Course Notes 4 – Correlation

4.0 Introduction

In some applications information can be obtained by comparing a reference signal with one or more signals (*i.e.* determining the correlation between the signals)

Correlation closely resembles convolution as it involves two signals. Correlation is used to determine the similarity between the two signals over all delay shifts.

Correlation is often encountered in radar, sonar, communications, and other areas of science and engineering.

For example, consider the received radar signal

$$y(n) = \alpha\, x(n - D) + w(n)$$

where $x(n)$ was the transmitted signal, D is the round-trip delay of the received signal, α is some attenuation factor, and $w(n)$ is additive noise. Since we know $x(n)$, we can correlate $x(n)$ with $y(n)$ to determine the delay D and thereby obtain the range to some particular object.

Energy signals vs. Power signals

The energy E_x of a signal $x(n)$ is defined as

$$E_x \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The energy of a signal can be finite or infinite. If E_x is finite, then $x(n)$ is called an "energy signal".

Many signals with infinite energy have finite average power defined as

$$P_x \equiv \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

If E_x is finite, then $P_x = 0$.

If E_x is infinite, P_x may be either finite or infinite. If P_x is finite, the signal $x(n)$ is called a "power signal".

If a signal $x(n)$ is periodic with fundamental period N, it has infinite energy and its power is given by

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

Consequently, periodic signals are "power signals".

4.1 Cross-correlation and Auto-correlation

A measure of similarity between a pair of sequences $x(n)$ and $y(n)$ is given by the cross-correlation sequence

$$r_{xy}(\ell) = \sum_{n=-\infty}^{\infty} x(n)y(n-\ell), \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

where the lag parameter ℓ indicates the time-shift between the pair of signals.

The cross-correlation can equivalently be expressed as

$$r_{xy}(\ell) = \sum_{n=-\infty}^{\infty} x(n+\ell)y(n), \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

Similarly, we have

$$r_{yx}(\ell) = \sum_{n=-\infty}^{\infty} y(n)x(n-\ell), \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

or

$$r_{yx}(\ell) = \sum_{n=-\infty}^{\infty} y(n+\ell)x(n), \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

> For complex signals, the 1st sequence is usually complex-conjugated by convention. Note: the Matlab 'xcorr' command does this automatically … but conjugates the 2nd sequence (presumably due to "outer product notation").

Cross-correlation is similar to the convolution of $x(n)$ and $y(n)$ except that

1) Correlation does <u>not</u> involve the time reversal of one of the two sequences, and
2) Convolution is a sum over all delay shifts ℓ yielding an output indexed in discrete-time n, while correlation is a sum over the discrete-time index n with the output a function of delay shift ℓ.

Compare the cross-correlation

sum over time index

delay indexed output

$$r_{xy}(\ell) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-\ell), \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

no time-reversal

with the convolution sum

sum over delay index

time indexed output

$$z(n) = x(n) * y(n) = \sum_{\ell=-\infty}^{\infty} x(\ell)\,y(n-\ell)$$

time-reversal

Note that $r_{xy}(n) = x(n) * y(-n)$

a double time-reversal means <u>no</u> time-reversal

This process is referred to as "correlation filtering" or simply "matched filtering" and is a widely used form of *coherent processing*.

Auto-correlation

Auto-correlation is a special case of cross-correlation in which $y(n) = x(n)$. The auto-correlation sequence is defined as

$$r_{xx}(\ell) = \sum_{n=-\infty}^{\infty} x(n)x(n-\ell), \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

Again, with the 1st sequence complex-conjugated by convention if the signal is complex.

or equivalently as

$$r_{xx}(\ell) = \sum_{n=-\infty}^{\infty} x(n+\ell)x(n), \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

Finite length sequences

For causal, finite length sequences, the cross-correlation may be expressed as

$$r_{xy}(\ell) = \sum_{n=i}^{N-|k|-1} x(n)y(n-\ell)$$

where $i = \ell$, $k = 0$ for $\ell \geq 0$, and $i = 0$, $k = \ell$ for $\ell < 0$.

Example: Given $x(n) = a^n u(n)$, determine $r_{xx}(\ell)$

$$\text{for } \ell \geq 0, \quad r_{xx}(\ell) = \sum_{n=-\infty}^{\infty} a^n u(n)\, a^{n-\ell} u(n-\ell) = \sum_{n=\ell}^{\infty} a^{2n-\ell}$$

from previous page,
$i = \ell$, $k = 0$, $N = \infty$

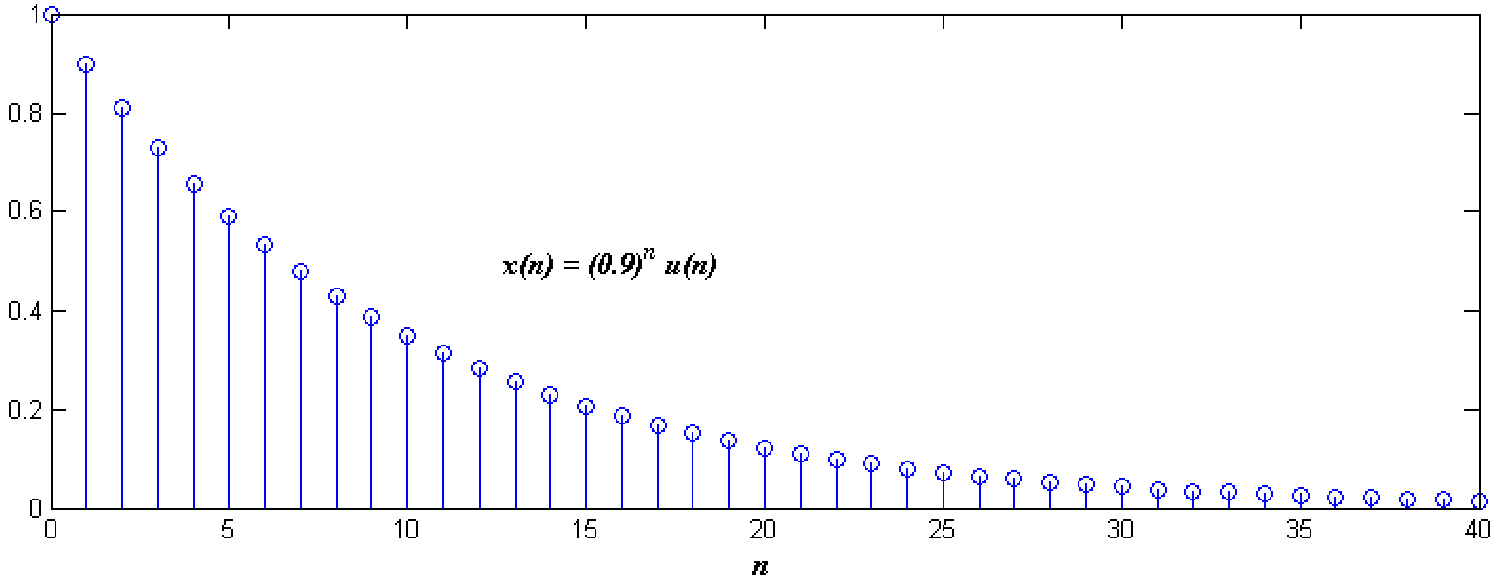
$$= a^{-\ell} \sum_{n=\ell}^{\infty} a^{2n} = a^{-\ell} \frac{a^{2\ell}}{1-a^2} = \boxed{\frac{a^{\ell}}{1-a^2}}$$

$$\text{for } \ell < 0, \quad r_{xx}(\ell) = \sum_{n=0}^{\infty} a^{2n-\ell} = \frac{a^{-\ell}}{1-a^2} = \boxed{\frac{a^{|\ell|}}{1-a^2}}$$

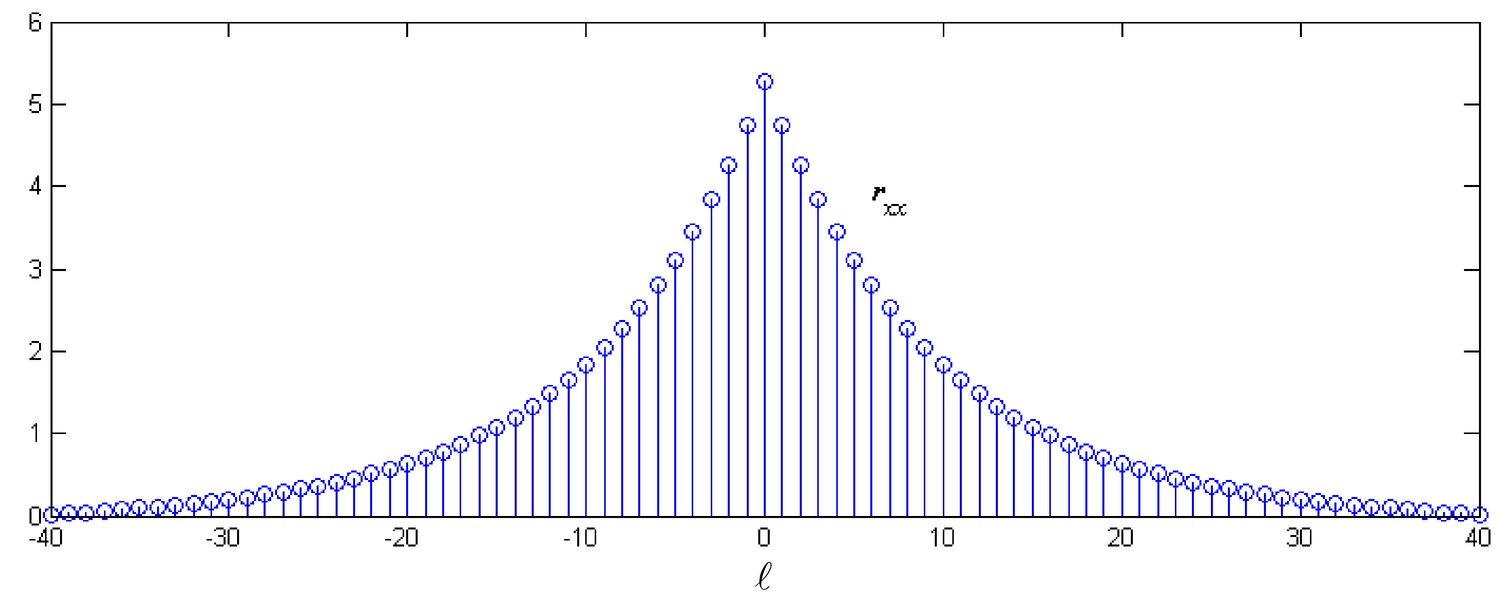
now,
$i = 0$, $k = \ell$, $N = \infty$

$$\text{Hence,} \quad r_{xx}(\ell) = \boxed{\frac{a^{|\ell|}}{1-a^2}}, \quad \text{for } \ell = -\infty, \cdots, \infty$$

Note that $r_{xx}(\ell) = r_{xx}(-\ell)$, *i.e.* symmetric



x(n) = (0.9)^n u(n)
n



r_xx
ℓ

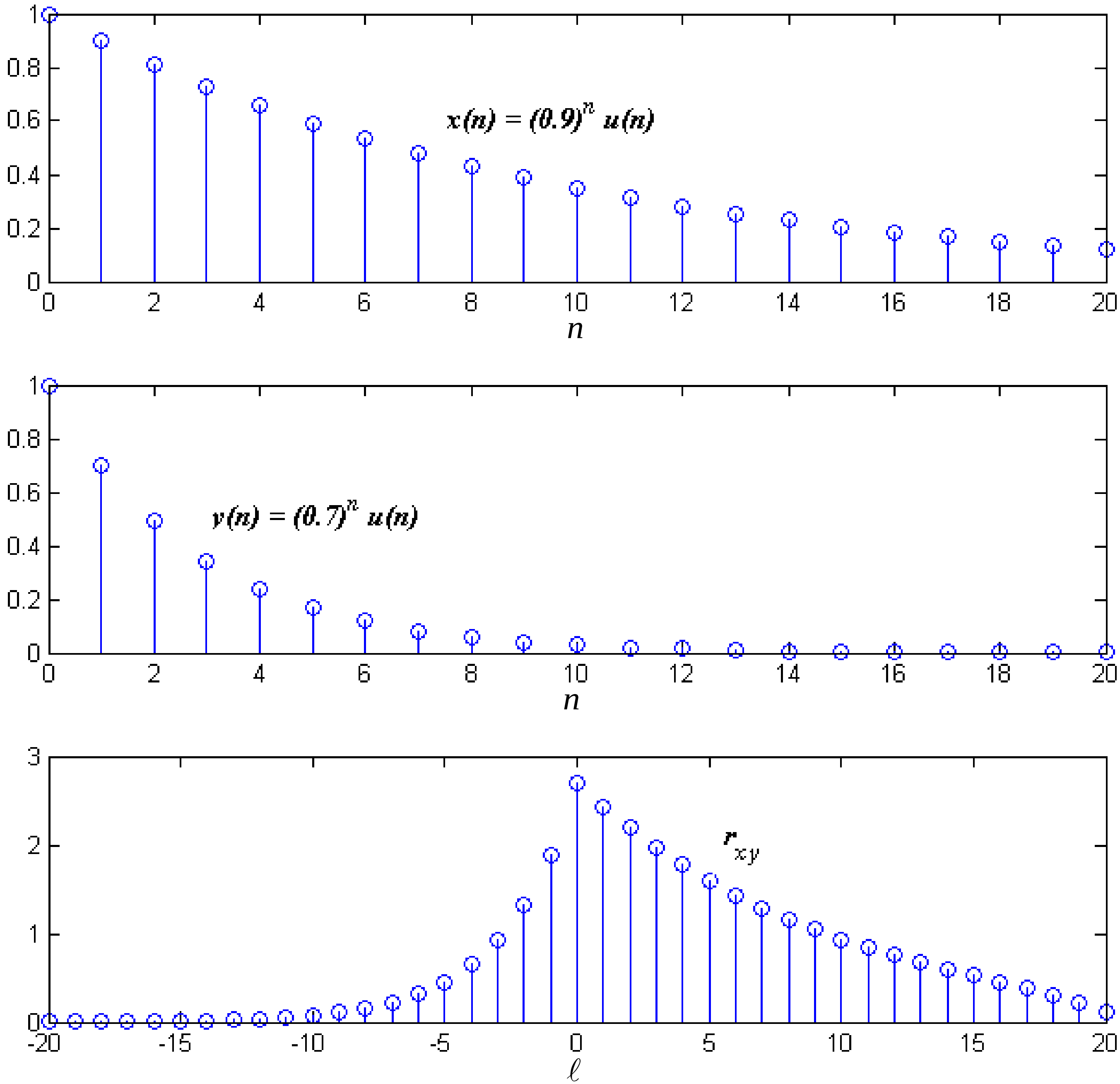
Example: Given $x(n) = a^n u(n)$ and $y(n) = b^n u(n)$, determine $r_{xy}(\ell)$

for $\ell \geq 0$, $r_{xy}(\ell) = \sum_{n=-\infty}^{\infty} a^n u(n)\, b^{n-\ell} u(n-\ell) = \sum_{n=\ell}^{\infty} (ab)^n b^{-\ell}$ $\quad i = \ell, k = 0, N = \infty$

$$= b^{-\ell} \frac{(ab)^\ell}{1-ab} = \boxed{\frac{a^\ell}{1-ab}}$$

for $\ell < 0$, $r_{xy}(\ell) = \sum_{n=0}^{\infty} (ab)^n b^{-\ell} = \boxed{\frac{b^{-\ell}}{1-ab}}$ $\quad i = 0, k = \ell, N = \infty$

Note that $r_{xy}(\ell) \neq r_{xy}(-\ell)$, *i.e.* not symmetric



x(n) = (0.9)^n u(n)
n
y(n) = (0.7)^n u(n)
n
r_xy
ℓ

4.2 Properties of correlation

1) $r_{xx}(0)$ is the total energy of the sequence $x(n)$

$$r_{xx}(0) = \sum_{n=-\infty}^{\infty} x(n)\, x(n) = E_x$$

2) $r_{xx}(0)$ is the autocorrelation peak value

$$\left| r_{xx}(\ell) \right| \le r_{xx}(0) = E_x$$

3) The <u>normalized autocorrelation</u> $\rho_{xx}(\ell)$ is bounded between –1 and 1 where

$$\rho_{xx}(\ell) = \frac{r_{xx}(\ell)}{r_{xx}(0)}$$

4) The <u>normalized cross-correlation</u> $\rho_{xy}(\ell)$ is defined as

$$\rho_{xy}(\ell) = \frac{r_{xy}(\ell)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}}$$

is bounded between –1 and 1 where

$$|r_{xy}(\ell)| \le \sqrt{r_{xx}(0)\, r_{yy}(0)} = \sqrt{E_x\, E_y}$$

5) The cross-correlation sequence satisfies the condition

$$r_{xy}(\ell) = r_{yx}(-\ell)$$

Note reverse order

If complex, then $r_{xy}(\ell) = r_{yx}^{*}(-\ell)$

6) The cross-correlation of two power sequences $x(n)$ and $y(n)$ is defined as

for autocorrelation set $y(n) = x(n)$

$$r_{xy}(\ell) \equiv \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)\, y(n-\ell)$$

If $x(n)$ and $y(n)$ are periodic, each with period N, the cross-correlation can also be expressed as

$$r_{xy}(\ell) \equiv \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, y(n-\ell)$$

because $r_{xy}(\ell)$ will repeat every kN samples (for k an integer).

7) Wiener-Khintchine theorem

Energy Spectral Density

$$r_{xx}(\ell) \overset{F}{\longleftrightarrow} S_{xx}(\omega) \qquad \text{for energy signals}$$

$$r_{xx}(\ell) \overset{F}{\longleftrightarrow} \Gamma_{xx}(\omega) \qquad \text{for power signals}$$

Power Spectral Density

In general, spectral density estimation is an entire research topic in SP. The “cross-spectral density” is also useful for signal separability.