Course Notes 3 – The *z*-Transform

3.0 Introduction

We begin the study of *z*-transforms by taking a step back and examining the concept of integral transforms. An integral transform has the form:

$$x(\alpha) = \int_a^b x(\beta)\underbrace{K(\alpha;\beta)}\,d\beta$$

This is the "kernel" of the transform

The integral transform is a mathematical device which maps a function from one space (or domain) into another. The most common domains of interest in the study of linear systems theory are, of course, time and frequency:

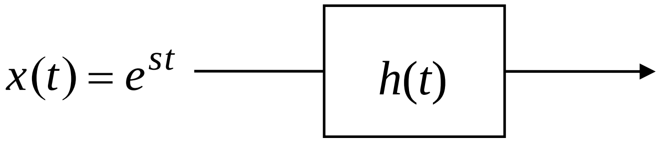
with:

$$\left.\begin{array}{l} \alpha = F \;\; (\text{or } \Omega) \\ \\ \beta = t \end{array}\right\} \longrightarrow \quad x(F) = \int_a^b x(t)K(F;t)\,dt$$

The choice of the kernel is influenced by the particular application. Since we are primarily interested in LTI systems, the kernel is chosen to be the eigenfunction of these systems, that is,

$$K\left(F;t\right) \rightarrow e^{j\zeta\left(F;t\right)}$$

For a continuous linear time-invariant system with impulse response *h*(*t*), the response of the system *y*(*t*) to a complex exponential input is:



$$x(t) = e^{st}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

then: $$y\left(t\right) = \int_{-\infty}^{\infty} h\left(\tau\right) e^{s\left(t-\tau\right)} d\tau$$

$$y(t) = e^{st} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \quad \longrightarrow \quad H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

the “Transfer Function”

Injecting the eigenfunction into the LTI system results in an output consisting of the eigenfunction multiplied by a complex constant. This complex constant is an integral transform which maps the system impulse response into the system transfer function.

We can recognize this mapping process (where $s = \sigma + j\Omega$) to be the Laplace Transform:

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

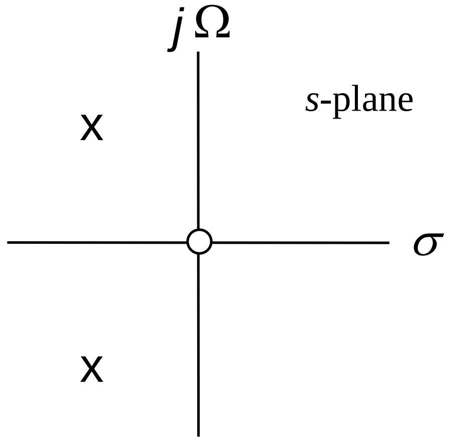
where $L\{\bullet\}$ is the continuous time Laplace transform

Let's examine the kernel of the Laplace Transform:

$$e^{-st} = e^{-\sigma t} e^{-j\Omega t}$$

complex sinusoid

real exponential convergence factor

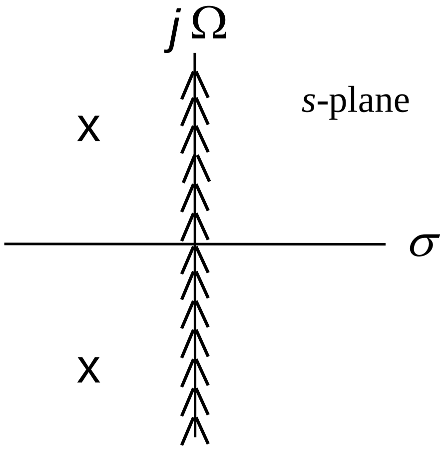


and the poles and zeros of $X(s)$ can be located in the complex s-plane when $\sigma = 0$, then $s = j\Omega$ and

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \Rightarrow CFT\{x(t)\}$$

where $CFT\{\bullet\}$ is the continuous-time Fourier Transform.

which you should recognize as the Fourier Transform. Therefore, the Fourier transform of a function is its Laplace transform evaluated on the imaginary axis in the complex-s plane



3.1 The *z*-Transform

Where does the *z*-transform fit into the integral transform approach? To answer this question, let's begin with the Laplace transform.

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

but now allow $x(t)$ to be sampled:

$$x(t)\Big|_{t = nT} = x(nT) \Rightarrow x(n) \qquad \text{for } -\infty<n<\infty$$

substituting the "time-series" superposition of impulses to represent the sampled signal:

$$X(s) = \int_{-\infty}^{\infty} \underbrace{\left[\sum_{n=-\infty}^{\infty} x(n)\delta(t-nT)\right]}_{\text{sampled } x(t)} e^{-st}dt$$

e^{-snT} by sifting property

$$X(s)=\sum_{n=-\infty}^{\infty} x(n)\int_{-\infty}^{\infty}\delta(t-nT)\,e^{-st}dt$$

$$X(s)=\sum_{n=-\infty}^{\infty} x(n)\;e^{-snT}$$

Recall, $\omega = \Omega T$

Now we can define the polar complex variable, “z” as

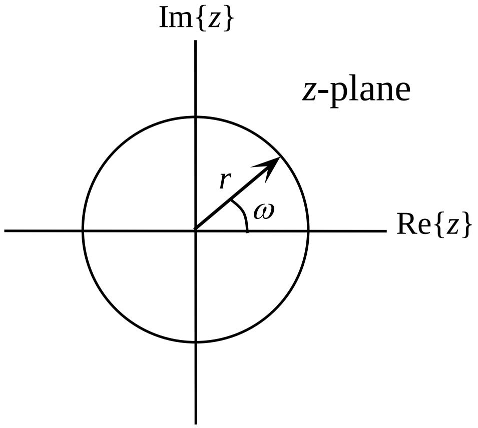
$$z=e^{sT}=e^{(\sigma+j\Omega)T}$$

$$=e^{\sigma T}e^{j\Omega T}=e^{\sigma T}e^{j\omega}$$

Note that $e^{j\omega}$ is a periodic function of ω. Therefore, let

$$r = e^{\sigma T}$$

so that $$z = r \cdot e^{j\omega}$$



It is obvious that "*z*" is a complex variable in polar form. In contrast, the Laplace transform kernel "*s*" is a complex variable in rectangular coordinates.

Therefore, the *z*-transform of the sequence *x*(*n*) is

$$\boxed{X\left(z\right) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = Z\left\{x\left(n\right)\right\}}$$

Note the special case where $r = 1$

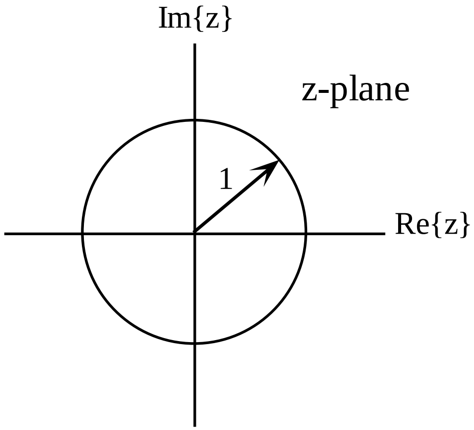
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = DTFT\left\{x(n)\right\}$$

So the *z*-transform evaluated on the unit circle is the Fourier transform of the sequence.

In summary:

Continuous **Discrete**

$$\begin{tikzcd}
L\{\bullet\} \arrow[rr, leftrightarrow, "e^{-st} \to z^{-n}"] \arrow[d, "\sigma = 0"'] & & Z\{\bullet\} \arrow[d, "r = 1"] \\
CFT\{\bullet\} & & DTFT\{\bullet\}
\end{tikzcd}$$



$Z\{x(n)\}$ as defined above is called the two sided *z*-transform. There is also a one-sided *z*-transform given by:

$$X_I(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = Z_I\{x(n)\}$$

For causal sequences, the one sided and two sided *z*-transforms are the same. We will primarily focus on the two-sided case.

For certain sequences, $x(n)$ and/or for particular values of z, the summation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

may diverge.

The z-transform is said to be uniformly convergent if it is absolutely summable. That is, if

$$\sum_{n=-\infty}^{\infty} \left| x(n) z^{-n} \right| < \infty$$

We obtain an equivalent condition for uniform convergence of the z-transform by letting

$$z = re^{j\omega}$$

and noting that $|e^{j\omega}| = 1$, this condition is

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

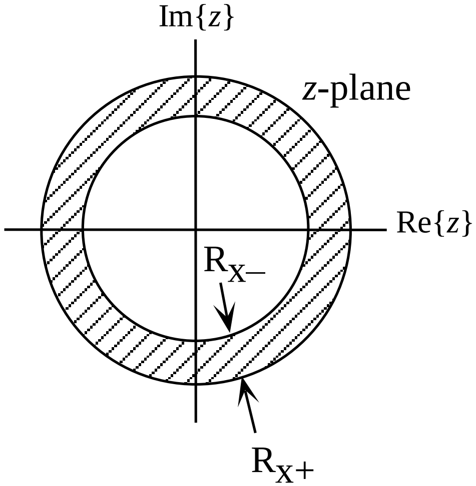
For any sequence *x*(*n*), the set of values of *z* for which the *z*-transform converges is called the region of convergence (ROC).

A typical region of convergence is an annular region of the *z*-plane described by

$$R_{x-} < |z| < R_{x+}$$

R_{x-} may be as small as zero

R_{x+} may be as large as ∞



A summation of the form of the *z*-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

is called a power series and more specifically in complex variable theory it is called a Laurent series. Any Laurent series is an analytic function at every point inside its region of convergence. The significance of this is that the *z*-transform and its derivatives of all orders are continuous functions of *z* throughout the region of convergence.

Note the following important points:

- The *z*-transform may exist for sequences that have no Fourier transform.

$$x(n) = u(n)$$

- Some sequences do not have a *z*-transform

$$x(n) = k \qquad \text{(a constant)}$$

- If the region of convergence (ROC) of the *z*-transform of the function includes |*z*| = 1 (the unit circle) then the Fourier transform of the function exists.

- When specifying the *z*-transform of a sequence, the ROC <u>must</u> be stated!

When using the *z*-transform it is often useful to express sequences in closed form. To accomplish this we make use of Geometric Sum Formulas.

Some others you may find useful:

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \qquad a \neq 1,0$$

$$\sum_{n=0}^{\infty} n a^n = \frac{a}{(1-a)^2} \qquad |a| < 1$$

$$\sum_{n=\alpha}^{\infty} a^n = \frac{a^\alpha}{1-a} \qquad |a| < 1$$

$$\sum_{n=-N}^{N} e^{j\alpha n} = \frac{\sin\left(N + {}^1\!/_2\right)\alpha}{\sin\left({}^{\alpha}\!/_2\right)}$$

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

$$\sum_{n=0}^{N-1} e^{j\alpha n} = \frac{\sin\left(\alpha\, {}^{N}\!/_2\right)}{\sin\left({}^{\alpha}\!/_2\right)}\, e^{j\alpha\left(\frac{N-1}{2}\right)}$$

Now we will examine the convergence of several special cases:

First of all, if $X(z)$ is a rational function (and most cases of interest meet this condition), we can write $X(z)$ as a ratio of polynomials in inverse powers of z.

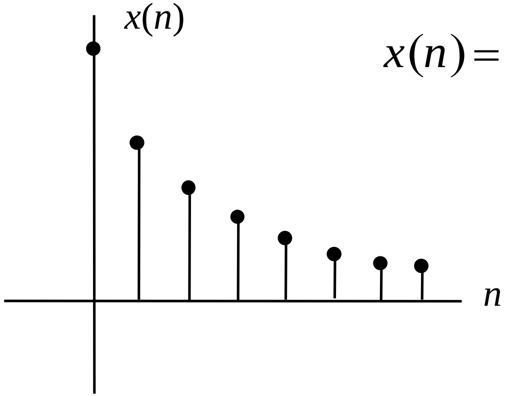
$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_o + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_o + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{\ell=1}^{M} \left(1 - p_\ell z^{-1}\right)}{\prod_{\ell=1}^{N} \left(1 - q_\ell z^{-1}\right)}$$

Values of z for which $P(z) = 0$, and consequently $X(z) = 0$, are called **zeros** of $X(z)$.

Values of z for which $Q(z) = 0$, and consequently $X(z)$ is infinite are called **poles** of $X(z)$

Example:

Given the following function, determine the z-transform and its ROC



$$x(n) = a^n u(n)$$

$$X(z) = \frac{z}{z-a} \qquad \text{ROC: } |z| > |a|$$

A “look up table” result … but we will derive shortly.

We can learn a lot about the *z*-transform of discrete functions by looking at several special cases, which together represent all possible types of functions that we may encounter. They are:

- right-sided sequences
- left-sided sequences
- two-sided sequences
- finite sequences

Recall that the *z*-transform is defined as:

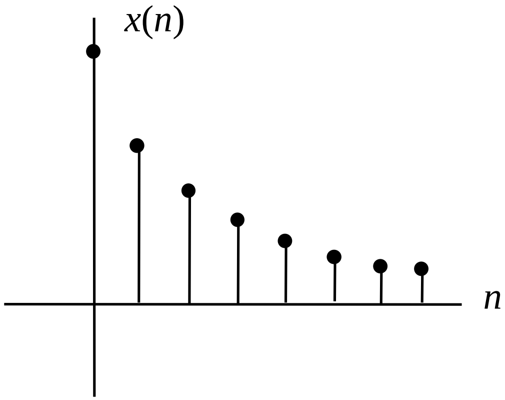
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Which has a result that is a ratio of polynomials for most functions of interest:

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_o + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_o + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} \qquad \text{with ROC } R_- < |z| < R_+$$

Right-Sided Sequences:

Let’s use a familiar example for this case: $x(n) = a^n u(n)$



$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

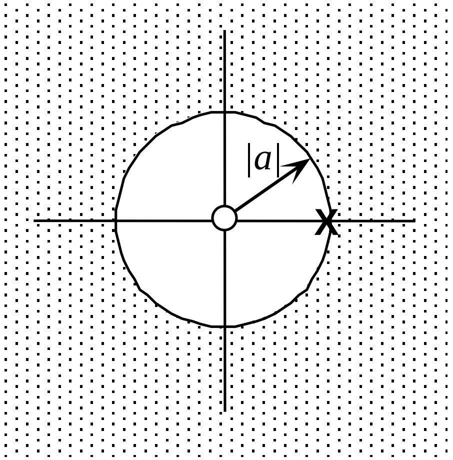
$|az^{-1}| < 1$

Using the appropriate geometric sum formula

$$X(z) = \frac{(az^{-1})^0 \left(1 - (az^{-1})^\infty\right)}{1 - az^{-1}} = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

Express this result in “pole-zero” form (i.e., in positive powers of ‘z’)

example for
a positive-valued



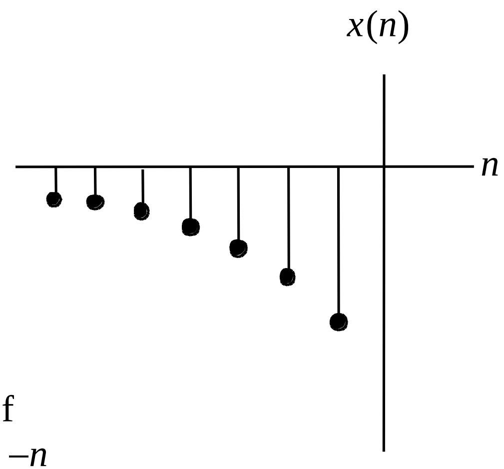
$$X(z) = \frac{z}{z - a} \qquad |z| > |a|$$

$X(z)$ has one pole at $z = a$
$X(z)$ has one zero at $z = 0$

The region of convergence (ROC) is the exterior of a circle in the *z*-plane

Left Sided Sequence:

Consider the discrete function: $x(n) = -b^n u(-n-1)$



$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left[-b^n u(-n-1) \right] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} b^n z^{-n} = -\sum_{n=1}^{\infty} b^{-n} z^n$$

inverting the order of summation and $n \rightarrow -n$

$$= -\sum_{n=1}^{\infty} \left(b^{-1} z \right)^n$$

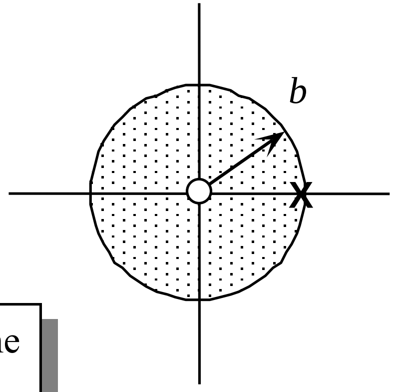
now apply Geometric Sum identity to get

$|b^{-1}z| < 1$

$$= \frac{-\left(b^{-1}z\right)^1 \left(1-(b^{-1}z)^{\infty}\right)}{1-\left(b^{-1}z\right)} = \frac{-\left(b^{-1}z\right)}{1-\left(b^{-1}z\right)}$$

$$|z| < |b|$$

ROC is the interior of a circle in the z-plane

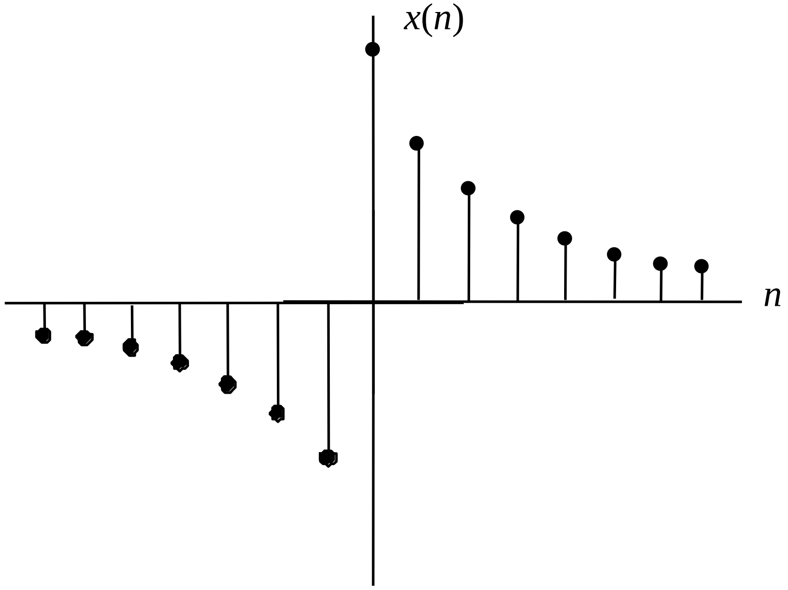


In pole-zero form:

$$X(z) = \frac{z}{z-b}$$

Note: Given z-transform alone we can't determine which sequence it corresponds to.

Two-Sided Sequence:



$$x(n) = a^n u(n) - b^n u(-n-1)$$

$$X(z) = \sum_{n=-\infty}^{-1} \left(-b^n z^{-n}\right) + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$\frac{z}{z-b} \qquad |z| < |b|$$

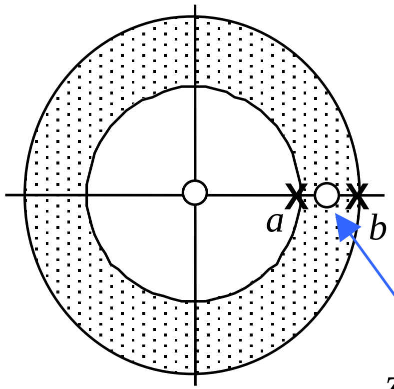
$$\frac{z}{z-a} \qquad |z| > |a|$$

Therefore: $X(z) = \dfrac{1}{1 - az^{-1}} + \dfrac{1}{1 - bz^{-1}}$ $\qquad |a| < |z| < |b|$

In pole-zero form:

$$X(z) = \frac{z}{z-a} + \frac{z}{z-b} = \frac{z(2z\text{-}a\text{-}b)}{(z\text{-}a)(z-b)}$$

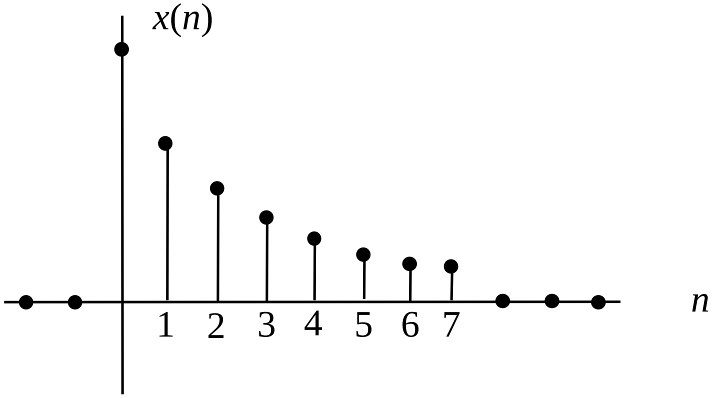
The poles form the boundary of the ROC



$$z = \frac{a+b}{2}$$

Finite Sequence:

$$x(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-8)]$$



$$X(z) = \sum_{n=0}^{7} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{7} \left(\frac{1}{2z}\right)^n$$

Again apply the Geometric Sum identity to yield:

$$X(z) = \frac{\left(\frac{1}{2z}\right)^0 \left(1 - \left(\frac{1}{2z}\right)^8\right)}{1 - \left(\frac{1}{2z}\right)} \qquad \text{for } z \neq \frac{1}{2}, 0$$

In pole-zero form:

$$X(z) = \frac{(2z)^8 - 1}{(2z)^8 - (2z)^7} = \frac{(2z)^8 - 1}{(2z)^7 (2z - 1)}$$

Examine for poles and zeros

Zero locations can be found by solving the following for z:

$$(2z)^8 - 1 = 0$$

$$z^8 = \left(\frac{1}{2}\right)^8 \quad \text{and since } \; z = re^{j\omega}$$

so $r^8 \, e^{\,j8\omega} = (1/2)^8$

$$\left(re^{j\omega}\right)^8 = \left(\frac{1}{2}\right)^8 \quad \text{where } \; r = \left(\frac{1}{2}\right) \quad \text{when } \; e^{j8\omega} = 1$$

But when does this occur?

To answer this question we use the concept of the “nth roots of unity”

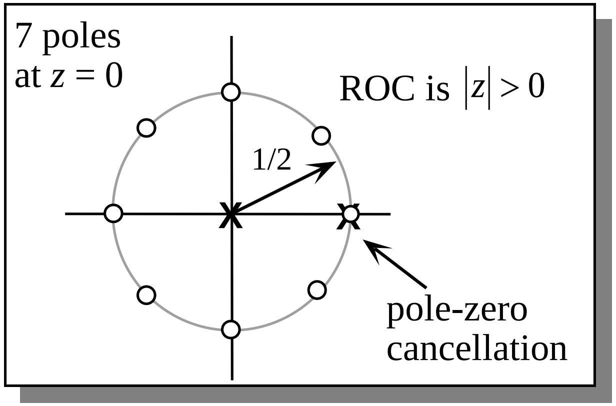
$$e^{j8\omega} = \cos(8\omega) + j\sin(8\omega) = 1$$

This is real.

1

0

$$\text{when } \; 8\omega = 2\pi k, \quad \text{or } \; \omega = \frac{2\pi k}{8} \qquad k = 1, 2, \ldots, 8$$



$$\text{so } \; \omega = \frac{\pi}{4}, \; \omega = \frac{2\pi}{4}, \; \omega = \frac{3\pi}{4}, \; \omega = \pi, \; \omega = \frac{5\pi}{4}, \; \omega = \frac{6\pi}{4}, \; \omega = \frac{7\pi}{4}, \; \omega = 2\pi$$

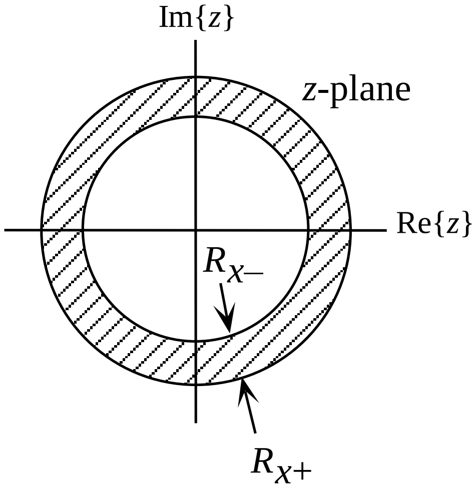
3.2 Properties of the ROC of the *z*-Transform

Some important notes regarding the Region of Convergence of the *z*-Transform

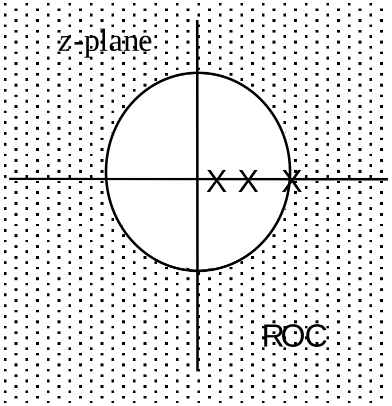
1. If the ROC includes the unit circle then the system is stable.
2. For a causal system the ROC of $H(z)$ is the exterior of a circle (including $z = \infty$) bounded by the outermost pole.
3. For a stable causal system, the ROC includes the unit circle and exterior *z*-plane, including $z = \infty$. All poles lie inside the unit circle.

R_{X-} can extend to zero

R_{X+} can extend to infinity

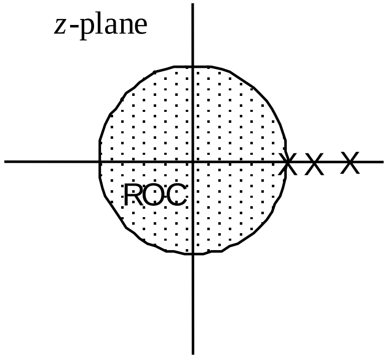


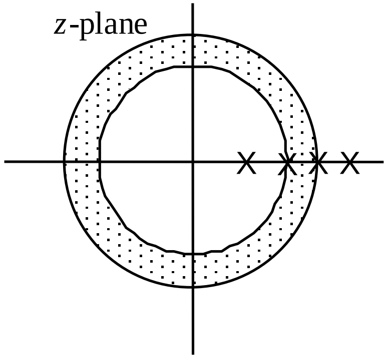
Summary of Convergence Conditions



For a right-sided sequence, the region of convergence is bounded on the inside by the pole with the largest magnitude and on the outside by infinity.

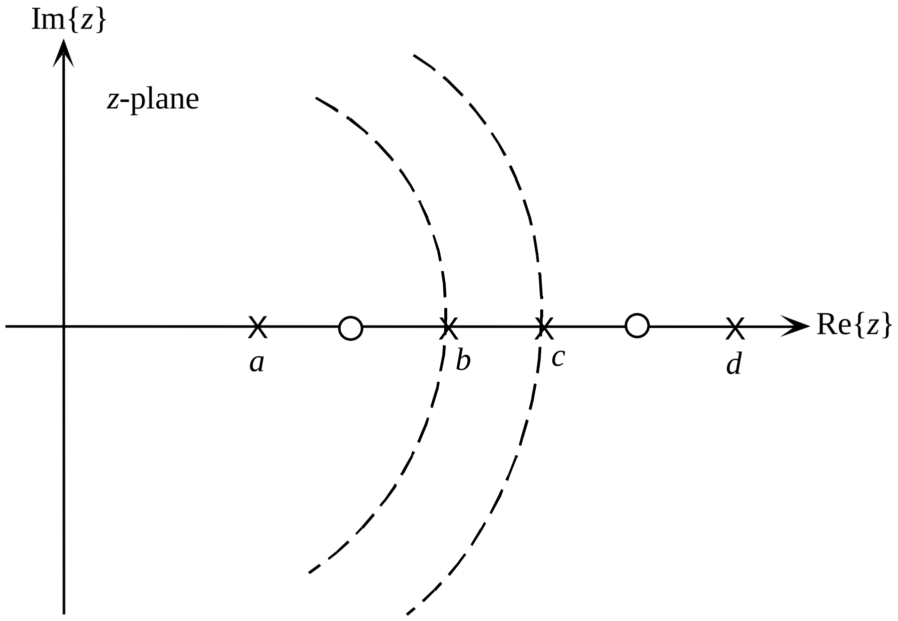
For a left-sided sequence, the region of convergence is bounded on the inside by zero and on the outside by the pole with the smallest magnitude.





For a two-sided sequence where some of the poles contribute only for $n \geq 0$ and the remainder only for $n \leq 0$, the region of convergence is bounded on the inside by the pole with the largest magnitude that contributes for $n \geq 0$ and on the outside by the pole with the smallest magnitude that contributes for $n \leq 0$.

Example:



Given the pole-zero plot on the right for a system function $H(z)$ with region of convergence:

$$b < |z| < c$$

and with $a < b < c < d$.

a. What is the requirement on b and c such that the system is stable? $b < 1$ and $c > 1$

b. If d is less than 1, is the system stable? Why? No, because then $c < d < 1$

c. If d is less than 1, is the system causal? No, since the ROC is not the exterior of a circle including $z = \infty$

d. Which poles contribute to the positive time part of the inverse transform? a and b

e. Which poles contribute to the negative time part of the inverse transform? c and d

3.3 The Inverse *z*-Transform

Given $X(z)$ we can determine the sequence $x(n)$ such that

$$Z^{-1}\{X(z)\} = x(n)$$

There are three common methods for evaluating the inverse *z*-transform;

- Direct evaluation by contour integration
- Partial Fraction Expansion (PFE)
- Power Series Expansion (PSE)

Let's look at each of these techniques:

Direct Evaluation by Contour Integration

Formally, $x(n)$ can be determined by taking the inverse *z*-transform, which is given by the contour integral

$$x(n) = \frac{1}{2\pi j} \oint_{c} X(z)\, z^{n-1} dz = Z^{-1}\{X(z)\}$$

where 'c' is a counterclockwise closed contour in the region of convergence of $X(z)$ and encircles the origin of the z-plane.

For most $X(z)$, evaluation of the inverse z-transform via the contour integral is quite difficult. For the case when $X(z)$ is a rational function, the integral can be evaluated using the residue theorem of complex variables.

Fortunately, for special cases (which are often the cases of interest) there are techniques which allow easy evaluation of the inverse z-transform. The next of these is partial fraction expansion.

Partial Fraction Expansion

If $X(z)$ is rational, *i.e.* if it can be expressed as

$$X(z) = \frac{P(z)}{Q(z)}$$

we can obtain an expression for $X(z)$ which is the sum of terms of the form,

$$\frac{K}{1 - az^{-1}}$$

that is, a partial fraction expansion. Let's introduce this technique through an example.

Example: Given $X(z)$ shown below, determine $x(n)$

$$X(z)=\frac{z^{-1}}{3-4z^{-1}+z^{-2}}$$

ROC unknown

First, express $X(z)$ in pole-zero form and locate poles and zeros

multiply by $\frac{z^2}{z^2}$

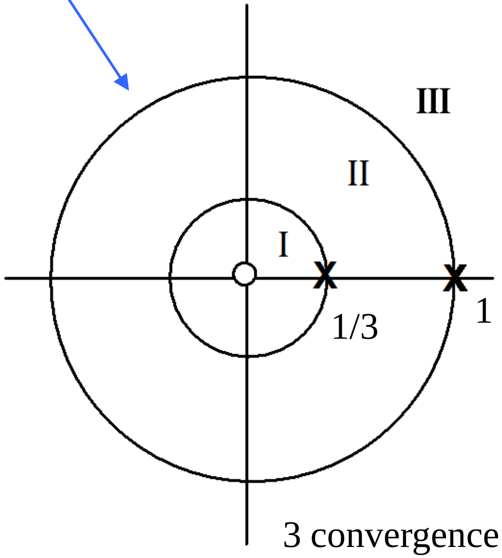
$$X(z)=\frac{z}{3z^2-4z+1}=\frac{z}{3\left(z^2-\frac{4}{3}z+\frac{1}{3}\right)}=\frac{z}{3\left(z-\frac{1}{3}\right)(z-1)}$$

When expanding $X(z)$, it is often convenient to define

$$F(z)=\frac{X(z)}{z}$$

and expand $F(z)$, as shown below:

$$F(z)=\frac{1}{3\left(z-\frac{1}{3}\right)(z-1)}=\frac{A}{\left(z-\frac{1}{3}\right)}+\frac{B}{(z-1)}$$



3 convergence
regions of interest

$$A = F(z)\left(z - \tfrac{1}{3}\right)\Bigg|_{z=\frac{1}{3}} = \frac{\left(z-\tfrac{1}{3}\right)}{3\left(z-\tfrac{1}{3}\right)(z-1)}\Bigg|_{z=\frac{1}{3}} = -\frac{1}{2}$$

$$B = F(z)(z-1)\Bigg|_{z=1} = \frac{(z-1)}{3\left(z-\tfrac{1}{3}\right)(z-1)}\Bigg|_{z=1} = \frac{1}{3\left(\frac{2}{3}\right)} = \frac{1}{2}$$

Therefore $\dfrac{X(z)}{z} = \dfrac{-\tfrac{1}{2}}{z-\tfrac{1}{3}} + \dfrac{\tfrac{1}{2}}{z-1}$ or $X(z) = \dfrac{-\tfrac{1}{2}\,z}{z-\tfrac{1}{3}} + \dfrac{\tfrac{1}{2}\,z}{z-1}$

Expressed in inverse powers of z

$$X(z) = \frac{-\tfrac{1}{2}}{\left(1-\frac{1}{3}z^{-1}\right)} + \frac{\tfrac{1}{2}}{\left(1-z^{-1}\right)}$$

remember the following transform pairs

$$a^n u(n) \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad |z| > |a|$$

$$-a^n u(-n-1) \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad |z| < |a|$$

both anti-causal

for <u>Region I</u>:

|z| < 1/3

$$x(n) = -\frac{1}{2}\left[-\left(\frac{1}{3}\right)^n u(-n-1)\right] + \frac{1}{2}\left[-(1)^n u(-n-1)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}\right)^n - 1\right] u(-n-1)$$

both causal

for <u>Region III</u>:

|z| > 1

$$x(n) = -\frac{1}{2}\left[\left(\frac{1}{3}\right)^n u(n)\right] + \frac{1}{2}\left[(1)^n u(n)\right]$$

$$= \frac{1}{2}\left[1-\left(\frac{1}{3}\right)^n\right] u(n)$$

for <u>Region II</u>:

1/3 < |z| < 1

$$X(z) = \frac{-1/2}{\left(1-\frac{1}{3}z^{-1}\right)} + \frac{1/2}{\left(1-z^{-1}\right)}$$

must be right sided sequence

must be left sided sequence

Therefore:

$$x(n) = -\frac{1}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{1}{2}u(-n-1)$$

causal

anti-causal

Multiple Order Poles

For example: $$X(z) = \frac{1}{\left(1+z^{-1}\right)\left(1-z^{-1}\right)^2}$$

The appropriate Partial Fraction Expansion, in positive powers of *z*, is

$$F(z) = \frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{(z-1)^2} + \frac{C}{z-1}$$

which is solved as

$$A = (z+1)F(z)\Big|_{z=-1} = \frac{1}{4}$$

$$B = (z-1)^2 F(z)\Big|_{z=1} = \frac{1}{2}$$

$$C = \frac{d}{dz}\left[(z-1)^2 F(z)\right]\Big|_{z=1} = \frac{3}{4}$$

$$\begin{tikzcd} & \dfrac{az^{-1}}{\left(1-az^{-1}\right)^2} \arrow[dl] \arrow[dr, "Z^{-1}\{\bullet\}"'] & \\ na^n u(n) \quad \text{if } |z| > |a| & & -na^n u(-n-1) \quad \text{if } |z| < |a| \end{tikzcd}$$

In general, a rational function can be expressed as

$$X(z)=\frac{b_o z^{-m}+b_1 z^{-(m+1)}+b_2 z^{-(m+2)}+\ldots+b_M z^{-M}}{a_o z^{-n}+a_1 z^{-(n+1)}+a_2 z^{-(n+2)}+\ldots+a_N z^{-N}}=\frac{P(z)}{Q(z)}$$

multiply by $\frac{z^K}{z^K}$

if $(M-m)<(N-n)$, we can apply Partial Fraction Expansion (PFE) to

order of the polynomials

$$F(z)=\frac{X(z)}{z}=\frac{1}{z}\frac{b_o z^{K-m}+b_1 z^{K-(m+1)}+b_2 z^{K-(m+2)}+\ldots+b_M z^{K-M}}{a_o z^{K-n}+a_1 z^{K-(n+1)}+a_2 z^{K-(n+2)}+\ldots+a_N z^{K-N}}$$

where $K=\max\{M+1, N\}$. The rational function in this case is “proper”.

Otherwise, an “improper” rational function can, by Power Series Expansion (PSE), be expressed as

$$X(z)=\frac{P(z)}{Q(z)}=\underbrace{c_0+c_1 z^{-1}+\cdots+c_{M-N}z^{-(M-N)}}+\underbrace{\frac{P_1(z)}{Q(z)}}$$

power series,
inverse *z*-transform can
be obtained by inspection

proper,
perform PFE upon $\frac{1}{z}\frac{P_1(z)}{Q(z)}$

Note: right-sided signal

Example:

$$X(z) = \frac{1 + 1.5z^{-1} + 0.75z^{-2} + 0.125z^{-3}}{1 - z^{-2}} \qquad |z| > 1$$

Power Series Expansion (PSE) of $X(z)$

$$\begin{array}{r|l l l l}
 & 1 + & 1.5z^{-1} & & \\
\cline{2-5}
1 - z^{-2} & 1 + & 1.5z^{-1} & + \ 0.75z^{-2} & + \ 0.125z^{-3} \\
 & -(1 & & - \ z^{-2}) & \\
\cline{2-4}
 & & 1.5z^{-1} & + \ 1.75z^{-2} & + \ 0.125z^{-3} \\
 & & -(1.5z^{-1} & & - \ 1.5z^{-3}) \\
\cline{3-5}
 & & & 1.75z^{-2} & + 1.625z^{-3}
\end{array}$$

so $X(z)$ can be expressed as

$$X(z) = 1 + 1.5z^{-1} + \underbrace{\frac{1.75z^{-2} + 1.625z^{-3}}{1 - z^{-2}}}$$

Expand this part using PFE

Power Series Expansion

Rational forms of $X(z)$ can often be expanded in a power series. Then $x(n)$ can be obtained from the coefficients of the series.

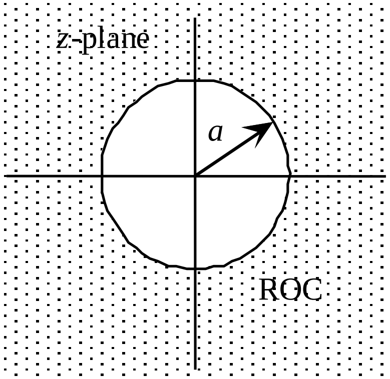
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \ldots + x(-2)z^{2} + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots$$

Often, the progression of the coefficients can be written in closed form.

Let’s look at an example:

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$



The ROC is the exterior of a circle, therefore the sequence *x*(*n*) is right-sided.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \ldots$$

Note sequence of z^{-n} with increasing *n* for a causal, right-sided sequence

Therefore, divide to obtain a series in powers of z^{-n}

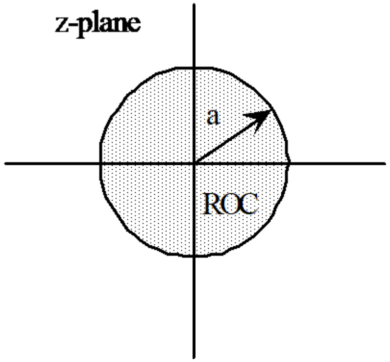
recognize this series as $x(n) = a^n u(n)$

$$1 - az^{-1} \overline{\smash{\big)}\,1\quad} \; 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \ldots$$

This approach is valid for <u>any</u> right sided sequence

Another Example:

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| < |a|$$



ROC is the interior of a circle, therefore the sequence $x(n)$ is left sided.

$$\lim_{z \to 0} X(z) = 0$$

(i.e., finite so ROC includes $z = 0$. Therefore, the sequence is anti-causal.)

$$X(z) = \sum_{n=-\infty}^{0} x(n) z^{-n} = x(0) + x(-1)z + x(-2)z^2 + x(-3)z^3 + \ldots$$

Note sequence of z^n for decreasing $x(n)$ index for an anti-causal, left-sided sequence

Reverse order of terms then divide to obtain a series in powers of z^n

$$\frac{1}{1 - az^{-1}} = \frac{z}{-a + z} \longrightarrow -a + z\overline{)z} \quad \overline{-a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \ldots}$$

recognize this series as: $x(n) = -a^n u(-n-1)$

Note: This approach is valid for any left sided sequence

3.4 Z-Transform Properties

If $Z\{x(n)\} = X(z)$ with ROC $R_{x-} < |z| < R_{x+}$

and $Z\{y(n)\} = Y(z)$ with ROC $R_{y-} < |z| < R_{y+}$

1. Linearity (superposition)

$$Z\{ax(n) + by(n)\} = aX(z) + bY(z) \quad \text{with ROC at least } R_x \cap R_y$$

The ROC may be larger if some poles are canceled out by zeros (after cross-multiplication)

2. Translations (shift of a sequence)

$$Z\{x(n - m)\} = z^{-m} X(z) \quad \text{with ROC } R_{x-} < |z| < R_{x+}$$

No change in the ROC results from translation except possibly at $z = 0$ or $z = \infty$

- $m > 0$ results in poles appearing at the origin (delay)
- $m < 0$ results in poles appearing at infinity (advance)

example: if $Z\{x(n)\} = \dfrac{1}{\left(1 - az^{-1}\right)\left(1 - bz^{-1}\right)}$ and $Z\{x_1(n)\} = \dfrac{z^{-5}}{\left(1 - az^{-1}\right)\left(1 - bz^{-1}\right)}$

then $x_1(n) = x(n-5)$

let's prove this property in general by letting

$$g(n) = x(n-m)$$

$$G(z) = \sum_{n=-\infty}^{\infty} g(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

let $k = n - m$
so $n = k + m$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-(k+m)} = z^{-m} \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

$$\therefore \quad Z\{x(n-m)\} = z^{-m} X(z)$$

3. <u>Scaling in *z*-domain</u>

$$a^n x(n) \xleftrightarrow{Z} X\left(a^{-1} z\right) \qquad \text{ROC } |a| R_{x-} < |z| < |a| R_{x+}$$

Proof:

$$Z\left\{a^n x(n)\right\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) \left(a^{-1} z\right)^{-n} = X\left(a^{-1} z\right)$$

Since the ROC of $X(z)$ is: $R_{x-} < |z| < R_{x+}$

The ROC of $X(a^{-1}z)$ is: $R_{x-} < \left|a^{-1}z\right| < R_{x+}$

$|a|R_{x-} < |z| < |a|R_{x+}$ (i.e., the pole/zero locations are scaled by a factor a)

4. Convolution

$Z\{x(n) * y(n)\} = X(z) \cdot Y(z)$ with ROC at least $R_x \cap R_y$

Proof:

$$g(n) = x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k) y(n-k)$$

$$G(z) = \sum_{n=-\infty}^{\infty} g(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k) y(n-k) \right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\, z^{-n}$$

and let $m = n - k$

$$G(z) = \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m)\, z^{-(m+k)}$$

since $m = n - k$
then $n = m + k$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{m=-\infty}^{\infty} y(m) z^{-m} = \underline{X(z)\cdot Y(z)}$$

note inversion
and reversal

5. Time Reversal

$$x(-n) \xleftrightarrow{Z} X\left(z^{-1}\right) \qquad \text{ROC} \quad \frac{1}{R_{x+}} < |z| < \frac{1}{R_{x-}}$$

6. Differentiation in the *z*-Domain

$$nx(n) \xleftrightarrow{Z} -z\,\frac{dX(z)}{dz} \qquad \text{ROC} \quad R_{x-} < |z| < R_{x+}$$

7. Initial Value Theorem

If $x(n)$ is causal, then $x(0) = \lim_{z\to\infty} X(z)$

$$\textit{proof}: \quad X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

$$\text{As } \; z \to \infty, \;\; z^{-n} \to 0 \;\; \text{since} \;\; n > 0$$

causal

8. Final Value Theorem

For the one-sided z-transform defined as $X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$,

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1} \left[(z-1) X^{+}(z) \right]$$

If the ROC of $[(z - 1)\, X^{+}(z)]$ includes the unit circle.

Cancels pole @ z = +1 (if it exists), which would correspond to a step function

System Function Properties - Poles and Zeros

Three important ways of representing digital systems

1. Difference equation (and algorithm)
2. Impulse response
3. Transfer function

Let's begin with the difference equation:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$Z\left\{\sum_{k=0}^{N} a_k y(n-k)\right\} = Z\left\{\sum_{k=0}^{M} b_k x(n-k)\right\}$$

Take *z*-transform of both sides

$$\sum_{k=0}^{N} a_k Z\{y(n-k)\} = \sum_{k=0}^{M} b_k Z\{x(n-k)\}$$

By linearity of *z*-transform

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Using the shifting property

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = H(z)$$

Defined as the “system function” or transfer function of the digital system.

$$\underbrace{Y(z) = H(z)X(z)}_{z\text{-domain}} \Leftrightarrow \underbrace{y(n) = h(n) * x(n)}_{\text{time (data) domain}}$$

Note that $H(z)$ is the z-transform of the system impulse response:

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

And $H(\omega)$ is the Fourier Transform of the system impulse response:

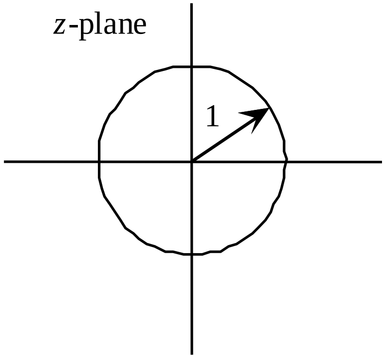
$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

which implies:

$$H(\omega) = H(z)\Big|_{z = e^{j\omega}}$$

Since

$$z = re^{j\omega}$$



this means that the DTFT is the z-transform evaluated at r = 1 (i.e., the unit circle).

We can assess the stability of a system by observing the ROC of $H(z)$, the system function.

We know a system is stable if

$$\sum_{n=-\infty}^{\infty} \left|h(n)\right| < \infty$$

Similarly, we note that the system function $H(z)$ converges if

$$\sum_{n=-\infty}^{\infty} \left|h(n) z^{-n}\right| < \infty \qquad \text{or} \qquad \sum_{n=-\infty}^{\infty} \left|h(n) \cdot r^{-n}\right| \cancelto{1}{\left|e^{-j\omega n}\right|} < \infty$$

Therefore, if the system is stable it follows that the unit circle (r = 1) must lie in the ROC of $H(z)$

$$\textit{i.e.} \text{ if } \sum_{n=-\infty}^{\infty} \left|h(n)\right| < \infty, \quad \text{then} \quad \sum_{n=-\infty}^{\infty} \left|h(n) \cdot r^{-n}\right| < \infty \quad \text{for} \quad r = 1$$

Let’s examine an important characteristic of the system function - the location of system poles and zeros.

Begin with the system function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m\, z^{-m}}{\sum_{n=0}^{N} a_n\, z^{-n}}$$

Factor numerator and denominator polynomials

$$H(z) = A \cdot \frac{\prod_{m=1}^{M} (z - c_m)}{\prod_{n=1}^{N} (z - d_n)}$$

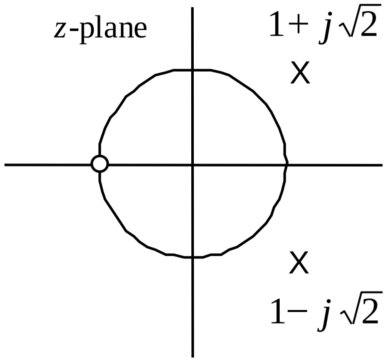
$H(z)$ has <u>zeros</u> at $z = c_m$
$H(z)$ has <u>poles</u> at $z = d_n$

Given a pole/zero diagram we can, to within a constant, determine *H*(*z*) and therefore the difference equation of the system.

Example:

Given the following pole/zero diagram:

Determine the system function and diff equation.



$$H(z)=\frac{z+1}{\left(z-1-j\sqrt{2}\right)\left(z-1+j\sqrt{2}\right)}$$

$$=\frac{z+1}{z^2-2z+3}$$

correct to within a constant

Next convert the $H(z)$ expression to inverse powers of z.

$$H(z)=\frac{z^{-1}+z^{-2}}{1-2z^{-1}+3z^{-2}}=\frac{Y(z)}{X(z)}$$

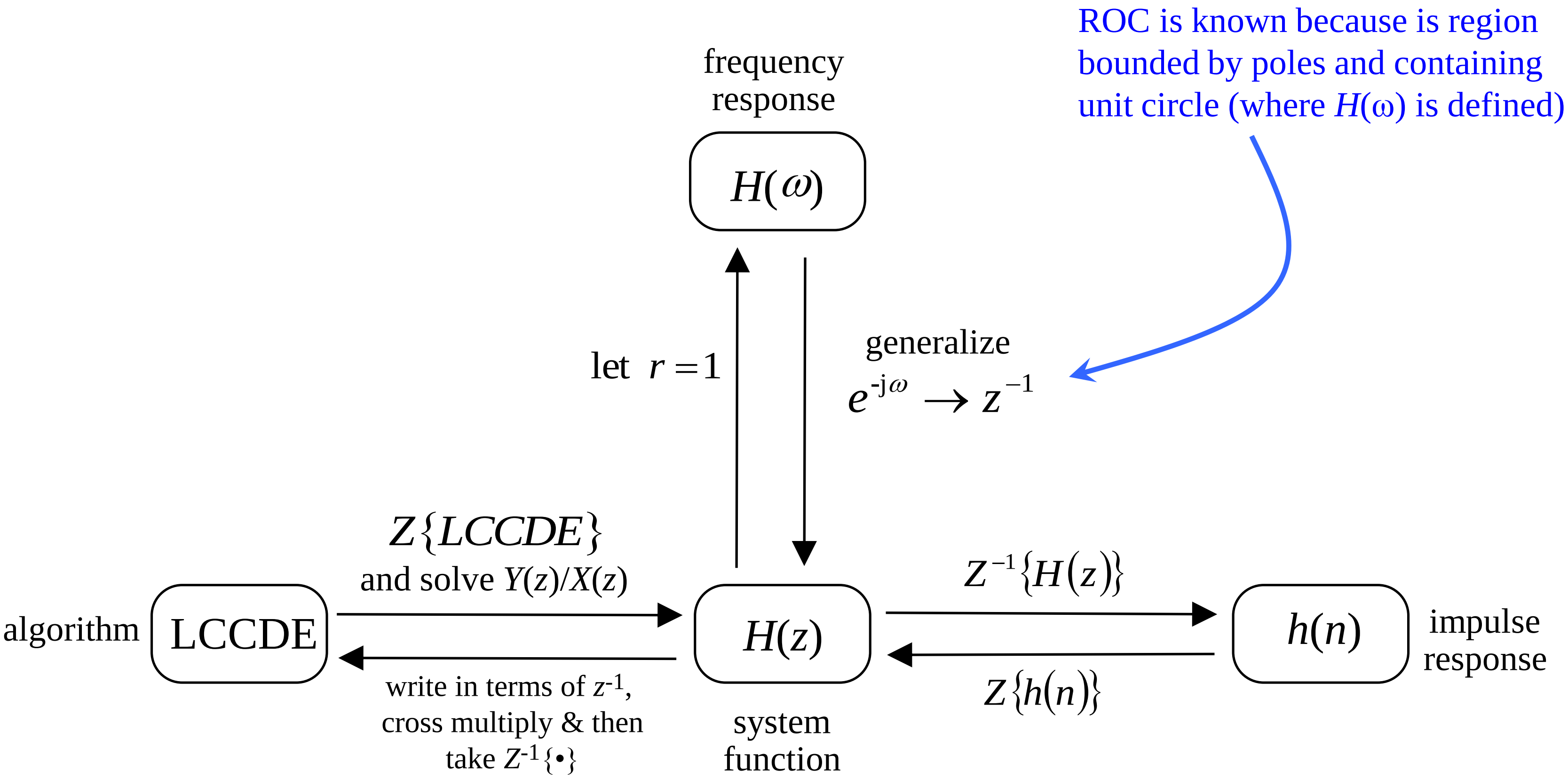
$$Y(z)\left(1-2z^{-1}+3z^{-2}\right)=X(z)\left(z^{-1}+z^{-2}\right)$$

$$Y(z)-2z^{-1}Y(z)+3z^{-2}Y(z)=z^{-1}X(z)+z^{-2}X(z)$$

Difference Equation: $y(n)-2y(n-1)+3y(n-2)=x(n-1)+x(n-2)$ taking the inverse z-transform

Algorithm: $y(n)=2y(n-1)-3y(n-2)+x(n-1)+x(n-2)$

A review and summary of the concepts



ROC is known because is region bounded by poles and containing unit circle (where $H(\omega)$ is defined)

frequency response

$H(\omega)$

let $r=1$

generalize
$e^{-j\omega} \to z^{-1}$

algorithm

LCCDE

$Z\{LCCDE\}$
and solve $Y(z)/X(z)$

write in terms of z^{-1},
cross multiply & then
take $Z^{-1}\{\bullet\}$

$H(z)$

system function

$Z^{-1}\{H(z)\}$

$Z\{h(n)\}$

$h(n)$

impulse response