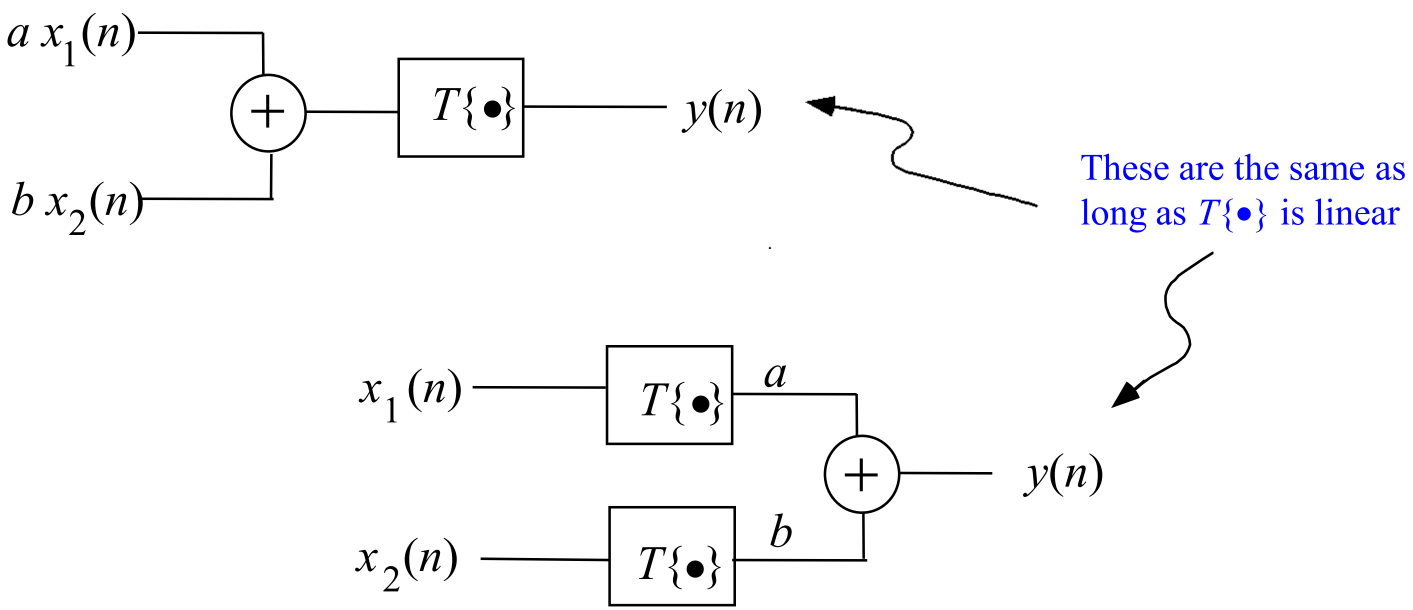
The system contains memory

for example: $y(n) = x(n) + 3\,x(n-1)$

2. Linearity - The most significant characteristic of discrete-time systems is linearity. The principle of superposition as it applies to discrete-time systems can be stated as follows:

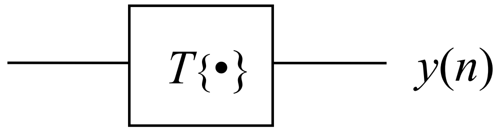
$$T\{a\,x_1(n) + b\,x_2(n)\} = a\,T\{x_1(n)\} + b\,T\{x_2(n)\}$$
$$= a\,y_1(n) + b\,y_2(n)$$

Stated graphically, linearity is a relatively straightforward concept



In order to more precisely characterize the transformation $T\{\bullet\}$ for linear systems, let the input signal be an arbitrary sequence.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$



$$y(n) = T\left\{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right\} = \sum_{k=-\infty}^{\infty} x(k)\underbrace{T\{\delta(n-k)\}}$$

response of system to an impulse at $n = k$

define: $h_k(n) = T\{\delta(n-k)\}$ as the "impulse response" of the linear system

therefore, $y(n) = \sum_{k=-\infty}^{\infty} x(k)h_k(n)$ is the "superposition sum"

With only linearity imposed, the impulse response is dependent upon both k and n.
Linear systems are completely specified by $h_k(n)$, the set of responses to $\delta(n-k)$

3. Shift (or Time) Invariance - Simply stated, shift (or time) invariance means that the system reacts identically to a particular input signal, no matter when the input signal is applied. This is a further restriction on the linear system, however it is quite significant since most linear systems have this property.

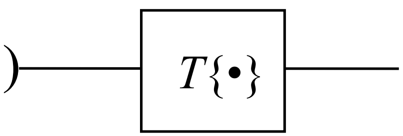
Time invariance implies that if $y(n) = T\{x(n)\}$

then $y(n-n_o) = T\{x(n-n_o)\}$

This in turn implies $T\{\delta(n-k)\} = h_k(n) = h(n-k)$ (That is, the impulse response does not change with time)

For a linear time invariant (LTI) system with arbitrary input:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$



$$y(n) = T\left\{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right\}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)T\{\delta(n-k)\}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

is called the "convolution sum"

We have discussed linearity and time invariance. There are two other properties that are important in our discussion of discrete-time systems:

4. Causality - A system is causal if $y(n_o)$, the sample output at n_o, depends upon the input sequence $x(n)$ only for $n \le n_o$. That is, the system cannot respond to an input signal before it is applied to the system.

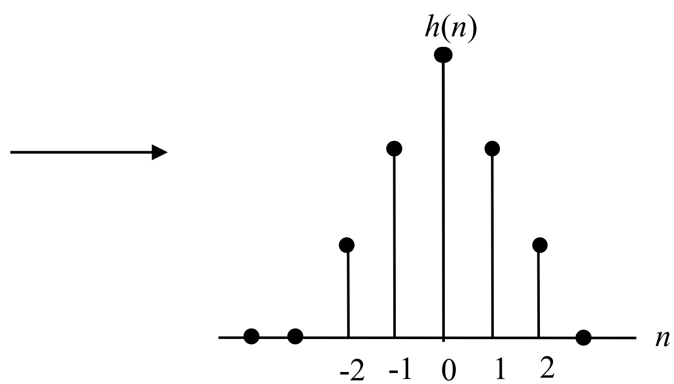
For LTI Systems, causality implies the impulse response has

$$h(n) = 0, \quad \text{for } n < 0$$

The output of a system is dependent upon present and past inputs, but not on future inputs.

> Note that causality is only inviolate in continuous time systems. It is less of an issue with discrete-time systems, especially non-real time systems.

A simple example of a non-causal system is shown on the right



5. <u>Stability</u> - A system $T\{\bullet\}$ is stable if every bounded input produces a bounded output (BIBO stability)

That is, if

$$|x(n)| \le B \qquad -\infty \le n \le \infty$$

where B is finite, then

$$|y(n)| \le C \qquad -\infty \le n \le \infty$$

where C is also finite.

As a consequence of this definition a system with impulse response,

$$h(n) = u(n)$$

is unstable, but a system with impulse response

$$h(n) = a^n u(n), \qquad 0 < \mathrm{a} < 1$$

is stable. Why?

For the special case of LTI systems, stability implies

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

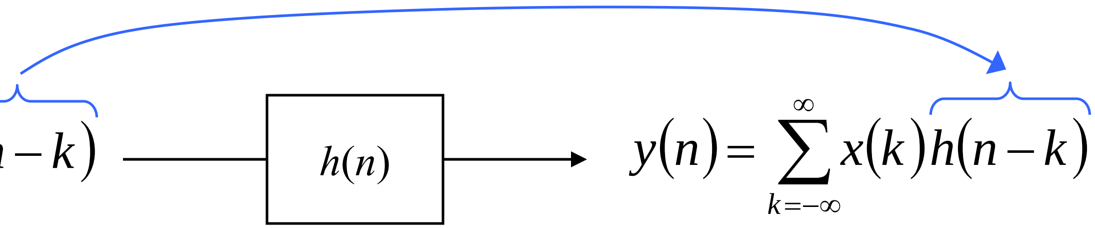
$$\sum_{k=0}^{\infty} a^n = \frac{a^0(1-a^\infty)}{1-a} = \frac{1}{1-a} \quad \text{only if}$$

2.3 Linear Time-Invariant Systems

Discrete-Time Convolution

We will focus attention almost exclusively on LTI systems. These systems are described by their impulse response and they obey convolution.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$



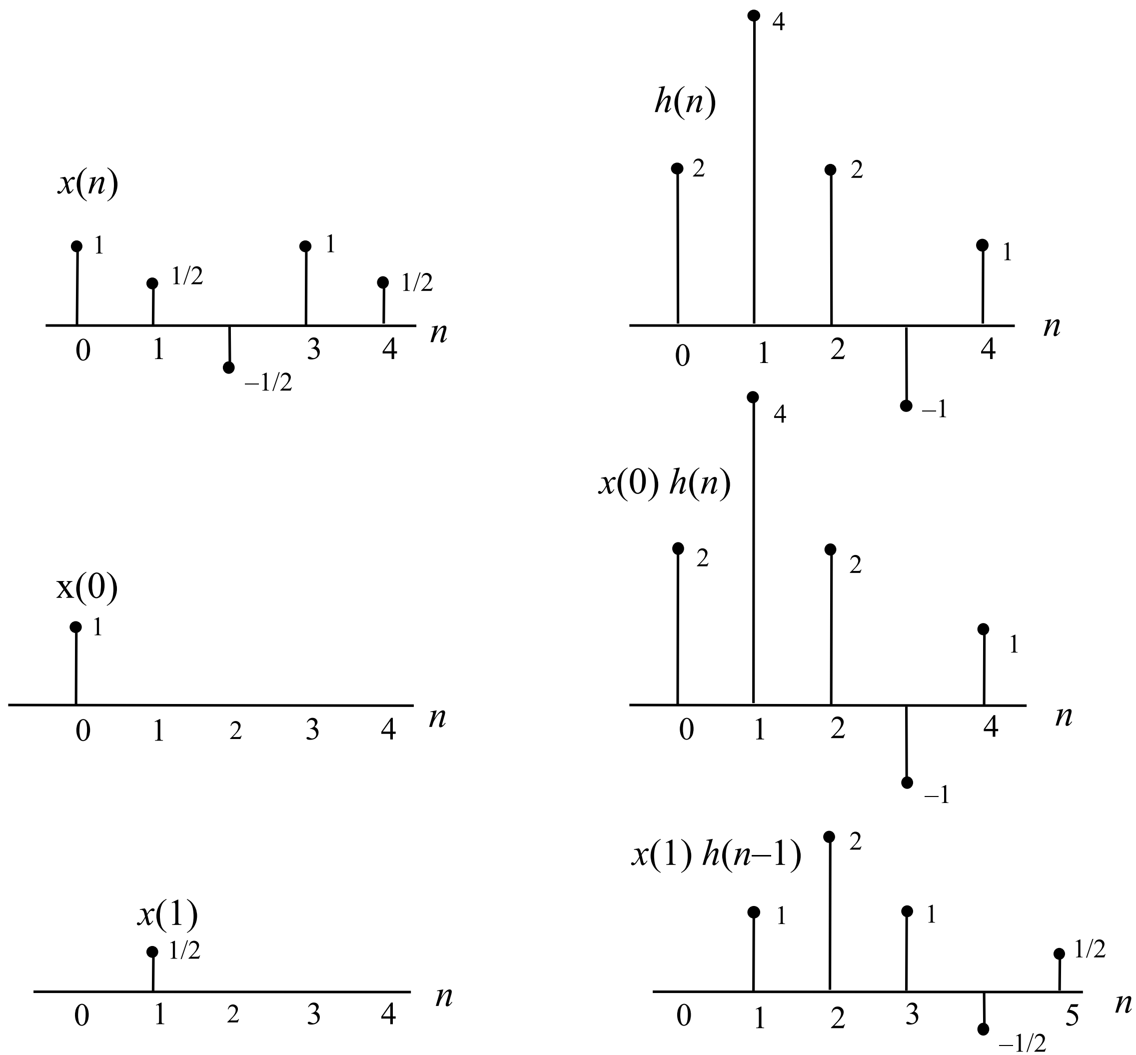
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

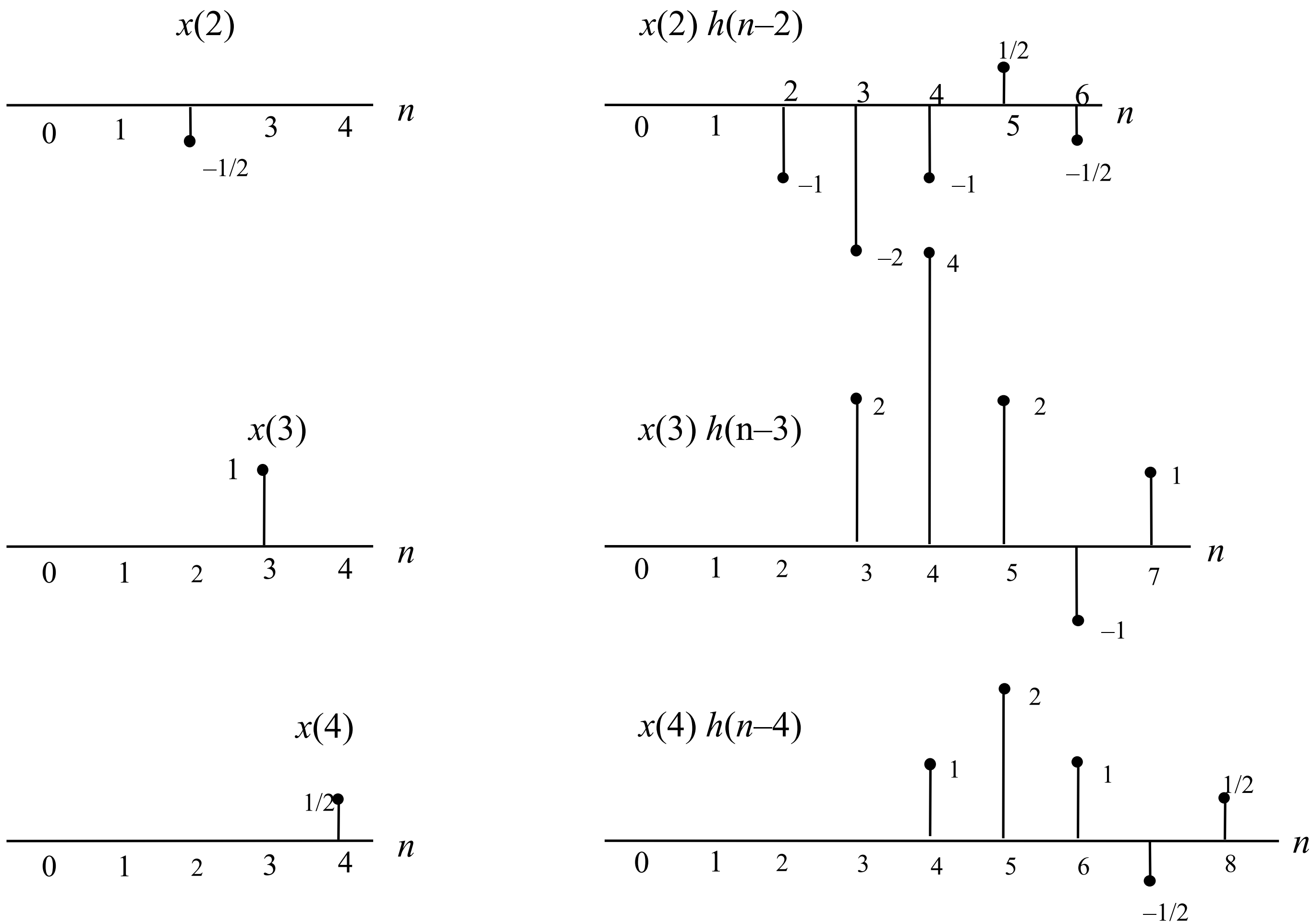
With a knowledge of the input sequence and the system impulse response, it is easy to compute the output sequence numerically using the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

In practical systems of interest the summation is over a finite region.

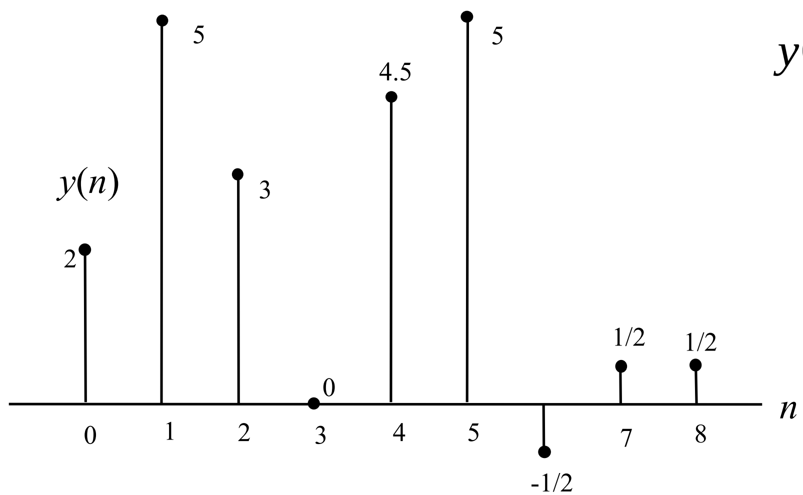
For insight into the significance of convolution consider the following example:





x(2)
x(2) h(n–2)
x(3)
x(3) h(n–3)
x(4)
x(4) h(n–4)
n
0
1
2
3
4
5
6
7
8
–1/2
1/2
–1
–2
1
2
4

The result of the convolution (summing together at the overlapping points in discrete time):



$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Note that the maximum nonzero length of the output sequence of a convolution operation is easily determined as:

$$\text{length}\{y(n)\} = \text{length}\{x(n)\} + \text{length}\{h(n)\} - 1$$

$$9 = 5 + 5 - 1$$

How do we actually perform the convolution operation? There are two approaches:

1. Numerical Approach - with knowledge of the system impulse response, compute the convolution sum as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

we will see later, after studying the z-transform, that convolution of two finite sequences can be accomplished as a polynomial multiplication.

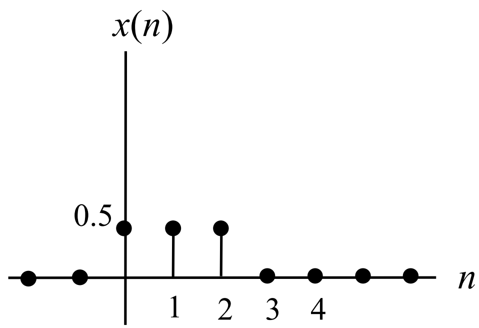
2. Graphical Approach - useful only for demonstrating the principles of convolution to gain a better understanding of the process.

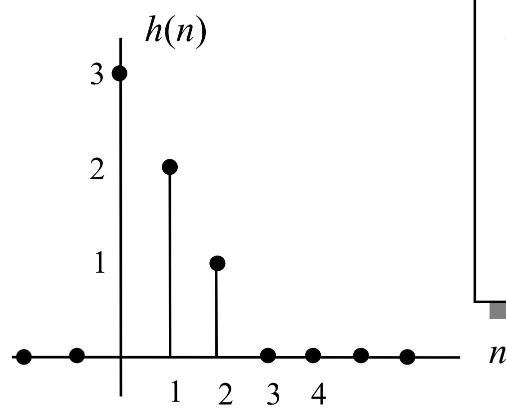
Steps in computing the convolution graphically:

1. express $x(n)$ as a function of k → $x(k)$
2. express $h(n)$ as a function of $-k$ for some n → $h(n-k)$
3. form the product sequence → $x(k)\ h(n-k)$
4. add samples of the product sequence → $\sum x(k)\ h(n-k)$

folded & translated

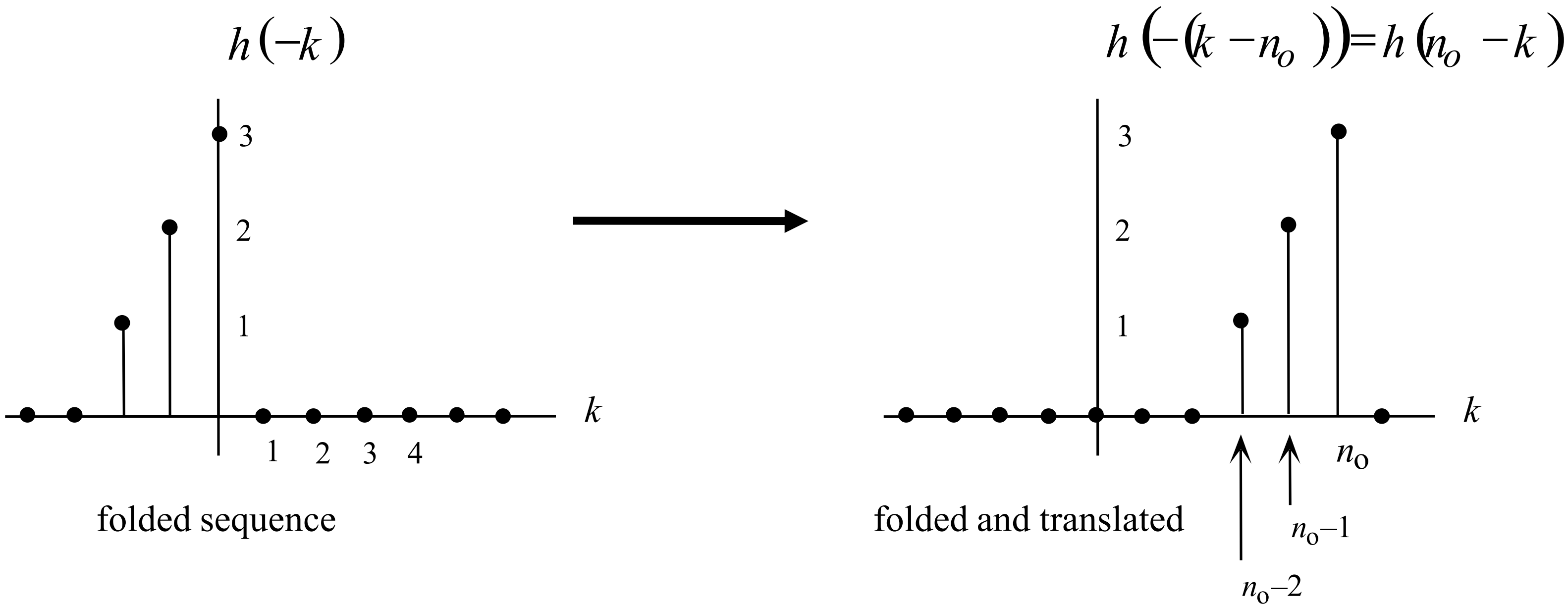
Example: Let $x(n)$ be the input to a LTI system characterized by unit sample response $h(n)$, and call $y(n)$ the corresponding output. For $x(n)$ and $h(n)$ given below, find the output $y(n)$.





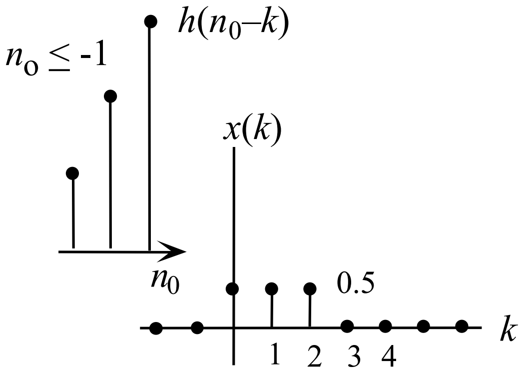
The output of the system can be found by convolving the input $x(n)$ with $h(n)$. It then becomes:

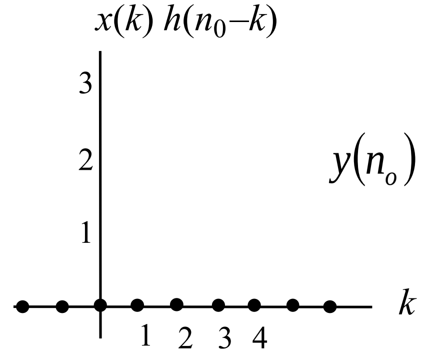
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$



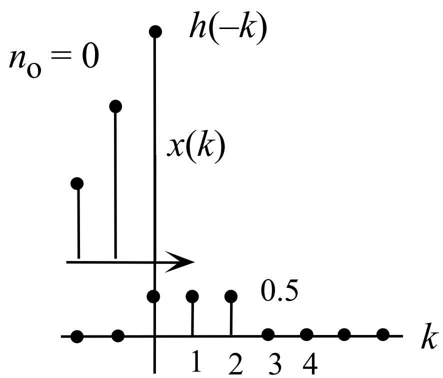
folded sequence

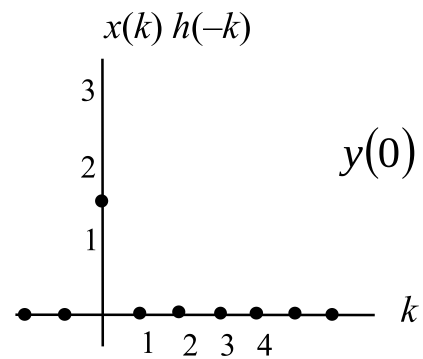
folded and translated



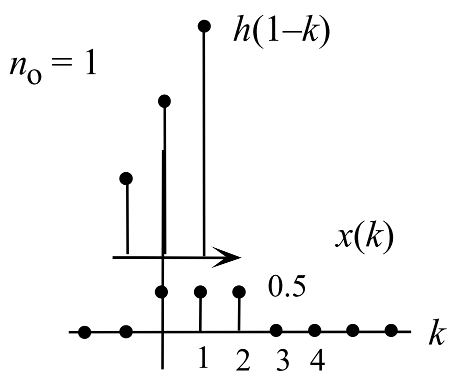


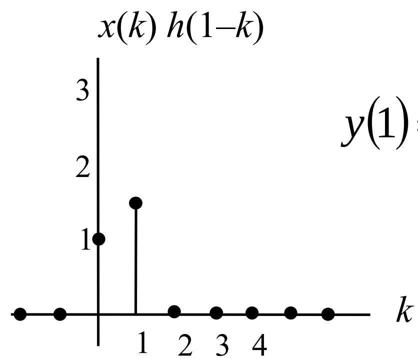
$$y(n_o) = \sum_{k=-\infty}^{\infty} x(k)\,h(n_o - k) = 0, \quad \text{for } n_o \le -1$$



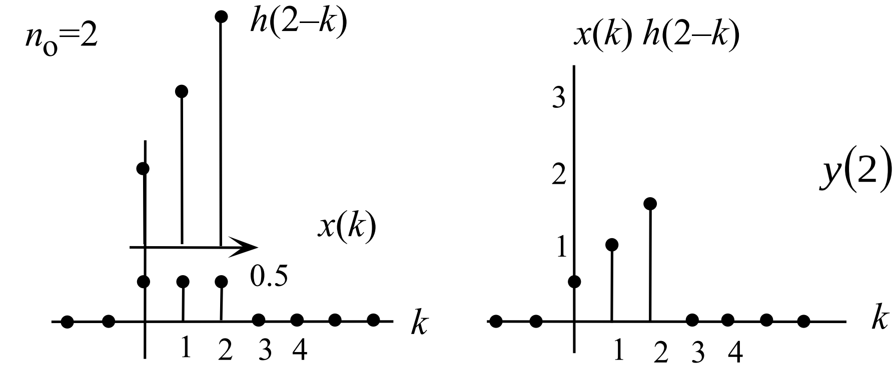


$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k) = 1.5, \quad \text{for } n = 0$$

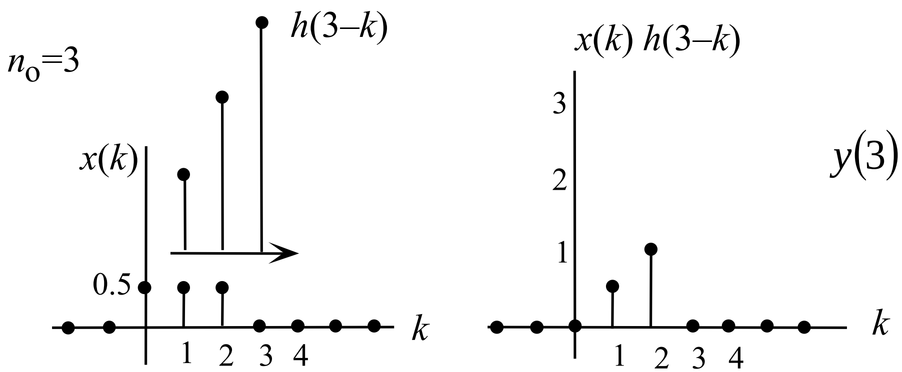




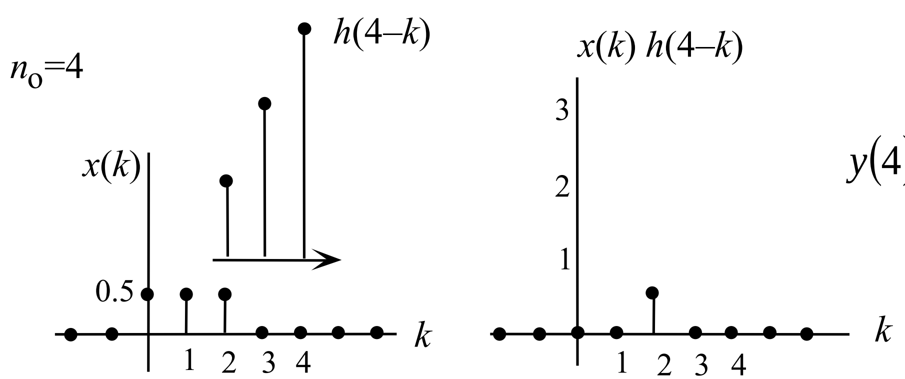
$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(1 - k) = 2.5, \quad \text{for } n_o = 1$$



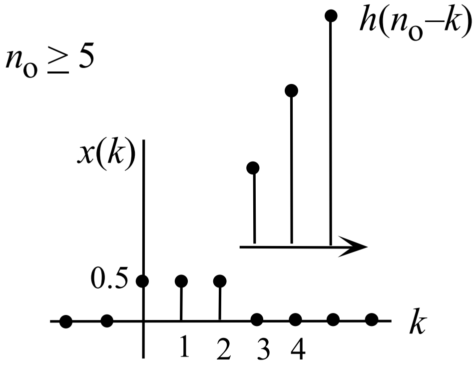
$$y(2)=\sum_{k=-\infty}^{\infty} x(k)\,h(2-k)=3,\quad \text{for } n_o=2$$

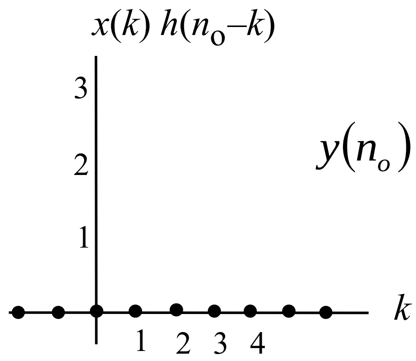


$$y(3)=\sum_{k=-\infty}^{\infty} x(k)\,h(3-k)=1.5,\quad \text{for } n_o=3$$



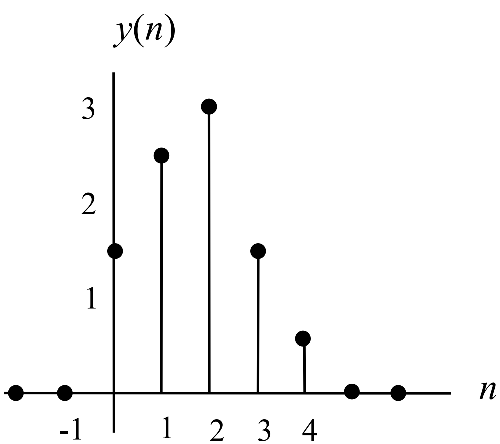
$$y(4)=\sum_{k=-\infty}^{\infty} x(k)\,h(4-k)=0.5,\quad \text{for } n_o=4$$





$$y(n_o) = \sum_{k=-\infty}^{\infty} x(k)\,h(n_o - k) = 0, \quad \text{for } n_o \geq 5$$

And the final result of the graphical convolution is shown below.



Verify this answer using MATLAB:

```
x = [.5 .5 .5]
y = [3 2 1]
conv(x,y)
```

2.4 Properties of Linear Time Invariant Systems

<u>Commutative Property</u>

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

Let's verify this:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

and let $n - k = m$
so that $k = n - m$

$$= \sum_{m=\infty}^{-\infty} x(n-m)\, h(m)$$

reverse order of summation

$$y(n) = \sum_{m=-\infty}^{\infty} x(n-m)\, h(m)$$

Note: $\sum_{n=a}^{b} = \sum_{n=b}^{a}$ however $\int_{a}^{b} = -\int_{b}^{a}$

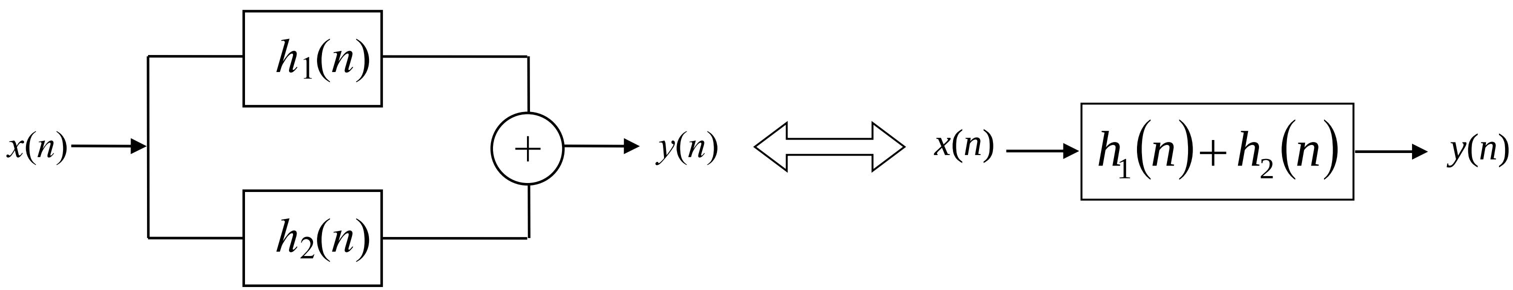
redefine $k = m$ and you have it!

$\therefore$ The order in which you perform the convolution is unimportant

Distributive property:

$$x(n)*[h_1(n)+h_2(n)] = x(n)*h_1(n)+x(n)*h_2(n)$$

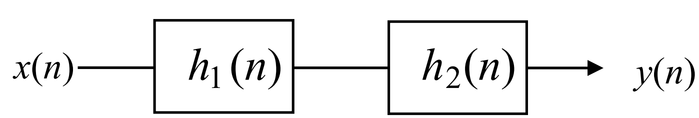
can be viewed as systems in parallel



Associative property:

$$y(n) = x(n)*[h_1(n)*h_2(n)] = [x(n)*h_1(n)]*h_2(n)$$

Which can be viewed as systems in series



2.5 Linear Constant-Coefficient Difference Equations

Discrete-time systems are completely described by difference equations. A difference equation is a discrete-time version of a differential equation. Difference equations (and differential equations) are the most fundamental mathematical models for systems of all types (linear or nonlinear). Continuous-time linear time-invariant (LTI) systems are completely characterized by a linear constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Remember that the impulse response characterization applies only to linear systems. The differential equation describes any class of system.

Discrete-time LTI systems are completely described by linear constant - coefficient difference equations (LCCDE), such as:

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

an "*N*th" order LCCDE

The LCCDE can also be written as an algorithm (*i.e.*, as an explicit function of $y(n)$):

$$y(n) = \frac{1}{a_o}\left[b_o\, x(n) + b_1\, x(n-1) + \ldots - a_1\, y(n-1) - a_2\, y(n-2) - \ldots\right]$$

In general, the algorithm can be expressed as:

$$y(n) = \sum_{k=0}^{M} \frac{b_k}{a_o} x(n-k) - \sum_{k=1}^{N} \frac{a_k}{a_o} y(n-k)$$

Often, a_o is taken to be 1.

The importance of the difference equation (and algorithm) representation:

1. provides a compact specification of the discrete system
2. provides insight into possible realizations
3. system properties can be inferred by inspection:

$\{a_k\}$ and $\{b_k\}$ along with M and N define the system

There are three ways in which we can solve the LCCDE:

1. Conventional approach
2. Direct approach
3. Transforms

Solutions to Difference Equations

- Conventional approach (analogous to solving differential equations)

  1. determine the homogeneous solution $y_h(n)$, with the input set to zero (*i.e.*, the "zero-input" solution)

  2. determine the particular solution $y_p(n)$, by any method (*i.e.*, the "zero-state" solution)

  3. form the general solution as the sum of the two

- Direct approach - plug in and solve by iteration

We will not be concerned with formally solving LCCDE in this course. However, it's important to keep the following points in mind:

- The LCCDE does not uniquely specify system behavior unless the appropriate initial (auxiliary) conditions are provided.

- If we make the following specifications:

  if $x(n) = 0$ for $n < n_o$
  then $y(n) = 0$ for $n < n_o$

  then these represent sufficient initial conditions to characterize the system as being causal

- We will assume that if a system satisfies a LCCDE (and has causal input) then it is LTI

The direct approach is often useful for determining the impulse response for simple systems. For example:

Find the impulse response for the causal LTI system described by the following LCCDE:

$$y(n) - y(n-2) = x(n) + 2x(n-1)$$

let $x(n)=\delta(n)$ then $y(n) \Rightarrow h(n)$ by definition

$$h(n) = h(n-2) + \delta(n) + 2\delta(n-1)$$

$$n=0 \Rightarrow \quad h(0) = \overset{0}{\cancel{h(-2)}} + \delta(0) + 2\overset{0}{\cancel{\delta(-1)}} = 1$$

$$n=1 \Rightarrow \quad h(1) = h(-1) + 0 + 2\delta(0) = 2$$

$$n=2 \Rightarrow \quad h(2) = h(0) + 0 + 0 = 1$$

$$n=3 \Rightarrow \quad h(3) = h(1) + 0 + 0 = 2$$

etc.

$$h(n) = \begin{cases} 1, & n \geq 0 \text{ and even} \\ 2, & n > 0 \text{ and odd} \\ 0, & n < 0 \end{cases}$$

IIR/FIR Systems

Two important classifications of discrete-time systems pertain to the nature of $h(n)$, the system impulse response.

Infinite Impulse Response (IIR) systems - $h(n)$ has an infinite number of non-zero sample values. IIR systems are usually represented in recursive form:

$$y(n) = \sum_{k=0}^{M} \frac{b_k}{a_o} x(n-k) - \sum_{k=1}^{N} \frac{a_k}{a_o} y(n-k)$$

The output consists of past output terms as well as input and possible past input terms. Note that not all IIR systems can be represented with a LCCDE.

Finite Impulse Response (FIR) systems - $h(n)$ has a finite number of non-zero sample values. FIR systems can always be represented as a non-recursive algorithm:

$$y(n) = \sum_{k=0}^{M} \frac{b_k}{a_o} x(n-k)$$

The output consists of only the present input and past input terms. Note that it is possible to represent FIR systems in a recursive form (e.g. M-sample moving average).

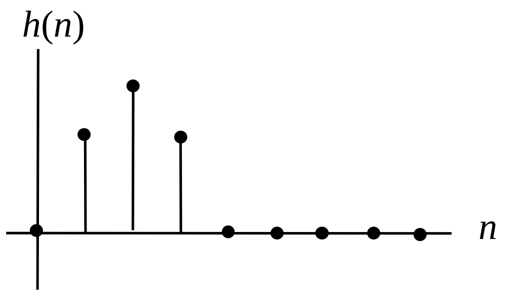
> Always think of recursive/non-recursive as <u>implementation</u> techniques and FIR/IIR as <u>classes</u> of LTI systems

Example: <u>a FIR system</u>

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) \qquad \text{and let } \; x(n) = \delta(n)$$

the output is:

$$y(0) = b_o$$
$$y(1) = b_1$$
$$y(2) = b_2$$
$$\cdots$$
$$y(M) = b_M$$



the impulse response is:

$$h(n) = \begin{cases} b_n & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

The impulse response of a non-recursive FIR system is evident by observing the coefficient sequence, b_k

Example: an IIR system

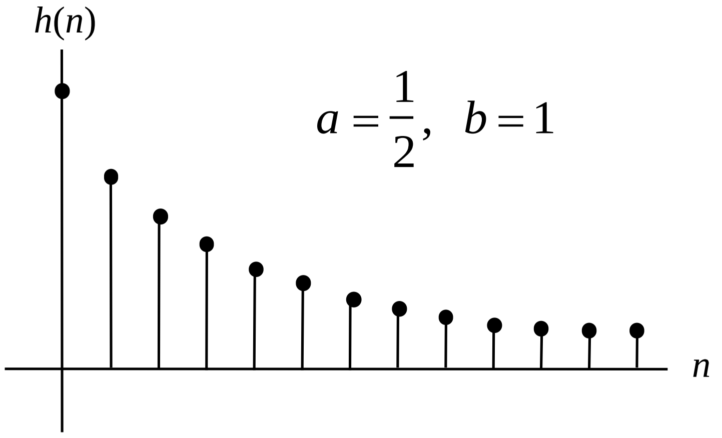
$$y(n) = ay(n-1) + bx(n)$$

A recursive system. The present output sample is created by scaling the present input by 'b' and adding it to the previous output (after scaling by 'a').

Let $x(n) = \delta(n)$ with initial conditions $y(-1) = 0$

The output is:

$$y(0) = b$$
$$y(1) = ab$$
$$y(2) = a^2 b$$
$$\cdots$$
$$y(N) = a^N b$$



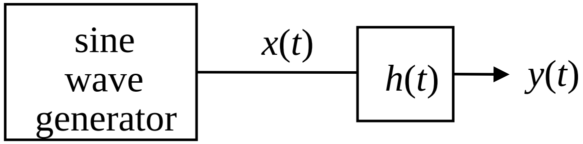
$$h(n) = a^n b, \quad n \geq 0$$

what happens if: $a = 2$? $a = 1$?

Note that the impulse response is not obvious from simple observation of 'a' and 'b'.

2.6 Frequency-Domain Representation of DT Signals & Systems

We have examined several representations for LTI systems (LCCDE, impulse response). Any continuous time system has a frequency response. How do we represent the frequency response of a digital system? Consider the following:



let $x(t) = e^{j\Omega_o t}$ (a complex sinusoid)

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{j\Omega_o (t-\tau)} d\tau = e^{j\Omega_o t} \int_{-\infty}^{\infty} h(\tau) e^{-j\Omega_o \tau} d\tau$$

$$y(t) = \boxed{e^{j\Omega_0 t}} \boxed{\int_{-\infty}^{\infty} h(\tau) e^{-j\Omega_0 \tau} d\tau}$$

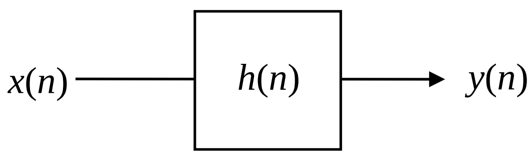
System eigenfunction

(a complex sinusoid)

System frequency response, $H(\Omega_0)$ -
also called the “eigenvalue” (a constant)

$H(\Omega_0)$ is a complex constant which alters the amplitude and phase of the input sinusoid at Ω_0

Now for the digital system:



and let $x(n) = e^{j\omega_o n}$ a sampled complex sinusoid

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k) \qquad \text{Convolution Sum}$$

$$= \sum_{k=-\infty}^{\infty} h(k) e^{j\omega_o (n-k)}$$

$$y(n) = \boxed{e^{j\omega_o n}} \boxed{\sum_{k=-\infty}^{\infty} h(k) e^{-j\omega_o k}}$$

System eigenfunction

System frequency response, $H(\omega_0)$

In summary:

for sinusoidal input:

$$y(t) = e^{j\Omega_o t} H(\Omega_o) \quad \rightarrow \quad H(\Omega) \text{ the frequency response of the analog system}$$

in general

$$y(n) = e^{j\omega_o n} H(\omega_o) \quad \rightarrow \quad H(\omega) \text{ the frequency response of the digital system}$$

therefore,

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

The frequency response of a digital system.

Let's use this approach to determine the frequency response of a digital filter described by the following difference equation:

$$y(n) = \frac{1}{2}x(n) + x(n-1) + \frac{1}{2}x(n-2)$$

Since this is an FIR filter, the impulse response can be determined by inspection:

$$h(n) = \frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2) \longrightarrow \{0.5,\ 1,\ 0.5\}$$

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2)\right]e^{-j\omega n}$$

$$= \frac{1}{2} + e^{-j\omega} + \frac{1}{2}e^{-j2\omega}$$

$$= e^{-j\omega}\left[\frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega}\right]$$

from Euler's identity

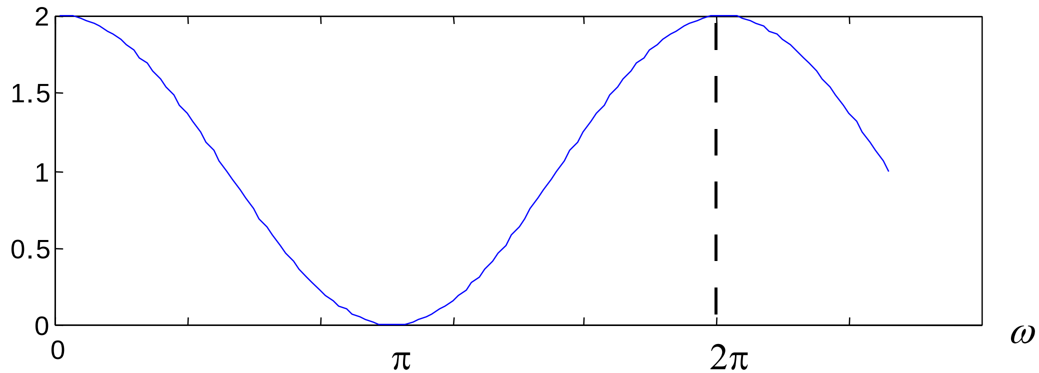
$$H(\omega) = e^{-j\omega}[1 + \cos\omega]$$

Note the magnitude and phase of $H(\omega)$:

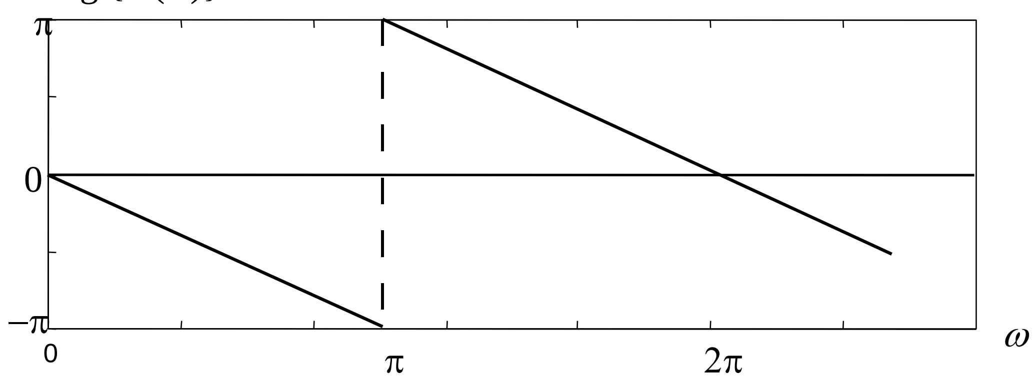
$$|H(\omega)| = 1 + \cos\omega$$

$$\arg\{H(\omega)\} = -\omega$$

$|H(\omega)|$



$\arg\{H(\omega)\}$



<u>Note the following points</u>:

- $H(\omega)$ is a continuous fcn of ω
- $H(\omega)$ is periodic in ω
- $H(\omega)$ is complex, in general

$$H(\omega) = H_r(\omega) + jH_i(\omega)$$

$$H(\omega) = |H(\omega)| e^{j\arg\{H(\omega)\}}$$

A brief review of complex numbers:

$c = a + jb$ c is complex, and b & a are real

$$|c| = \sqrt{a^2 + b^2} = \sqrt{cc^*} \rightarrow |c|^2 = cc^*$$

$$c = |c| e^{j\arg(c)} \quad \text{where:} \quad \arg\{c\} = \arctan\left(\frac{b}{a}\right) \rightarrow$$

$$e^{j\omega} = \cos(\omega) + j\sin(\omega)$$
$$= \Re\{e^{j\omega}\} + j\Im\{e^{j\omega}\}$$

thus

$$\arg\{e^{j\omega}\} = \tan^{-1}\left(\frac{\sin(\omega)}{\cos(\omega)}\right)$$
$$= \tan^{-1}(\tan(\omega))$$
$$= \omega$$

Since $H(\omega)$ is periodic, we can view the expression

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

as a Fourier Series expansion of $H(\omega)$

treat like Fourier coefficients

We can invert this expression by solving for $h(n)$ to obtain an inversion formula. The result should be the familiar expression for the inverse Fourier Series.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

m an integer

$$\int_{-\pi}^{\pi} H(\omega) e^{j\omega m} d\omega = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} e^{j\omega m} d\omega$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-j\omega(n-m)} d\omega = 2\pi \cdot h(m)$$

Only non-zero when *n* = *m*

$$h(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega m} d\omega$$

We can generalize these concepts so that they apply for any sequence

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

This is the **Discrete-Time Fourier Transform (DTFT)**. This is not the DFT or the FFT.

We have the dual relationship indicated by:

$$X(\omega) \Leftrightarrow x(n)$$

2.7 Representation of Sequences by Fourier Transforms

We have established that the Discrete-Time Fourier Transform pair is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$ direct transform (analysis)

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$ inverse transform (synthesis)

This dual relationship is often stated as:

$$X(\omega) \Leftrightarrow x(n)$$

Also, in general $X(e^{j\omega})$ is complex, so that:

$$X(e^{j\omega}) = X_R(e^{j\omega}) + j\, X_I(e^{j\omega})$$

$X(e^{j\omega})$ can alternatively be expressed as:

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg\{X(e^{j\omega})\} = \tan^{-1}\left\{\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right\}$$

• Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \leq \theta(\omega) < \pi$$

• The DTFT of a sequence $x(n)$ is a continuous function of ω

• The DTFT of a sequence $x(n)$ is also a periodic function of ω with a period 2π:

$$X(e^{j(\omega_o+2\pi k)}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega_o+2\pi k)n} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega_o n} e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega_o n} = X(e^{j\omega_o})$$

<u>Example</u>: The DTFT of the unit sample sequence (*i.e.*, impulse) is:

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n} = \delta(0) = 1$$

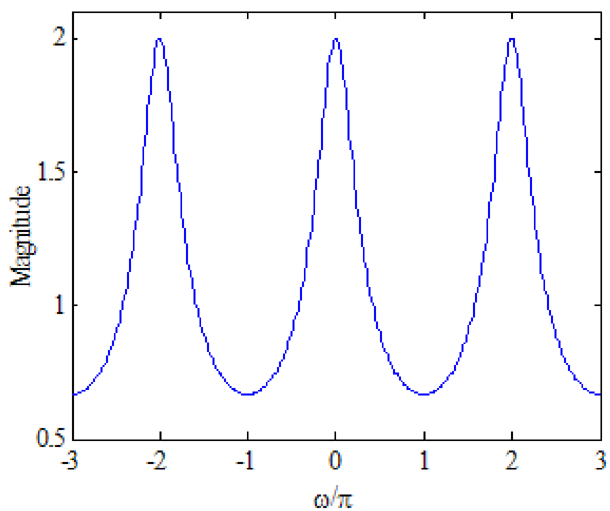
Example: Determine the DTFT of the decaying exponential sequence

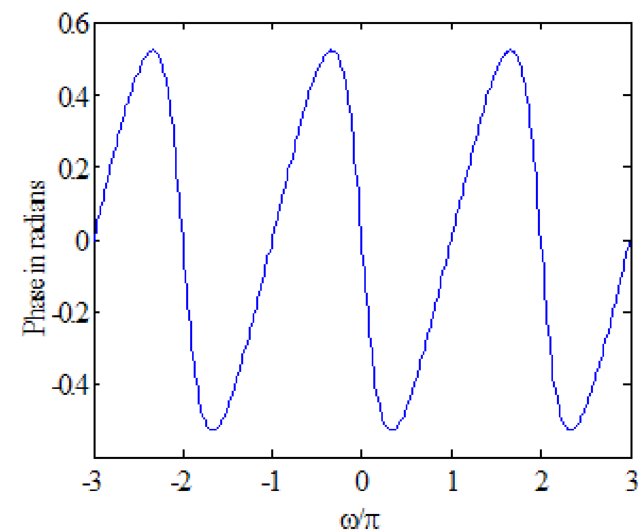
$$x(n) = \alpha^n u(n), \quad |\alpha| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}} \qquad \text{where} \quad |\alpha e^{-j\omega}| = |\alpha| < 1$$

plots for $\alpha = 0.5$





2.8 Symmetry Properties of the Fourier Transform

Another useful signal classification pertains to the symmetry of a sequence. Any sequence can be decomposed into two signal components, one having **even** symmetry, one having **odd** symmetry.

$$x(n) = x_e(n) + x_o(n)$$

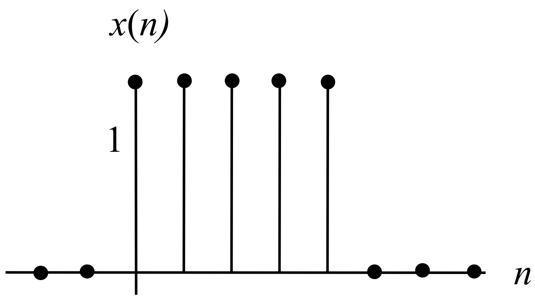
$$x_e(n) = x_e(-n) \qquad\qquad x_o(n) = -x_o(-n)$$

Each of these sequences can be expressed in terms of the original sequence $x(n)$ as

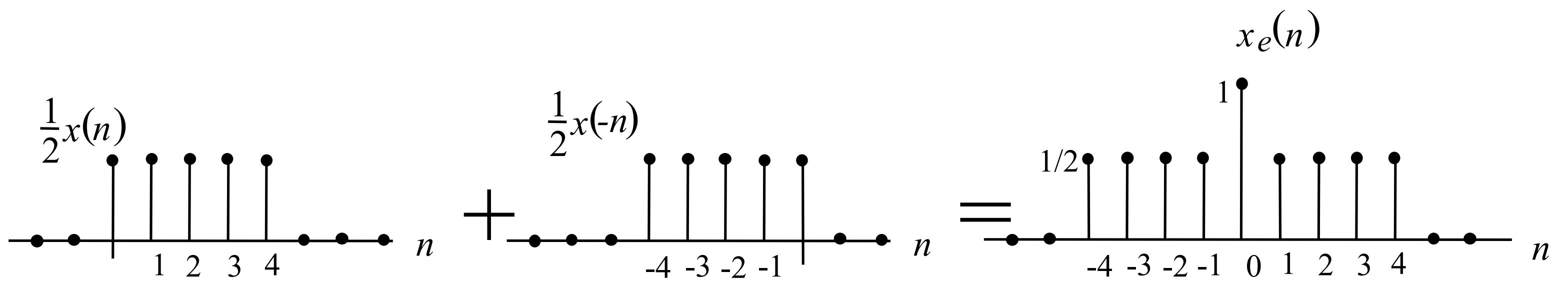
$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

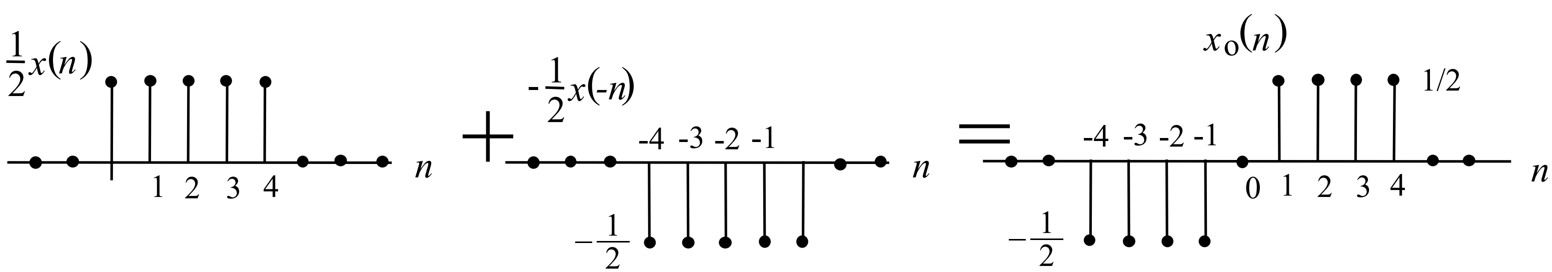
$$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

An example: resolve $x(n)$ into its even and odd components.



Note that $x(n) = x_e(n) + x_o(n)$

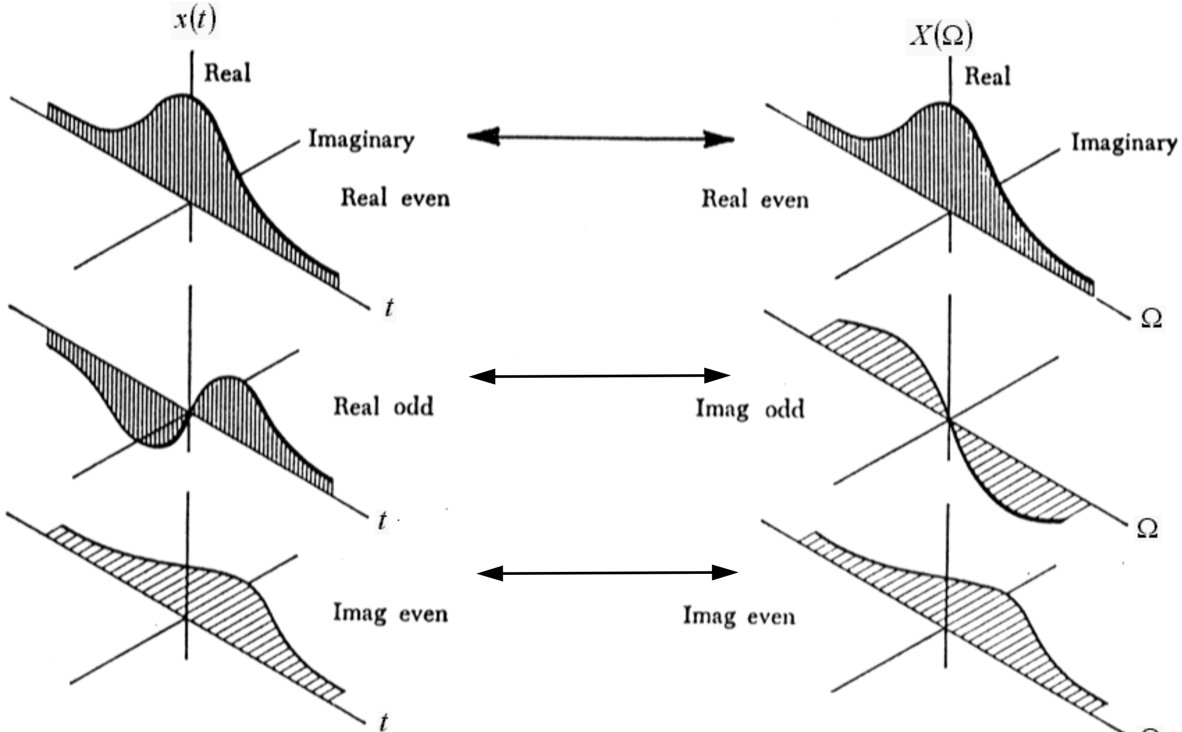


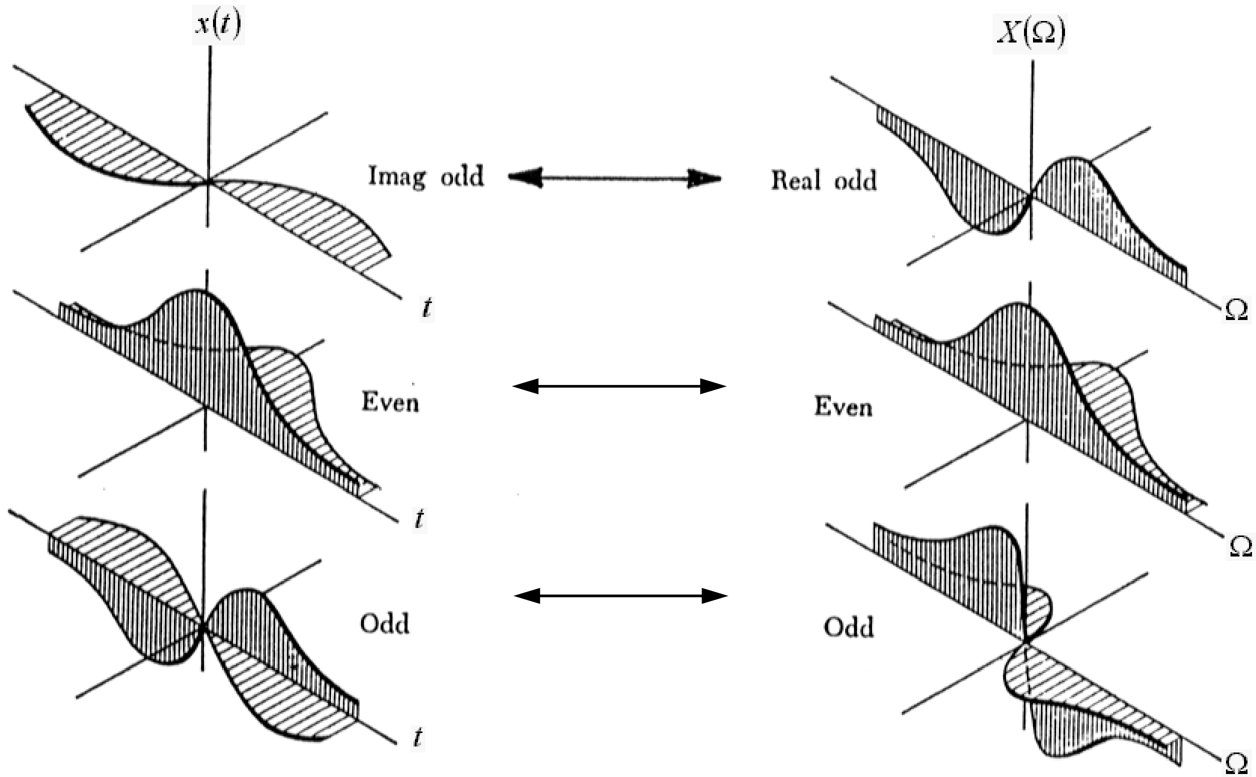


Symmetry properties of the Fourier transform are very important in understanding the dual time-frequency behavior of a signal:

$$x(t) = x_o(t) + x_e(t) = \mathrm{Re}[x_o(t)] + j\,\mathrm{Im}[x_o(t)] + \mathrm{Re}[x_e(t)] + j\,\mathrm{Im}[x_e(t)]$$

$$X(\Omega) = X_o(\Omega) + X_e(\Omega) = \mathrm{Re}[X_o(\Omega)] + j\,\mathrm{Im}[X_o(\Omega)] + \mathrm{Re}[X_e(\Omega)] + j\,\mathrm{Im}[X_e(\Omega)]$$





x(t)
X(Ω)
Imag odd
Real odd
t
Ω
Even
Even
t
Ω
Odd
Odd
t
Ω

2.9 Fourier Transform Theorems

Here are some Fourier transform properties in which $x(n)$ is a discrete-time sequence and $X(\omega)$ is the corresponding frequency spectrum. Take a closer look at each of these in the text.

Property	**Signal**	**Spectrum**
Time Shifting	$x(n-m)$	$e^{-j\omega m}X(\omega)$
Frequency Shifting	$e^{-j\alpha n}x(n)$	$X(\omega-\alpha)$
Time/Frequency Scaling	$x\left(\frac{n}{a}\right)$	$\lvert a\rvert X(a\omega)$
Convolution	$x(n)*y(n)$	$X(\omega)\cdot Y(\omega)$
Modulation	$x(n)\cdot y(n)$	$\frac{1}{2\pi}X(\omega)*Y(\omega)$

Property	**Signal**	**Spectrum**
Time Reversal	$x(-n)$	$X(-\omega)$

Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

$\left|X\left(e^{j\omega}\right)\right|^2$ Is called the Energy Density Spectrum of $x(n)$

In general, Parseval's theorem is:

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)d\omega$$