Course Notes 6 – Transform Analysis of Linear Time-Invariant Systems

6.0 Introduction

Linear shift invariant system: $x(n)$ input; $y(n)$ output
Linear constant coefficient difference equation representation:

$$a_N y(n-N) + a_{N-1} y(n-(N-1)) + \cdots + a_0 y(n) = b_M x(n-M) + b_{M-1} x(n-(M-1)) + \cdots + b_0 x(n)$$

System function:

$$Y(z)/X(z) = H(z) \triangleq \sum_{i=0}^{M} b_i z^{-i} \Big/ \sum_{i=0}^{N} a_i z^{-i}$$

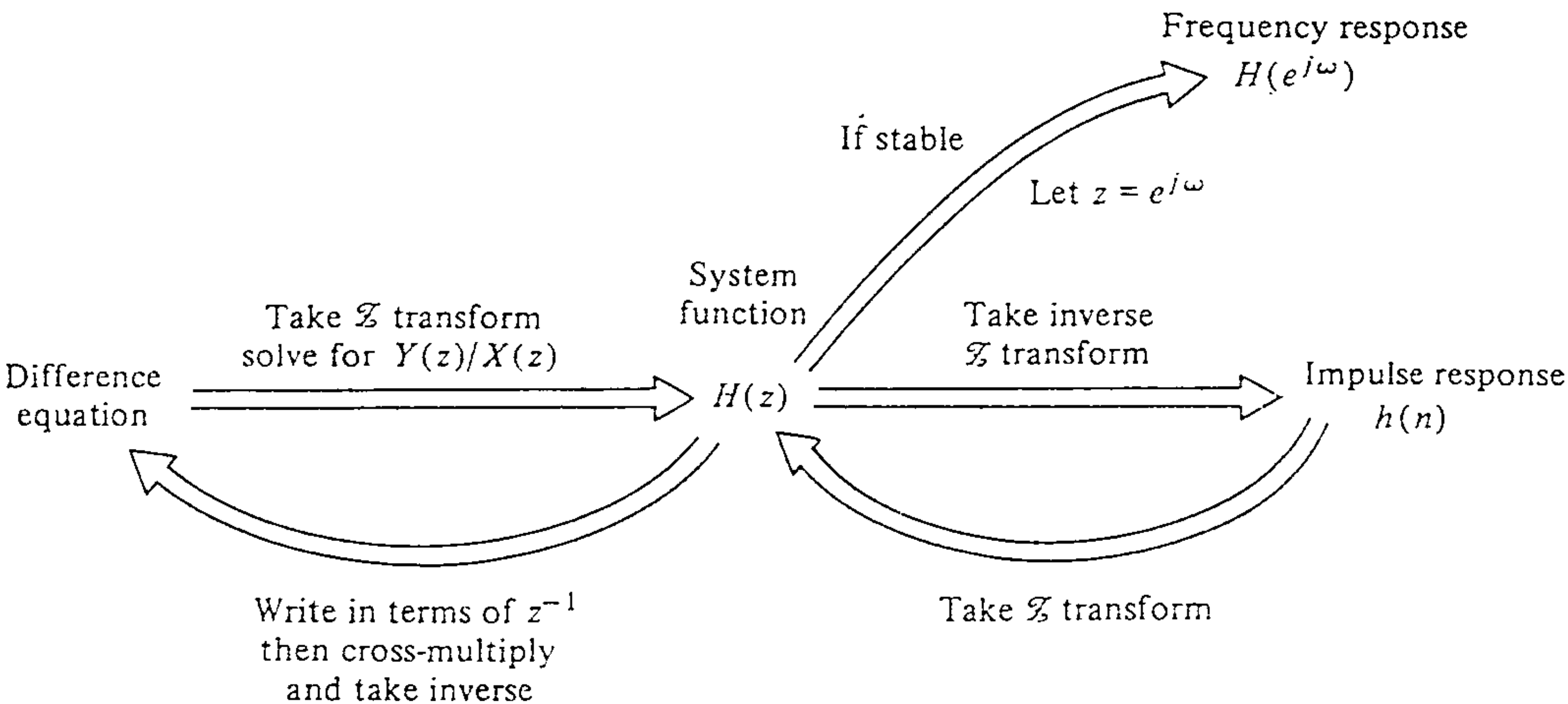
Frequency response: $H(e^{j\omega}) = H(z)\big|_{z \to e^{j\omega}}$

Unit sample response: $h(n) = \mathcal{Z}^{-1}[H(z)]$

Output as a convolution sum: $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$

Stability: if and only if $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$ (BIBO)

Causality: if and only if $h(n) = 0$ for $n < 0$



Frequency response
$H(e^{j\omega})$
If stable
Let $z = e^{j\omega}$
System
function
Take $\mathcal{Z}$ transform
solve for $Y(z)/X(z)$
Take inverse
$\mathcal{Z}$ transform
Difference
equation
$H(z)$
Impulse response
$h(n)$
Write in terms of z^{-1}
then cross-multiply
and take inverse
Take $\mathcal{Z}$ transform

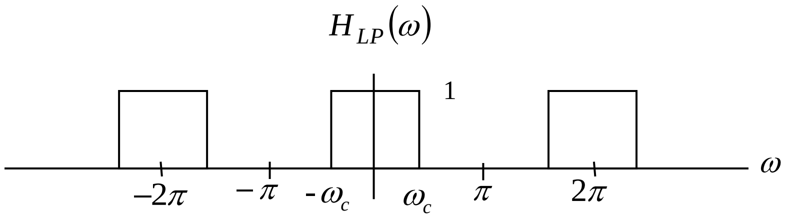
6.1 The Frequency Response of LTI Systems

$$Y(\omega) = H(\omega)X(\omega)$$

$$|Y(\omega)| = |H(\omega)||X(\omega)| \qquad \text{Magnitude}$$

$$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega) \qquad \text{Phase}$$

Ideal Frequency Selective Filters

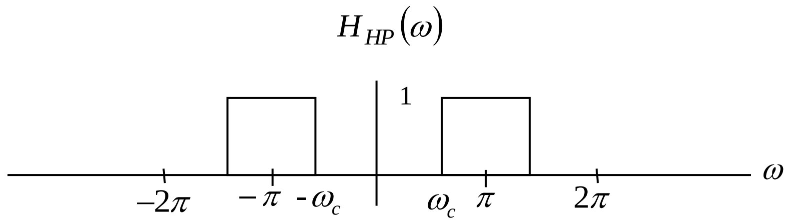


$$H_{LP}(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{LP}(n) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

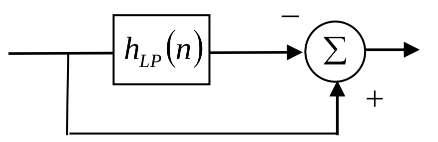
This is the ideal LPF – it is non-causal and hence not realizable

Ideal Highpass Filter:

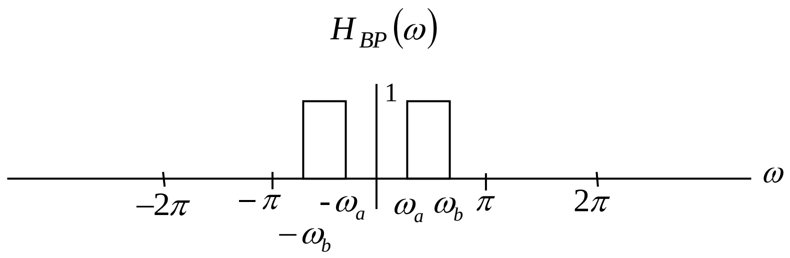


$$H_{HP}(\omega) = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & \omega_c < |\omega| \le \pi \end{cases}$$

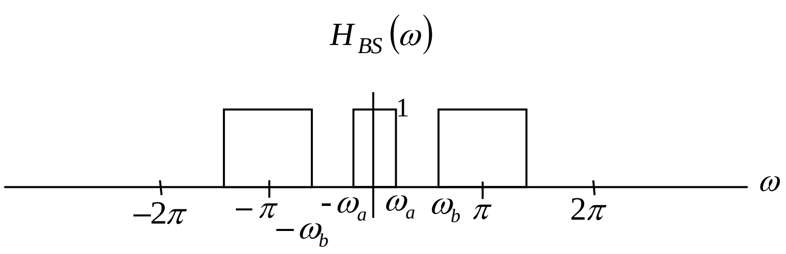
$$h_{HP}(n) = \delta(n) - h_{LP}(n) = \delta(n) - \frac{\sin \omega_c n}{\pi n}$$



Ideal Bandpass Filter:

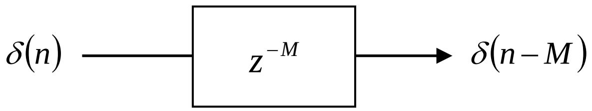


Ideal Bandstop (Notch) Filter:



Phase Distortion and Delay

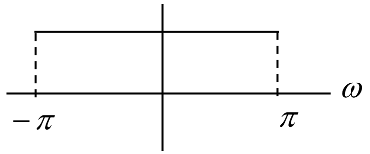
What is meant by "phase"? Consider the following system:



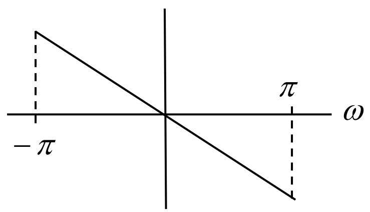
$$h(n) = \delta(n-M)$$

$$H(\omega) = e^{-j\omega M}$$

$|H(\omega)| = 1$



$\arg H(\omega) = -\omega M$



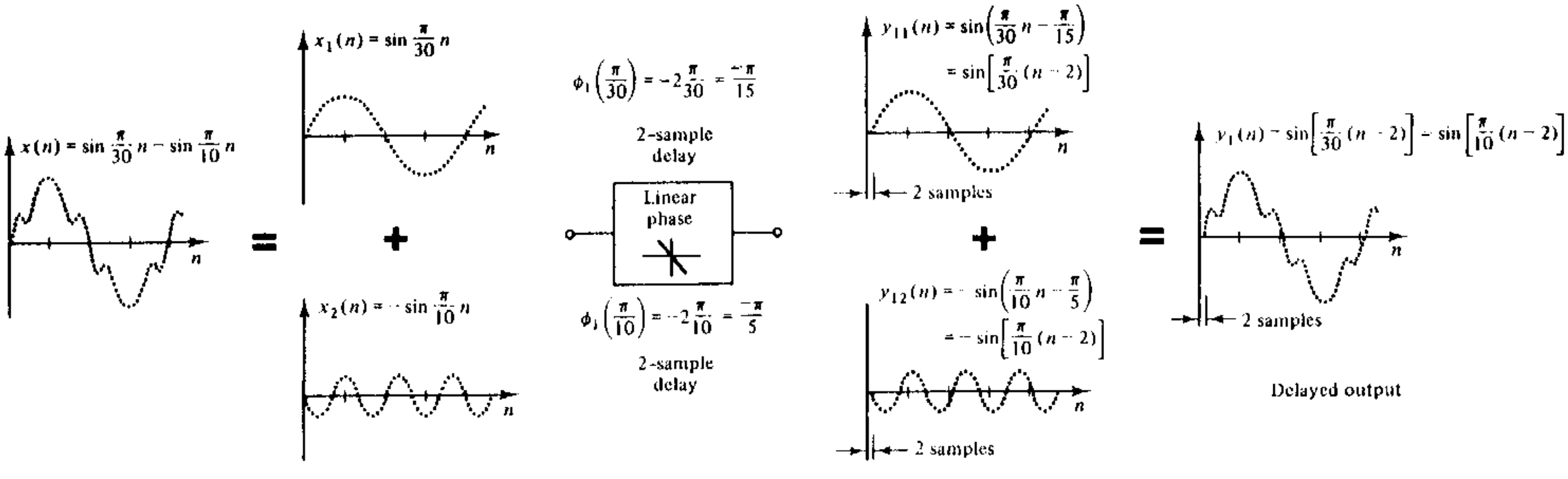
Phase delay is the propagation time of a particular frequency through a linear system.

A convenient measure of linearity is group delay (sometimes called "envelope delay"). Group delay is the derivative of phase delay.

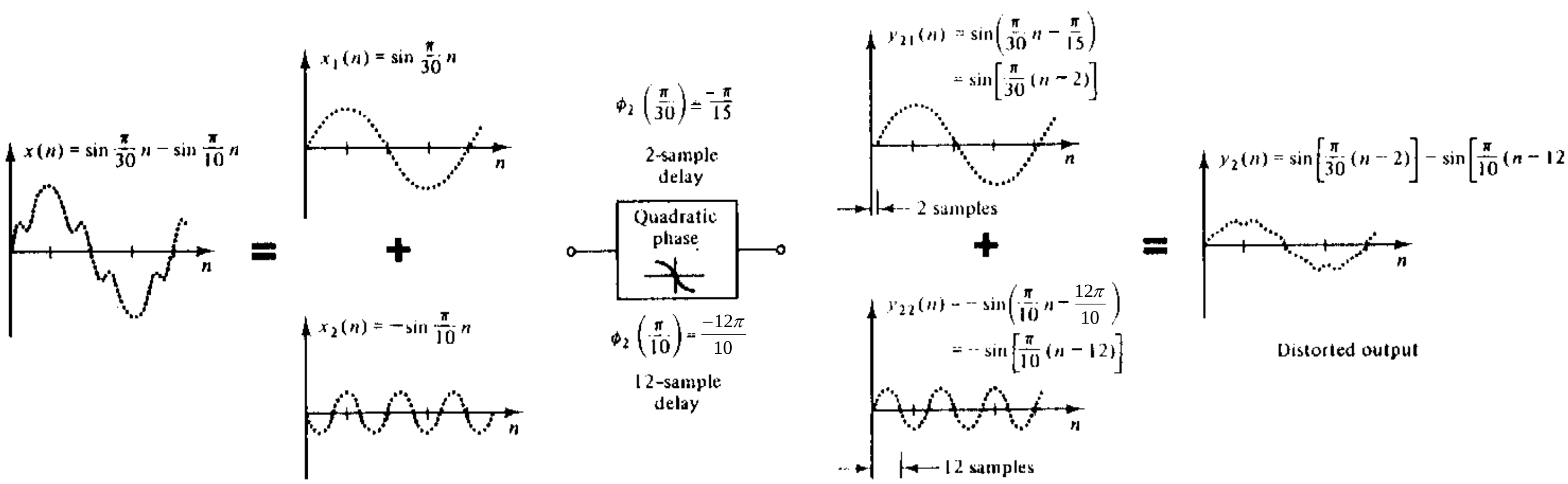
$$\tau(\omega) = -\frac{d}{d\omega}\left[\arg H(\omega)\right]$$

There will be distortion whenever the group delay is not constant over the bandwidth of a signal.

Here is the significance of delay distortion on a signal

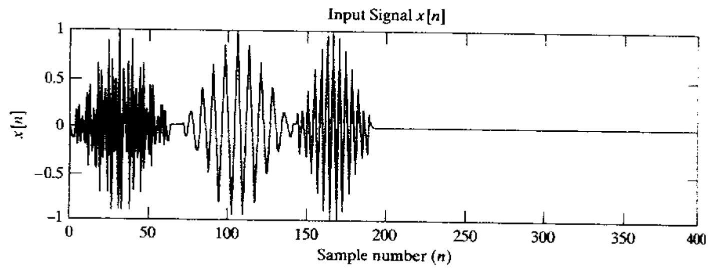


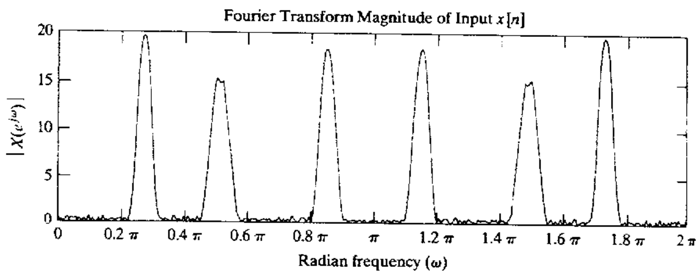
(a) *Linear phase:* $\phi_1(\omega) = -2\omega$

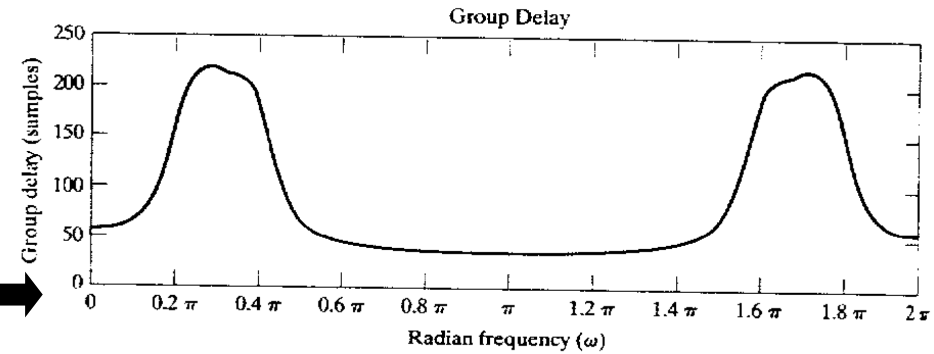


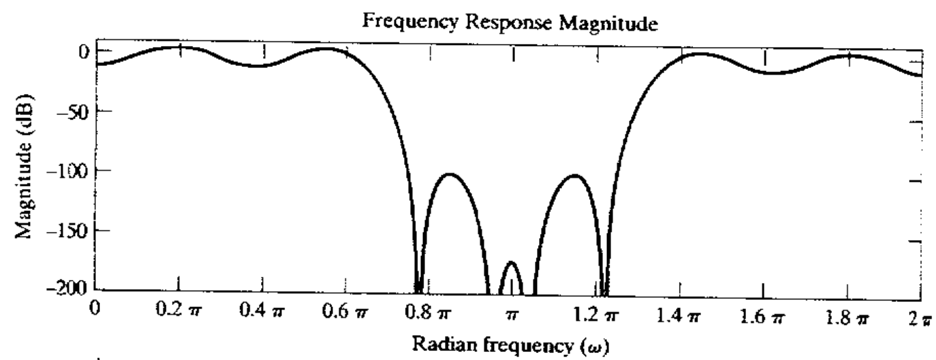
(b) *Quadratic phase:* $\phi_2(\omega) = \frac{-150}{\pi}\omega^2 + 3\omega$

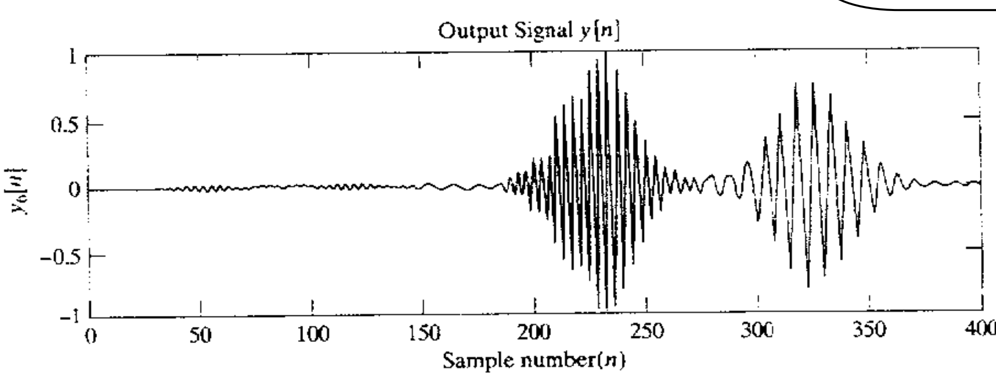
Example: A filter having group delay distortion





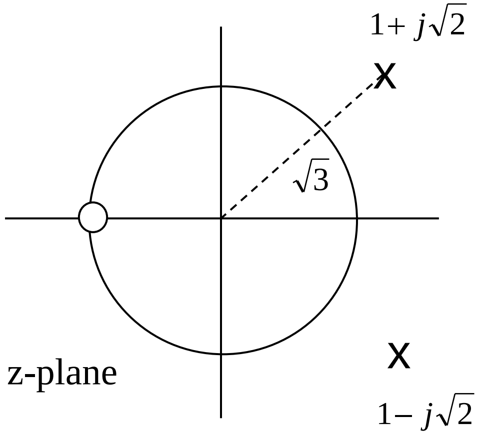






6.2 System Functions for Systems Characterized by LCCDE

Given a pole-zero diagram we can, to within a constant, determine $H(z)$, and therefore the difference equation of the system.



$$H(z)=\frac{z+1}{\left(z-1-j\sqrt{2}\right)\left(z-1+j\sqrt{2}\right)}=\frac{z+1}{z^2-2z+3}$$

In pole-zero form – correct to within a constant

Convert to inverse powers of z:

$$H(z)=\frac{z^{-1}+z^{-2}}{1-2z^{-1}+3z^{-2}}=\frac{Y(z)}{X(z)}$$

Finally, determine the difference equation by cross multiplying;

$$Y(z)\left(1-2z^{-1}+3z^{-2}\right)=X(z)\left(z^{-1}+z^{-2}\right)$$

Take the inverse z-transform and use the delay property to get the difference equation:

$$y(n)=2y(n-1)-3y(n-2)+x(n-1)+x(n-2)$$

6.3 Frequency Response for Rational System Functions

Frequency Response of a Single Zero or Pole

Let's look at how to estimate the frequency response of a system having the following system function:

$$H(z) = 1 - az^{-1}$$

This is a simple FIR filter. Let's put it in pole-zero form:

$$H(z) = \frac{z-a}{z}$$

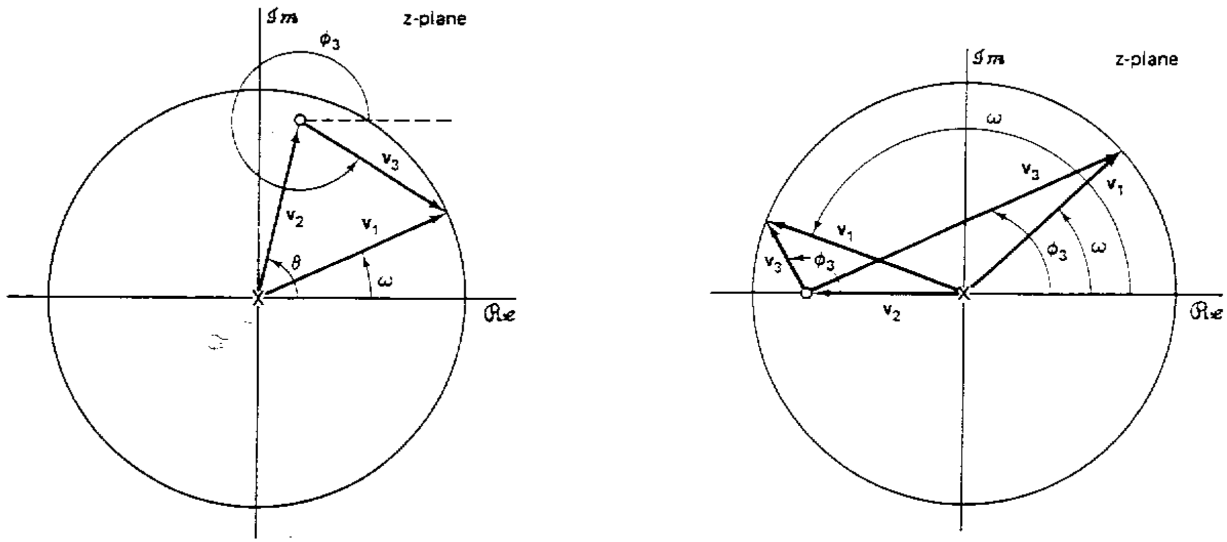
Finally, allow '*a*' to be the location of a complex zero, where

$$a = re^{j\theta}$$

In this case, we will assume that r < 1. The system function in pole-zero form is:

$$H(z) = \frac{z - re^{j\theta}}{z}$$

Two possible pole-zero diagrams for this system function (for two choices of r and θ) are shown below:



The frequency response of this system (assuming that the unit circle lies in the region of convergence) is:

$$H\ \ \omega\ = \frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}$$

The magnitude of the frequency response is:

$$\left|H(\omega)\right| = \left|\frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}\right| = \frac{\left|V_1 - V_2\right|}{\left|V_1\right|} = \frac{\left|V_3\right|}{\left|V_1\right|}$$

Now we can look at the pole-zero diagrams and interpret the vectors. Note that the significant vectors are the ones drawn from the pole and zero to some point on the unit circle

Frequency Response of Multiple Zeros or Poles

Let's look at another example. Consider a second order system:

$$H(z) = \frac{1}{\left(1 - az^{-1}\right)\left(1 - bz^{-1}\right)}$$

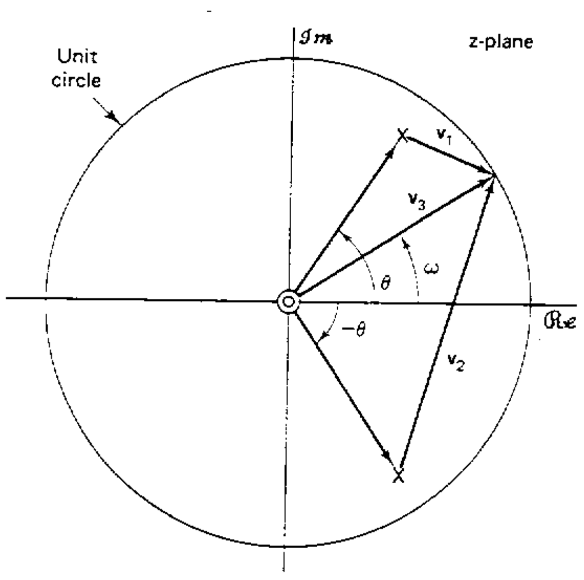
In this case we let 'a' and 'b' be complex conjugate poles – indicating a real system. Expressing $H(z)$ in pole-zero form, we have:

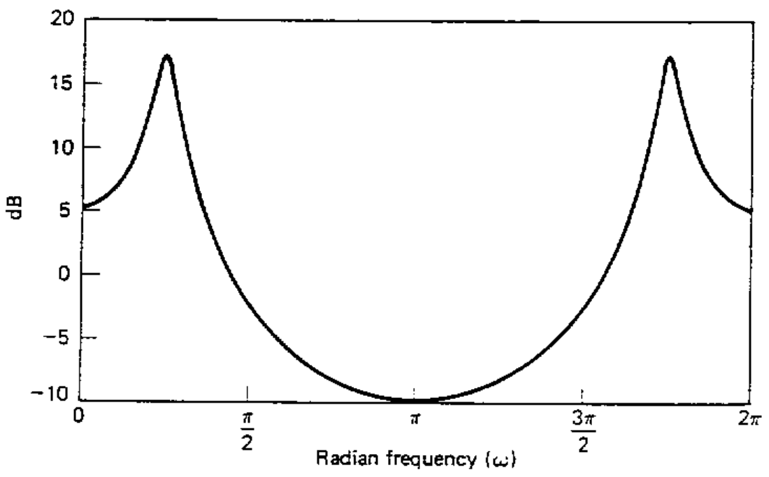
$$H(z) = \frac{z^2}{\left(z - re^{j\theta}\right)\left(z - re^{-j\theta}\right)}$$

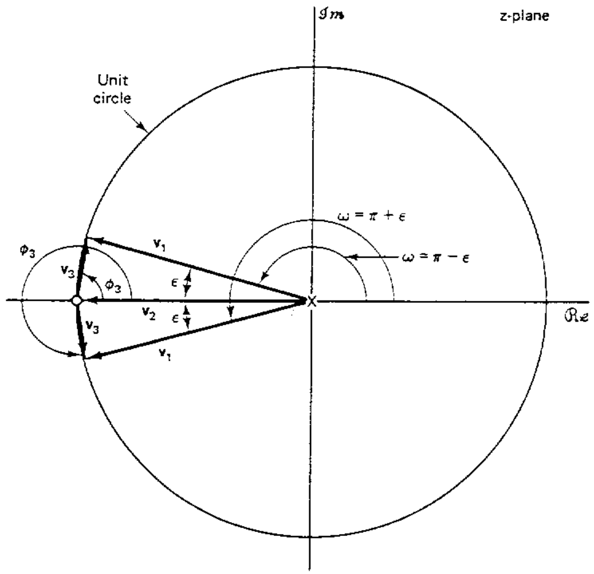
The magnitude of the frequency response is:

$$\left|H(\omega)\right| = \left|\frac{\left(e^{j\omega}\right)^2}{\left(e^{j\omega} - re^{j\theta}\right)\left(e^{j\omega} - re^{-j\theta}\right)}\right| = \frac{\left|V_3\right|^2}{\left|V_1\right|\left|V_2\right|}$$

Try to develop an intuitive feel for the appearance of the frequency response characteristic based on the pole-zero placement in the *z*-plane







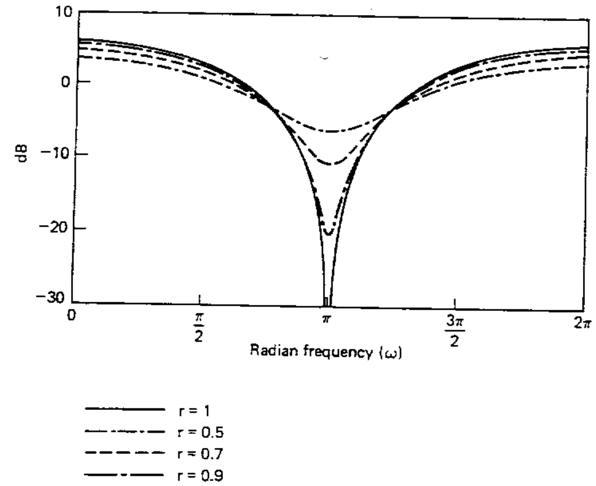
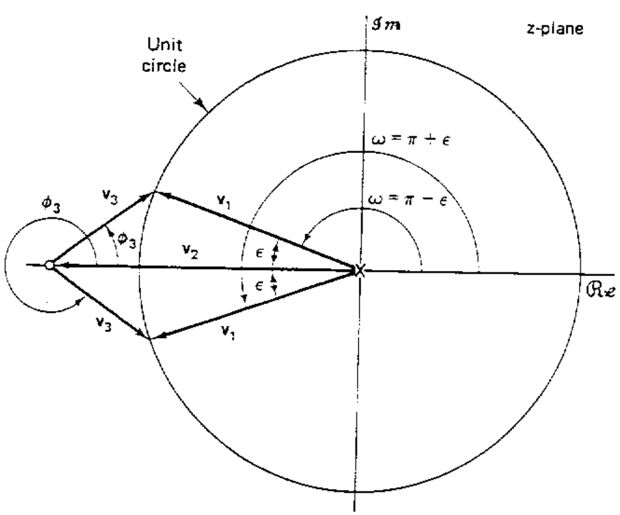


Figure 5.9 z-plane vectors for a zero at $z = -1$ for two different frequencies close to π



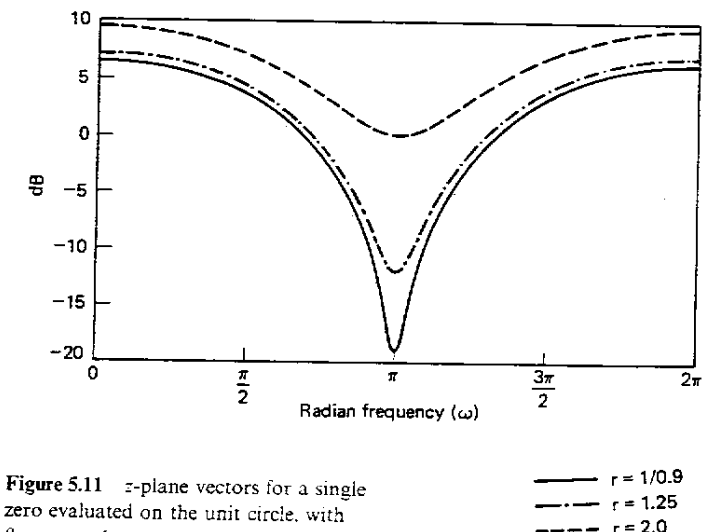


Figure 5.11 z-plane vectors for a single zero evaluated on the unit circle, with $\theta = \pi$, $r > 1$.

As an example of a system function with both poles and zeros, consider

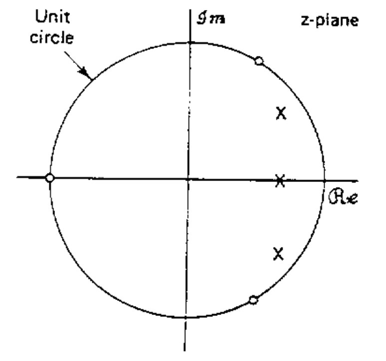
$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}.$$

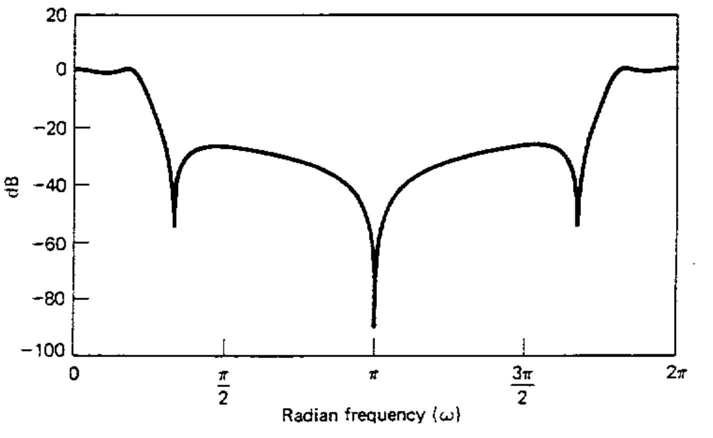
The zeros of this system function are at

Radius	Angle
1	π rad
1	$\pm$ 1.0376 rad (59.45°)

The poles are at

Radius	Angle
0.683	0
0.892	$\pm$ 0.6257 rad (35.85°)





Simple lowpass to highpass filter transformation

If $h_{LP}(n)$ is the impulse response of a lowpass filter with frequency response $H_{LP}(\omega)$, a highpass filter can be obtained by translating $H_{LP}(\omega)$ by π radians. Thus,

$$H_{HP}(\omega) = H_{LP}(\omega - \pi)$$

Since a frequency translation of π radians is equivalent to multiplication of the impulse response by $e^{j\pi n}$, the impulse response of the highpass filter is

$$h_{HP}(n) = (e^{j\pi})^n h_{LP}(n) = (-1)^n h_{LP}(n).$$

Also, it can be shown that for the lowpass filter described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

the highpass filter difference equation obtained by translating the lowpass filter by π radians is

$$y(n) = -\sum_{k=1}^{N} (-1)^k a_k y(n-k) + \sum_{k=0}^{M} (-1)^k b_k x(n-k)$$

6.4 Relationship Between Magnitude and Phase

We now investigate issues surrounding the magnitude squared function, which is valuable in filter design.

In general, we have

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = H(z)H^*\left(1/z^*\right)\Big|_{z=e^{j\omega}}$$

so if… $$H(z) = \left(\frac{b_o}{a_o}\right)\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)}$$

and… $$H^*\left(1/z^*\right) = \left(\frac{b_o}{a_o}\right)\frac{\prod_{k=1}^{M}\left(1-c_k^* z\right)}{\prod_{k=1}^{N}\left(1-d_k^* z\right)}$$

$$|H(z)|^2 = \left(\frac{b_o}{a_o}\right)^2 \frac{\prod_{k=1}^{M}\left(1-c_k^* z\right)\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k^* z\right)\left(1-d_k z^{-1}\right)}$$

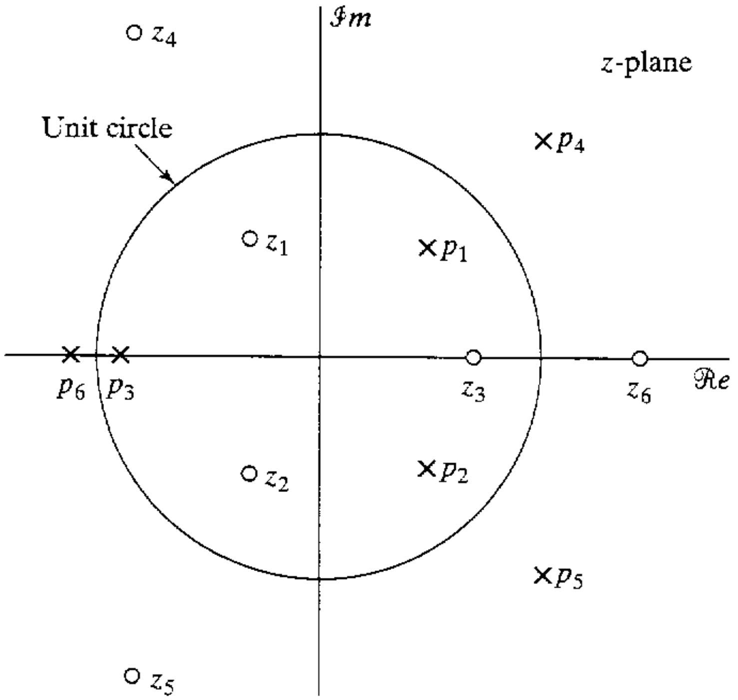
What do we know about $H(z)$ by observing $|H(z)|^2$?

By observing $|H(z)|^2$, we can conclude the following:

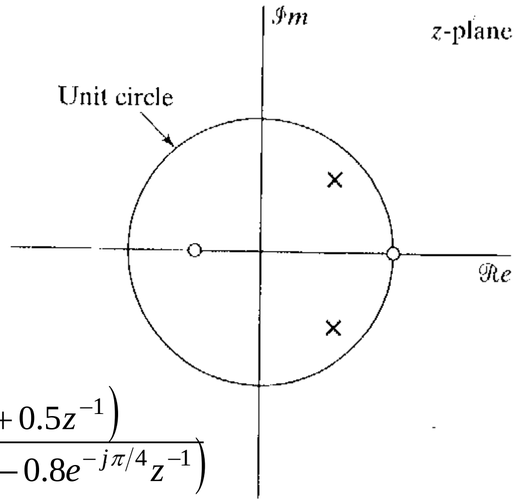
- Poles and zeros occur in conjugate reciprocal pairs (one associated with $H(z)$ and one with $H^*\left(1/z^*\right)$)
- If $H(z)$ is assumed to be causal and stable, then all poles of $H(z)$ must be inside the unit circle.
- Zeros of $H(z)$ cannot be uniquely identified

Example: Suppose we are given the pole-zero diagram shown below with p_1, p_2, and p_3 associated with $H(z)$. The zeros associated with $H(z)$ are:

$$\left\{(z_3 \text{ or } z_6), (z_2 \text{ or } z_5), (z_1 \text{ or } z_4)\right\}$$

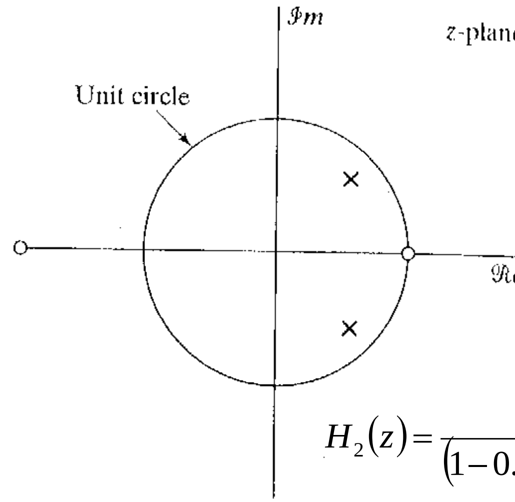


The example on the next page illustrates that different transfer functions can have the same magnitude squared function.



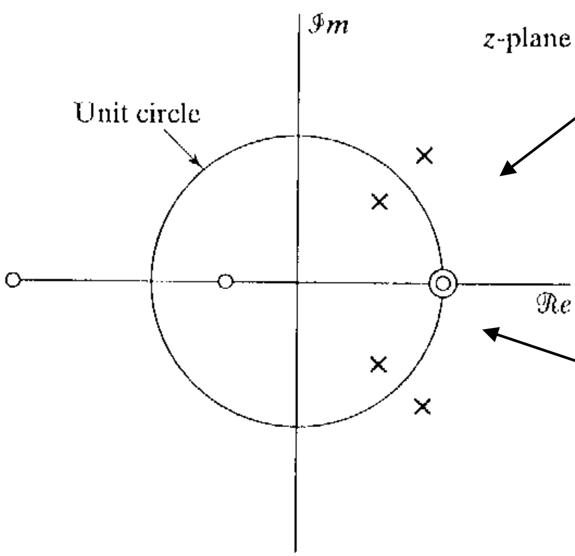
$$H_1(z) = \frac{2\left(1 - z^{-1}\right)\left(1 + 0.5z^{-1}\right)}{\left(1 - 0.8e^{j\pi/4}z^{-1}\right)\left(1 - 0.8e^{-j\pi/4}z^{-1}\right)}$$

(a)



$$H_2(z) = \frac{\left(1 - z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1 - 0.8e^{j\pi/4}z^{-1}\right)\left(1 - 0.8e^{-j\pi/4}z^{-1}\right)}$$

(b)



$$|H_1(z)|^2 = H_1(z)H_1^*\left(1/z^*\right)$$

$$= \frac{4\left(1 - z^{-1}\right)\left(1 + 0.5z^{-1}\right)\left(1 - z\right)\left(1 + 0.5z\right)}{\left(1 - 0.8e^{j\pi/4}z^{-1}\right)\left(1 - 0.8e^{-j\pi/4}z^{-1}\right)\left(1 - 0.8e^{j\pi/4}z\right)\left(1 - 0.8e^{-j\pi/4}z\right)}$$

The magnitude squared functions are identical.

$$|H_2(z)|^2 = H_2(z)H_2^*\left(1/z^*\right)$$

$$= \frac{\left(1 - z^{-1}\right)\left(1 + 2z^{-1}\right)\left(1 - z\right)\left(1 + 2z\right)}{\left(1 - 0.8e^{j\pi/4}z^{-1}\right)\left(1 - 0.8e^{-j\pi/4}z^{-1}\right)\left(1 - 0.8e^{j\pi/4}z\right)\left(1 - 0.8e^{-j\pi/4}z\right)}$$

(c)

6.5 All-Pass Systems

A stable system function of the form

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

has a frequency response magnitude that is independent of ω, as shown below:

$$\left|H_{ap}(z)\right|\Bigg|_{z=e^{j\omega}} = \left|\frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}\right| = \left|e^{-j\omega}\right| \frac{\left|1 - a^* e^{j\omega}\right|}{\left|1 - ae^{-j\omega}\right|} = 1$$

unity magnitude

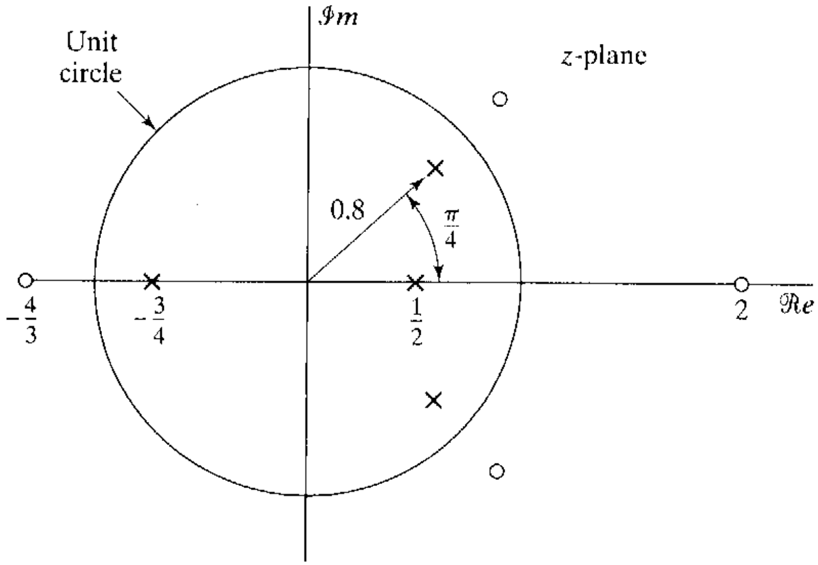
$$\left|1 - a^* e^{j\omega}\right| = \left|1 - ae^{-j\omega}\right|$$

This is an <u>all-pass</u> system. In general, for $h(n)$ real:

$$H_{ap}(z) = A \prod_{\ell=1}^{M_r} \frac{(z^{-1} - d_\ell)}{(1 - d_\ell z^{-1})} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

real poles/zeros

Complex-conjugate pairs of poles/zeros



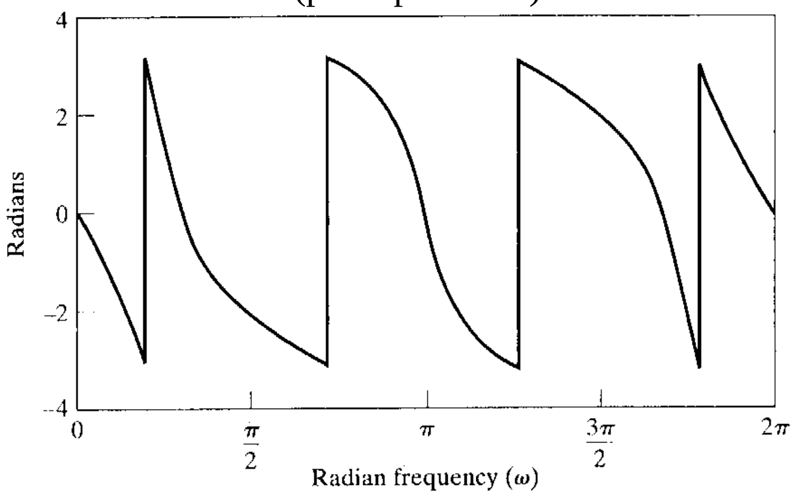
A typical pole-zero plot for an all-pass system

For a causal and stable all-pass system:

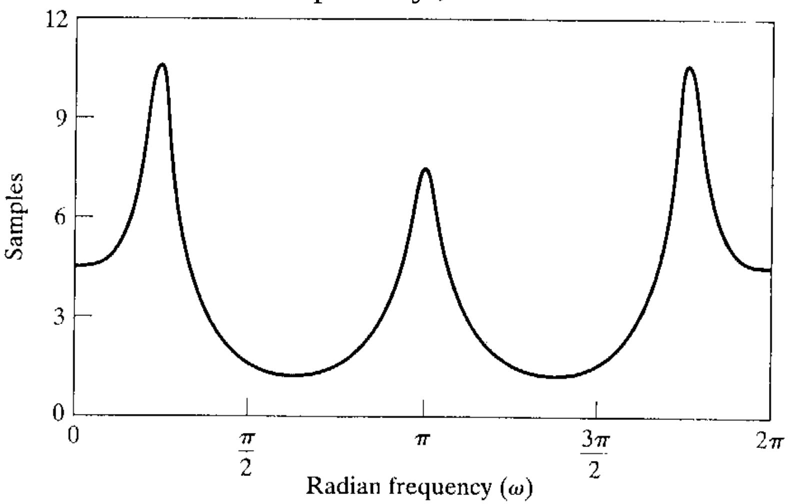
$$|e_k| < 1 \qquad |d_\ell| < 1$$

Useful as phase-equalizers to compensate for systems that have an undesired phase response, so the overall phase response is linear

Phase (principal value)



Group Delay



6.6 Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

For a given FIR system function $H(z)$ with M zeros, we denote it as *minimum-phase* if all of the zeros lie <u>inside</u> the unit circle. => the net phase change from $\omega = 0$ to $\omega = \pi$ is zero or

$$\angle H_{\omega}(\pi) - \angle H_{\omega}(0) = 0$$

For a given FIR system function $H(z)$ with M zeros, we denote it as *maximum-phase* if all of the zeros lie <u>outside</u> the unit circle. => the net phase change from $\omega = 0$ to $\omega = \pi$ is $M\pi$ or

$$\angle H_{\omega}(\pi) - \angle H_{\omega}(0) = M\pi$$

for example:

$$H_1(z) = 1 + \frac{1}{2}z^{-1} = z^{-1}(z + \frac{1}{2}) \longrightarrow \text{zero at -1/2}$$

$$H_2(z) = \frac{1}{2} + z^{-1} = z^{-1}(\frac{1}{2}z + 1) \longrightarrow \text{zero at -2}$$

reciprocal

The frequency responses of the individual functions can be expressed as

$$|H_1(\omega)| = |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos\omega} \longrightarrow \text{identical magnitude responses}$$

$$\theta_1(\omega) = \angle\left[e^{-j\omega}\left(e^{j\omega} + \frac{1}{2}\right)\right] \longrightarrow \theta_1(\pi) - \theta_1(0) = 0 - 0 = 0$$

$$\theta_2(\omega) = \angle\left[e^{-j\omega}\left(\frac{1}{2}e^{j\omega} + 1\right)\right] \longrightarrow \theta_2(\pi) - \theta_2(0) = \pi - 0 = \pi$$

Hence, $H_1(z)$ is minimum-phase and $H_2(z)$ is maximum-phase.

Minimum-phase implies minimum delay, while maximum-phase implies maximum delay.

An FIR system with zeros <u>both inside and outside</u> the unit circle is called a *mixed-phase system.*

A rational IIR system function

$$H(z) = \frac{B(z)}{A(z)}$$

is called *minimum-phase* if all of its poles and zeros are inside the unit circle.

For a stable & causal system (all roots of $A(z)$ lie inside the unit circle), it is called *maximum-phase* if all of the zeros are outside the unit circle, and *mixed-phase* if the zeros are both inside & outside the unit circle.

A stable, *minimum-phase* system has a stable inverse that is also *minimum-phase* with the system function

$$H^{-1}(z) = \frac{A(z)}{B(z)}$$

poles become zeros
and
zeros become poles

If a system is *minimum-phase*, we can uniquely determine $H(z)$ from $|H(z)|^2$.

Decomposition of a mixed-phase system

Any mixed-phase system can be expressed as the product

$$H(z) = H_{\min}(z) H_{ap}(z)$$

Minimum-phase system — All-pass system

Given the rational system function

$$H(z) = \frac{B(z)}{A(z)}$$

we can decompose $B(z) = B_1(z)\,B_2(z)$ and $A(z) = A_1(z)\,A_2(z)$ where $B_1(z)$ and $A_1(z)$ have all their roots <u>inside</u> the unit circle and $B_2(z)$ and $A_2(z)$ have all their roots <u>outside</u> the unit circle. Therefore,

$$H(z) = \frac{B_1(z)B_2(z)}{A_1(z)A_2(z)}$$

Since $B_2(z)$ and $A_2(z)$ have all their roots <u>outside</u> the unit circle, $B_2(z^{-1})$ and $A_2(z^{-1})$ must have all their roots <u>inside</u> the unit circle. As such, $H(z) = H_{\min}(z)\, H_{ap}(z)$ can be expressed as

$$H_{\min}(z) = \frac{B_1(z)}{A_1(z)} \frac{B_2(z^{-1})}{A_2(z^{-1})} \qquad H_{ap}(z) = \frac{B_2(z)}{B_2(z^{-1})} \frac{A_2(z^{-1})}{A_2(z)}$$

all poles and zeros inside the unit circle

for every pole/zero there is a reciprocal zero/pole

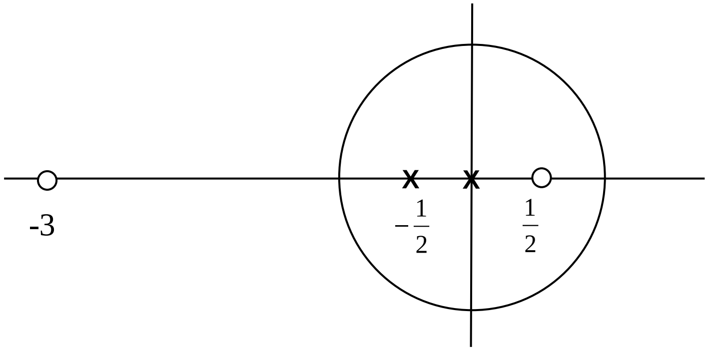
For the cases where $H(z)$ is causal and stable, this simplifies to

$$H_{\min}(z) = \frac{B_1(z) B_2(z^{-1})}{A(z)} \qquad H_{ap}(z) = \frac{B_2(z)}{B_2(z^{-1})}$$

there is no $A_2(z)$

The all-pass filter is now stable.

Example: $H(z)=\frac{(z+3)(z-\frac{1}{2})}{z(z+\frac{1}{2})} \quad |z|>\frac{1}{2}$



This stable system is neither minimum phase nor all-pass

However, we can represent this system as a cascade combination of a minimum phase system and an all-pass system as follows:

$$H(z)=H_{\min}(z)H_{ap}(z)$$

$$H(z)=\frac{(z+3)(z-\frac{1}{2})}{z(z+\frac{1}{2})}=\frac{(z+\frac{1}{3})(z-\frac{1}{2})}{z(z+\frac{1}{2})}\cdot\frac{z+3}{z+\frac{1}{3}}$$

Minimum-phase component

zeros @ $-\frac{1}{3}, \frac{1}{2}$

poles @ $-\frac{1}{2}, 0$

All-pass component

zero @ -3

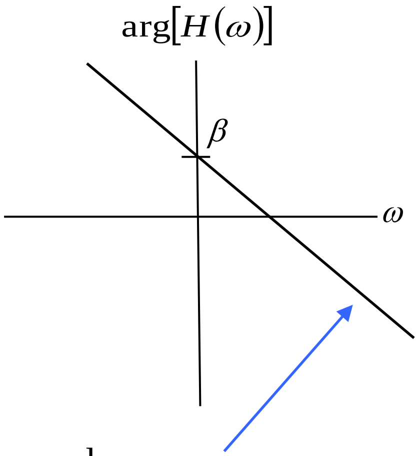
pole @ $-\frac{1}{3}$

6.7 Linear Systems with Generalized Linear Phase

When we discussed the advantages of linear phase earlier, we concluded that it is highly desirable to have the phase characteristic of a system to be linear, or as linear as possible.

Let's now turn our attention to recognizing and analyzing systems having linear phase. A system has linear phase if its frequency response can be expressed in the form:

$$H(\omega) = \underbrace{A(\omega)}_{\text{real-valued}} e^{-j\alpha\omega + j\beta}$$



slope = α

β = 0, π, or ±π/2

where

$$\arg[H(\omega)] = -\alpha\omega + \beta$$

$$-\frac{d}{d\omega}\left(\arg[H(\omega)]\right) = \alpha$$

Can we design systems that are guaranteed to have exactly linear phase? Yes.

The key lies in the symmetry of the impulse response of an FIR system.

It is always possible to design an FIR system with linear phase – while it is nearly impossible to design an IIR system with linear phase. We now show that a causal FIR transfer function of length N,

$$H(z)=\sum_{n=0}^{N-1}h(n)z^{-n}$$

has a linear phase if its impulse response $h(n)$ is either symmetric:

$$h(n)=h(N-1-n), \quad 0\le n\le N-1$$

or anti-symmetric:

$$h(n)=-h(N-1-n), \quad 0\le n\le N-1$$

Since the impulse response can be even or odd in length, we can define four types of symmetry as shown in the diagram:

(a) – odd symmetric
(b) – even symmetric
(c) – odd anti-symmetric
(d) – even anti-symmetric

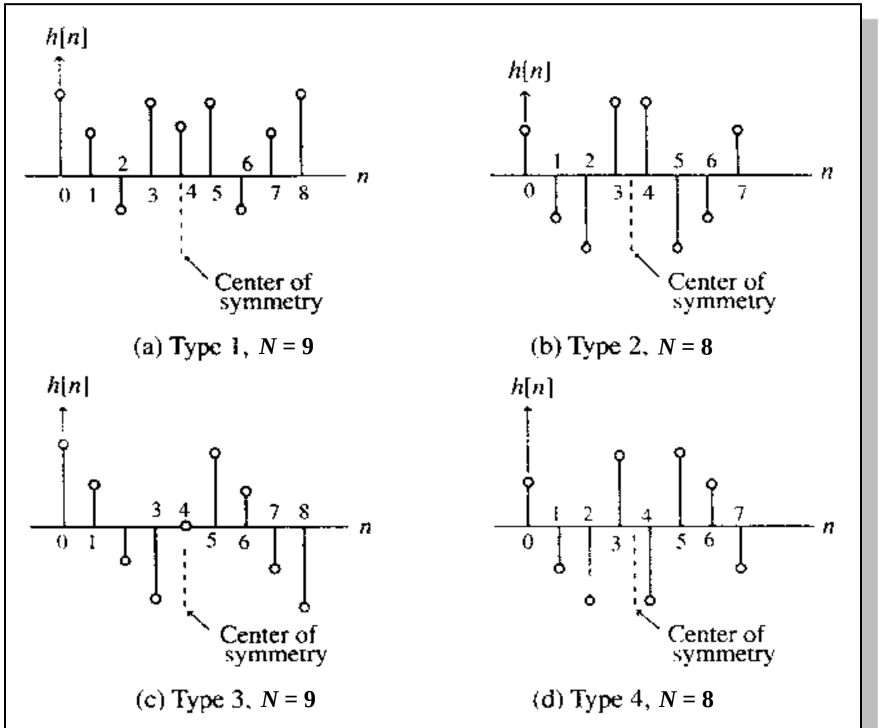


Illustration of the four types of impulse response symmetry.

We will demonstrate an FIR system can be designed to have exactly linear phase. We will use case (b) in which the impulse response is real and even symmetric.

Begin with $H(\omega)=\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$ for N even and break the summation into two equal-length parts

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \underbrace{\sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n}}$$

Next, use a substitution of variables with $m = N-1-n$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{m=\frac{N}{2}-1}^{0} h(N-1-m) e^{-j\omega(N-1-m)}$$

At this point we can employ the symmetry property of the impulse response: $h(m) = h(N-1-m)$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(m) e^{-j\omega(N-1-m)} = \sum_{n=0}^{\frac{N}{2}-1} h(n)\left[e^{-j\omega n} + e^{-j\omega(N-1-n)}\right]$$

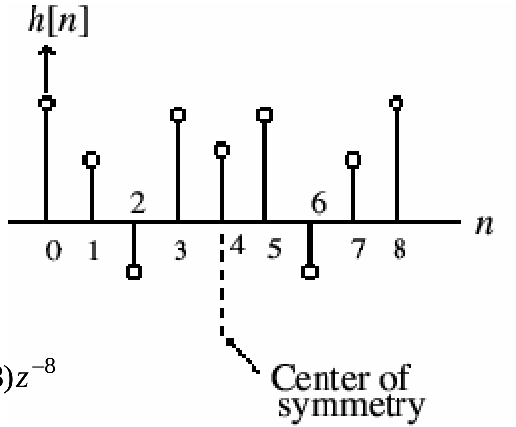
$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right] = \underbrace{e^{-j\omega\left(\frac{N-1}{2}\right)}}_{\text{linear phase term}} \underbrace{\sum_{n=0}^{\frac{N}{2}-1} 2h(n)\cos\left[\omega\left(\frac{N-1}{2}-n\right)\right]}_{\text{real amplitude term}}$$

N even

linear phase term

real amplitude term

Next we will examine the four types of FIR systems in more detail and examine their linear phase behavior and their pole/zero characteristics.



Type 1: Symmetric Filter with Odd Length

- In this case, N is odd
- Assume $N = 9$ for simplicity
- The transfer function $H(z)$ is given by

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

- Because of symmetry, we have $h(0) = h(8)$, $h(1) = h(7)$, $h(2) = h(6)$, and $h(3) = h(5)$
- Thus, we can write

$$H(z) = h(0)(1 + z^{-8}) + h(1)(z^{-1} + z^{-7}) + h(2)(z^{-2} + z^{-6}) + h(3)(z^{-3} + z^{-5}) + h(4)z^{-4}$$

$$= z^{-4}\left[h(0)(z^{4} + z^{-4}) + h(1)(z^{3} + z^{-3}) + h(2)(z^{2} + z^{-2}) + h(3)(z + z^{-1}) + h(4)\right]$$

$$z = e^{j\omega}$$

- The corresponding frequency response is then given by

$$H(\omega) = e^{-j4\omega}\left[2h(0)\cos(4\omega) + 2h(1)\cos(3\omega) + 2h(2)\cos(2\omega) + 2h(3)\cos(\omega) + h(4)\right]$$

- The quantity inside the brackets is a real function of ω.

- The phase function here is given by

$$\theta(\omega) = -4\omega + \beta$$

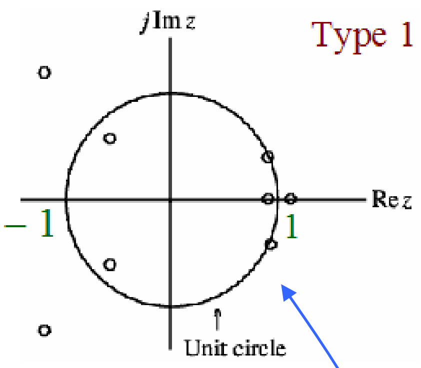
where β is either 0 or π, and hence, it is a linear function of ω in the generalized sense.

- The group delay is given by

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = 4$$

indicating a constant group delay of 4 samples

A typical pole-zero plot



Note that the zeros occur in reciprocal pairs

- In the general case for Type 1 (*i.e.* odd-length) FIR filters, the frequency response is of the form

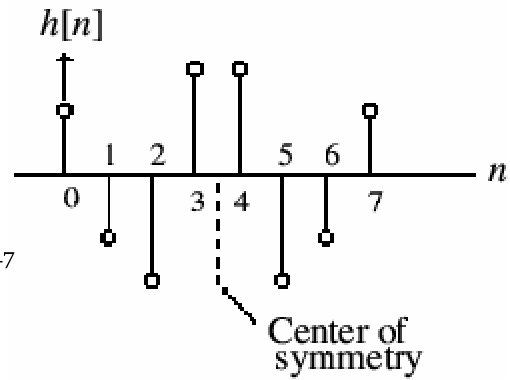
$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} A(\omega)$$

where the **amplitude response** $A(\omega)$, also called the **zero-phase response**, is of the form

$$A(\omega) = h\left(\tfrac{N-1}{2}\right) + 2\sum_{n=0}^{\frac{N-3}{2}} h(n)\cos\left[\omega\left(\tfrac{N-1}{2} - n\right)\right]$$

- This filter can be used for most applications in LP, BP, HP or BS form.

Type 2: Symmetric Filter with Even Length



- In this case, N is even
- Assume $N = 8$ for simplicity
- The transfer function is of the form

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}$$

- Making use of the symmetry of the impulse response coefficients, the transfer function can be written as

$$H(z) = h(0)(1 + z^{-7}) + h(1)(z^{-1} + z^{-6}) + h(2)(z^{-2} + z^{-5}) + h(3)(z^{-3} + z^{-4})$$

- The corresponding frequency response is given by

$$H(\omega) = e^{-j\omega\left(\frac{7}{2}\right)}\left[2h(0)\cos\left(\frac{7\omega}{2}\right) + 2h(1)\cos\left(\frac{5\omega}{2}\right) + 2h(2)\cos\left(\frac{3\omega}{2}\right) + 2h(3)\cos\left(\frac{\omega}{2}\right)\right]$$

- As before, the quantity inside the brackets is a real function of ω.
- Here the phase function is given by

$$\theta(\omega) = -\frac{7}{2}\omega + \beta$$

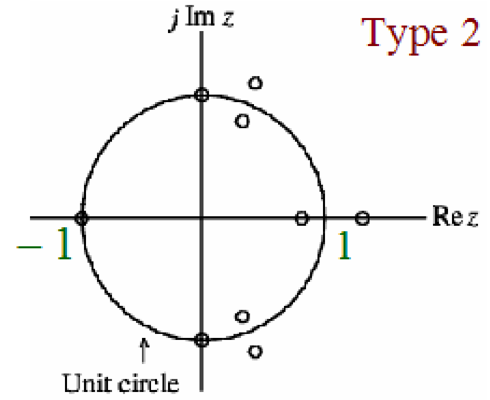
where again β is either 0 or π. As a result, the phase is also a linear function of ω in the generalized sense.

- The corresponding group delay is

$$\tau(\omega) = \frac{7}{2}$$

  indicating a group delay of 3.5 samples.

A typical pole-zero plot



- The expression for the frequency response in the general case for Type 2 (*i.e.* even-length) FIR filters is of the form

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} A(\omega)$$

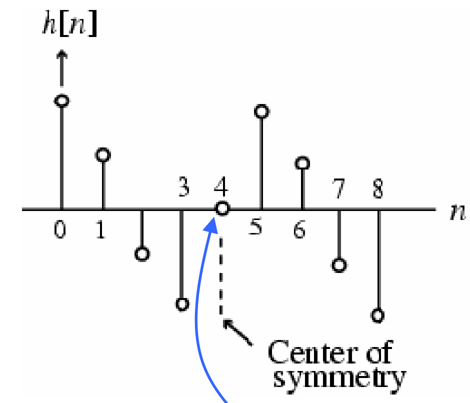
  where the amplitude response is given by

$$A(\omega) = 2\sum_{n=0}^{\frac{N}{2}-1} h(n)\cos\left[\omega\left(\frac{N-1}{2} - n\right)\right]$$

- For this filter there is always a zero at $\omega = \pi$. Hence, we cannot use systems of this type as HP or BS filters.

Type 3: Anti-symmetric Filter with Odd Length

- In this case, N is odd
- Assume $N = 9$ for simplicity
- Applying the anti-symmetry condition we get



by anti-symmetry

$$h\left(\frac{N-1}{2}\right) = -h\left(\frac{N-1}{2}\right) = 0$$

$$H(z) = z^{-4}\left[h(0)(z^4 - z^{-4}) + h(1)(z^3 - z^{-3}) + h(2)(z^2 - z^{-2}) + h(3)(z - z^{-1})\right]$$

- The corresponding frequency response is given by

$$H(\omega) = e^{-j4\omega} e^{j\frac{\pi}{2}}\left[2h(0)\sin(4\omega) + 2h(1)\sin(3\omega) + 2h(2)\sin(2\omega) + 2h(3)\sin(\omega)\right]$$

- It also exhibits a generalized phase response given by

$$\theta(\omega) = -4\omega + \beta$$

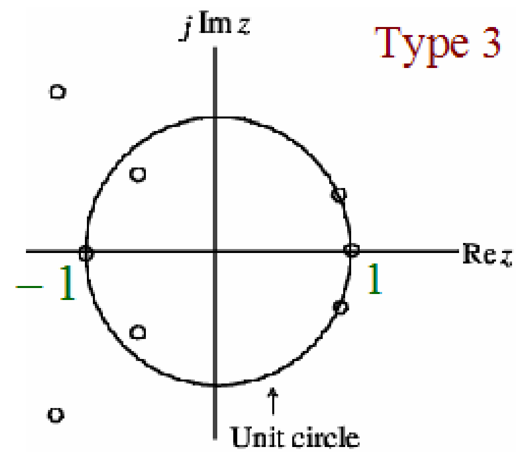
where β is either $\pm\frac{\pi}{2}$.

- The group delay here is

$$\tau(\omega) = 4$$

indicating a constant group delay of 4 samples.

A typical pole-zero plot



- In the general case

$$H(\omega) = e^{j\left[-\omega\left(\frac{N-1}{2}\right) \pm \frac{\pi}{2}\right]} A(\omega)$$

where the amplitude response is of the form

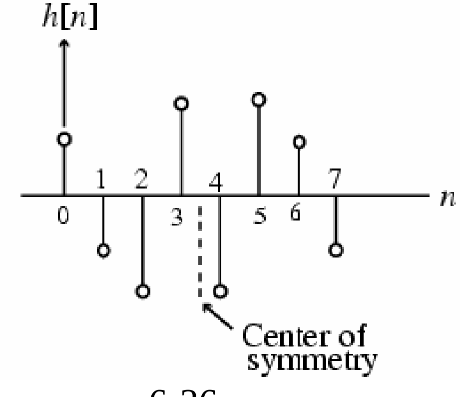
$$A(\omega) = 2\sum_{n=0}^{\frac{N-3}{2}} h(n)\sin\left[\omega\left(\frac{N-1}{2} - n\right)\right]$$

- For this system there is always a zero at $\omega = 0$ and $\omega = \pi$. This system cannot be used for HP, LP or BS filters. However can be used as a Hilbert transformer or differentiator.

Type 4: Anti-symmetric Filter with Even Length

- In this case, N is even
- Assume $N = 8$ for simplicity
- Applying the anti-symmetry condition we get

$$H(z) = z^{-\frac{7}{2}}\left\{h(0)(z^{\frac{7}{2}} - z^{-\frac{7}{2}}) + h(1)(z^{\frac{5}{2}} - z^{-\frac{5}{2}}) + h(2)(z^{\frac{3}{2}} - z^{-\frac{3}{2}}) + h(3)(z^{\frac{1}{2}} - z^{-\frac{1}{2}})\right\}$$



- The corresponding frequency response is given by

$$H(\omega) = e^{-j\omega\frac{7}{2}} e^{j\frac{\pi}{2}} \left[2h(0)\sin\left(\frac{7\omega}{2}\right) + 2h(1)\sin\left(\frac{5\omega}{2}\right) + 2h(2)\sin\left(\frac{3\omega}{2}\right) + 2h(3)\sin\left(\frac{\omega}{2}\right)\right]$$

- It again exhibits a generalized phase response given by

$$\theta(\omega) = -\frac{7}{2}\omega + \beta$$

where β is either $\pm\frac{\pi}{2}$.

- The group delay is constant and is given by

$$\tau(\omega) = \frac{7}{2}$$

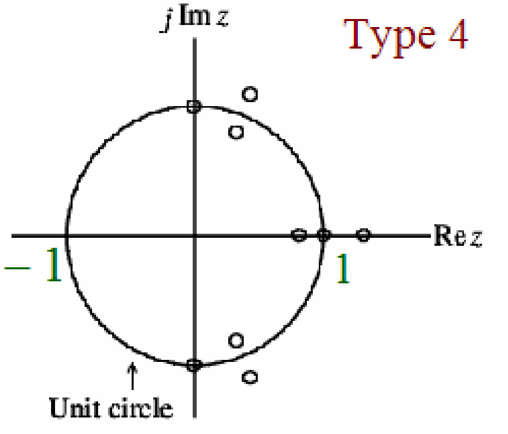
- In the general case we have

$$H(\omega) = e^{j\left[-\omega\left(\frac{N-1}{2}\right) \pm \frac{\pi}{2}\right]} A(\omega)$$

where now the amplitude response is of the form

$$A(\omega) = 2\sum_{n=0}^{\frac{N}{2}-1} h(n)\sin\left[\omega\left(\frac{N-1}{2} - n\right)\right]$$

A typical pole-zero plot



- For this system there is always a zero at $\omega = 0$. Cannot be used for LP filter but is suitable for Hilbert transformers and differentiators.

General Form of Frequency Response

In each of the four types of linear-phase FIR filters, the frequency response is of the form

$$H(\omega) = A(\omega)e^{-j\alpha\omega + j\beta}$$

The amplitude response, $A(\omega)$ for each of the four types of linear-phase FIR filters can become negative over certain frequency ranges, typically in the stopband.

The magnitude and phase responses of the linear-phase FIR are given by

$$|H(\omega)| = |A(\omega)|$$

$$\theta(\omega) = -\frac{(N-1)\,\omega}{2} + \beta \pm \pi$$

The group delay in each case is

$$\tau(\omega) = \frac{N-1}{2}$$

Note that, even though the group delay is constant, the output waveform is not a replica of the input waveform since in general $H(\omega)$ is not a constant.

Zero Locations of Linear-Phase FIR Transfer Functions

Consider first an FIR filter with a symmetric impulse response:

$$h(n) = h(N-1-n)$$

Its transfer function can be written as

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{N-1} h(N-1-n) z^{-n}$$

By making a change of variable $m = N-1-n$, we can write

$$\sum_{n=0}^{N-1} h(N-1-n) z^{-n} = \sum_{m=0}^{N-1} h(m) z^{-(N-1)+m} = z^{-(N-1)} \sum_{m=0}^{N-1} h(m) z^{m}$$

But,

$$\sum_{m=0}^{N-1} h(m) z^{m} = H(z^{-1})$$

Hence for an FIR filter with a symmetric impulse response of length *N* we have

$$H(z) = z^{-(N-1)} H(z^{-1})$$

A real-coefficient polynomial *H*(*z*) satisfying the above condition is called a **mirror-image polynomial** (MIP).

Now consider an FIR filter with an anti-symmetric impulse response:

$$h(n) = -h(N-1-n)$$

Its transfer function can be written as

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = -\sum_{n=0}^{N-1} h(N-1-n) z^{-n}$$

By making a change of variable $m = N-1-n$, we get

$$-\sum_{n=0}^{N-1} h(N-1-n) z^{-n} = -\sum_{m=0}^{N-1} h(m) z^{-(N-1)+m} = -z^{-(N-1)} H(z^{-1})$$

Hence, the transfer function $H(z)$ of an FIR filter with an anti-symmetric impulse response satisfies the condition

$$H(z) = -z^{-(N-1)} H(z^{-1})$$

A real-coefficient polynomial $H(z)$ satisfying the above condition is called a **antimirror-image polynomial** (AIP).

It follows from the relation $H(z) = \pm z^{-(N-1)} H(z^{-1})$ that if $z = a_o$ is a zero of $H(z)$, then so is $z = 1/a_o$.

Moreover, for an FIR filter with a real impulse response, the complex zeros of $H(z)$ must occur in complex conjugate pairs.

Hence, a complex zero at $z = a_o$ is associated with a zero at $z = a_o^*$.

Thus, a complex zero that is not on the unit circle belongs to a set of 4 zeros given by

complex-conjugate and reciprocal →

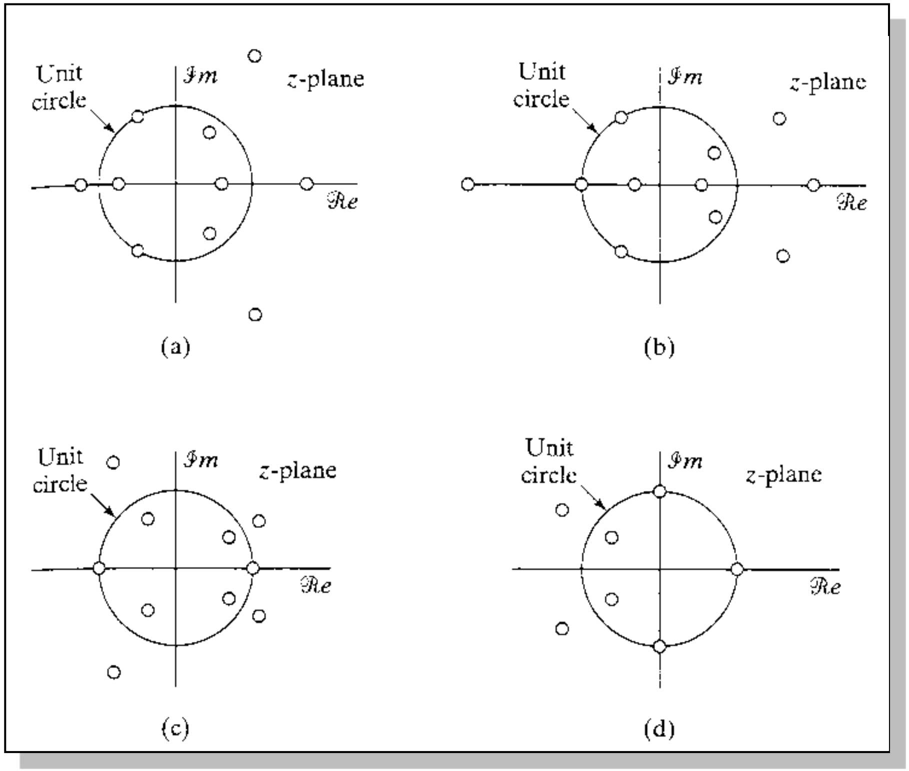
$$z = re^{\pm j\phi}, \qquad z = \frac{1}{r} e^{\pm j\phi}$$

A zero on the unit circle appears as a pair.

complex-conjugate →

$$z = e^{\pm j\phi}$$

Since a zero at either $z = \pm 1$ is its own reciprocal and is not complex, it appears singly.



A Type 2 FIR filter satisfies

$$H(z) = z^{-(N-1)} H(z^{-1}) \quad \text{with } N \text{ even}$$

Hence

$$H(-1) = (-1)^{-(N-1)} H(-1) = -H(-1)$$

implying $H(-1) = 0$, i.e., $H(z)$ must have a zero at $z = -1$.

Likewise, a Type 3 or 4 FIR filter satisfies

$$H(z) = -z^{-(N-1)} H(z^{-1})$$

Thus

$$H(1) = -(1)^{-(N-1)} H(1) = -H(1)$$ → $H(z)$ must have a zero at $z = 1$

Furthermore, the Type 3 FIR filter is restricted to have a zero at $z = -1$ since here the N is odd and hence,

$$H(-1) = -(-1)^{-(N-1)} H(-1) = -H(-1)$$

- **Summarizing**

  (1) Type 1 FIR filter: Either an even number or no zeros at $z = 1$ and $z = -1$

  (2) Type 2 FIR filter: Either an even number or no zeros at $z = 1$, and an odd number of zeros at $z = -1$

  (3) Type 3 FIR filter: An odd number of zeros at $z = 1$ and $z = -1$

  (4) Type 4 FIR filter: An odd number of zeros at $z = 1$, and either an even number or no zeros at $z = -1$

- The presence of zeros at $z = \pm 1$ leads to the following limitations on the use of these linear-phase transfer functions for designing frequency-selective filters
- A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero $z = -1$
- A Type 3 FIR filter has zeros at both $z = 1$ and $z = -1$, and hence cannot be used to design either a lowpass, highpass, or bandstop filter
- A Type 4 FIR filter is not appropriate to design a lowpass filter due to the presence of a zero at $z = 1$
- Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter