Course Notes 8 – IIR Filter Design Techniques

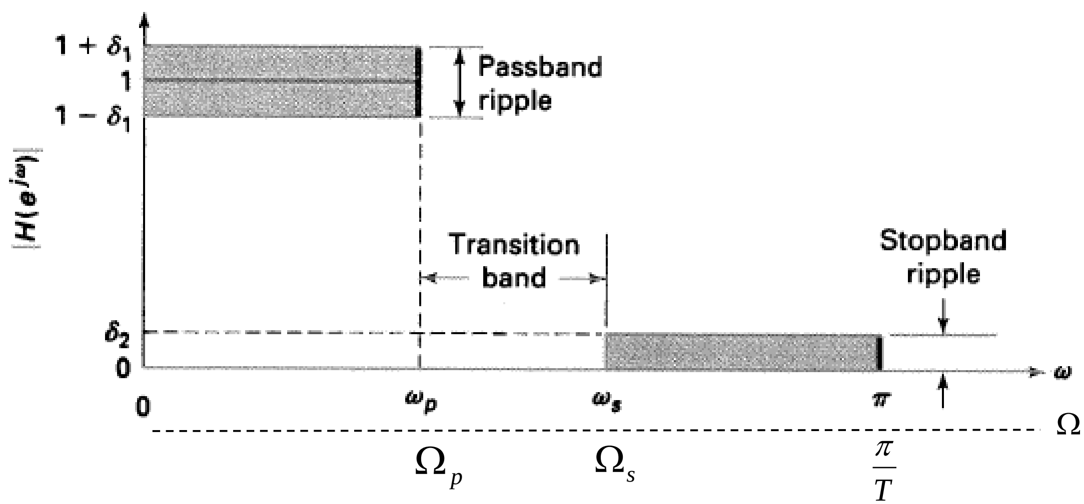
8.0 Introduction

The filter design process consists of three steps:

- Specification of the properties of the filter (i.e., pass band, transition band, stop band)
- Approximation of these specifications using a discrete system (the subject of this chapter)
- Implementation of the system with finite precision math.

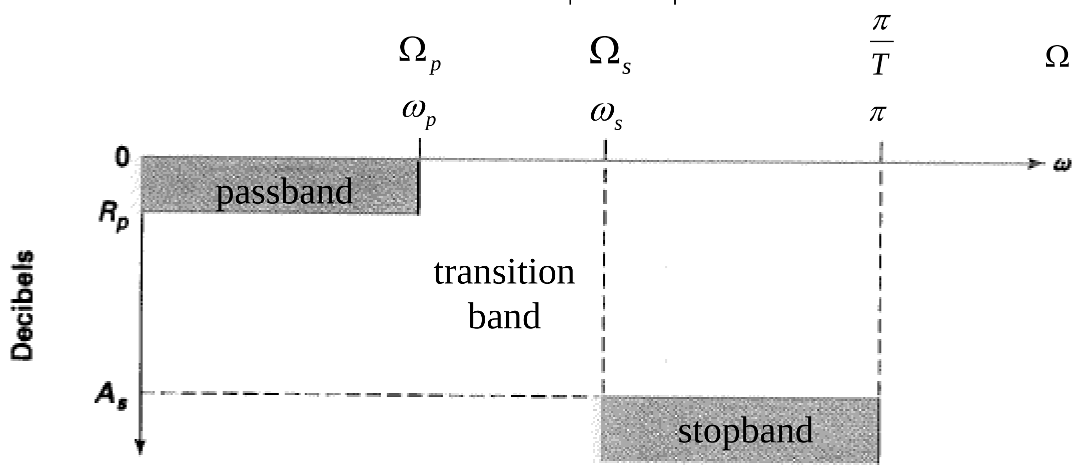
The magnitude specifications of filters are given in one of two ways:

1. <u>Absolute Specification</u>



2. Relative Specification – provides filter specs with respect to the maximum frequency response characteristic of the filter in decibels

$$\text{dB scale} = -20\log_{10}\frac{\left|H\left(e^{j\omega}\right)\right|_{\max}}{\left|H\left(e^{j\omega}\right)\right|} \geq 0$$



- $[0, \omega_p]$ is called the passband
- δ_1 is the passband tolerance (ripple)
- R_p is the passband ripple in dB
- $[\omega_s, \pi]$ is called the stopband
- δ_2 is the stopband tolerance (ripple)
- A_s is stopband attenuation in dB
- $[\omega_p, \omega_s]$ is called the transition band

Expressed as # dB below zero

Relating the parameters in these two specifications:

$$R_p = 20\log_{10}\left(\frac{1+\delta_1}{1-\delta_1}\right) \qquad \text{(usually a small number)}$$

$$A_s = 20\log_{10}\left(\frac{1+\delta_1}{\delta_2}\right) \qquad \text{(usually a large number)}$$

A complete specification of any filter should consist of:

- Amplitude vs. frequency characteristic
- Phase vs. frequency characteristic
- Transient response

In an actual design, however, because of the requirements of stability, causality, and simplicity, we often specify only an amplitude characteristic and accept the phase and transient response.

A typical problem statement might be as follows:

Design a LPF (i.e., obtain $H(z)$ or LCCDE, or algorithm) that has a passband $\left[0, \omega_p\right]$ with tolerance δ_1 (or R_p in dB) and a stopband $\left[\omega_s, \pi\right]$ with tolerance δ_2 (or A_s in dB).

8.1 Design of Discrete-Time IIR Filters from Continuous Time Filters

Design of IIR Filters from Analog Filters

The design philosophy with IIR filters is to adapt or transform analog filter designs into digital filters.

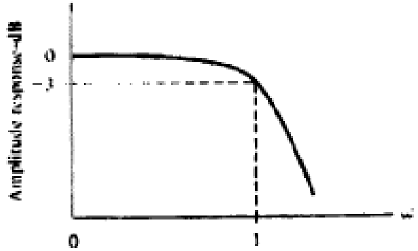
We begin with the traditional method of expressing the transfer function of an analog filter:

$$H_a(s) = \frac{\sum_{r=0}^{M} d_r s^r}{\sum_{k=0}^{N} c_k s^k} = \frac{Y_a(s)}{X_a(s)}$$

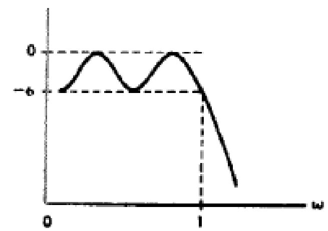
Implemented with resistors, capacitors, and inductors

Some common filter prototypes we will be interested in are briefly described below:

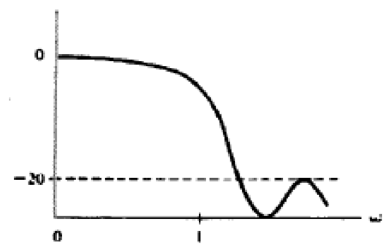
- Butterworth filter – Most popular of all filter prototypes. Has maximally flat amplitude response in the passband. The amplitude response decreases monotonically.



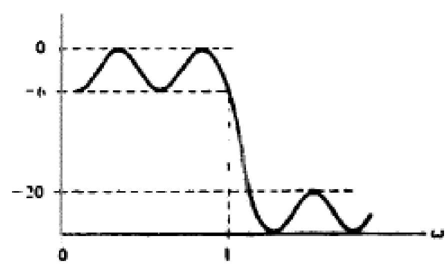
• Chebyshev I filter – Amplitude response has ripple in the passband. This allows the amplitude response to have a steeper rolloff outside the passband for a given filter order. Ripples in the passband have equal amplitude (equiripple). Response outside passband is monotonic.



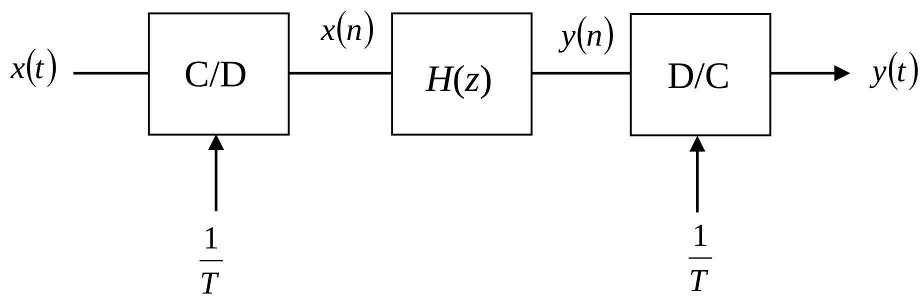
• Chebyshev II filter – Has a monotonic amplitude response in the passband and equal amplitude ripples in the stop band.



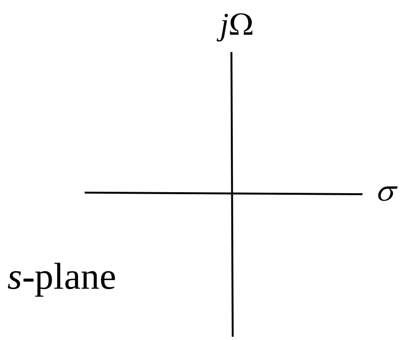
• Elliptic filter – Combination of Chebyshev I and II filters in that is has ripples in both the passband and the stopband. This filter minimizes the transition band for a given amount of ripple.



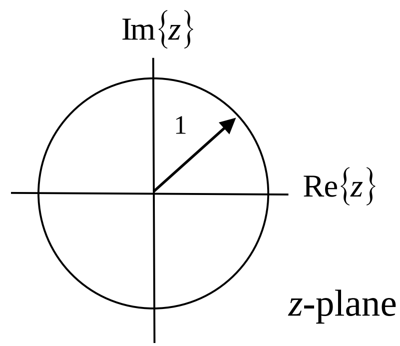
We would like to design digital filters to function in the following system:



To understand how this is accomplished let's first look at the complex frequency planes:



compare with →



- evaluate frequency response on $j\Omega$ axis.
- system defined by rational function of 's'

- evaluate frequency response on unit circle $(z = e^{j\omega})$
- system defined by rational function of 'z'

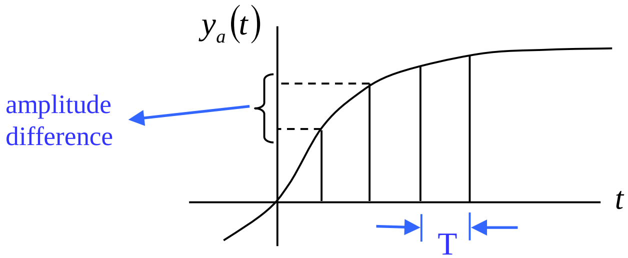
Several important considerations in transforming analog filters to discrete filters:

1. A stable/causal analog filter transforms to a stable/causal discrete filter
2. We want to preserve frequency response characteristic
3. Order of discrete filter related to order of analog filter

8.2 IIR Filter Design by Approximation of Derivatives

The most obvious approach is to 1) obtain the differential equation that describes the analog filter, 2) convert it to a difference equation, and then 3) implement the filter from the difference equation.

It is well known in numerical analysis that derivatives of continuous functions can be expressed in terms of finite differences as illustrated below:



where the first-order derivative of $y_a(t)$ can be approximated by

$$\left.\frac{d\,y_a(t)}{dt}\right|_{t=nT} \cong \frac{y(nT)-y(nT-T)}{T} = \frac{y(n)-y(n-1)}{T}$$ Backward Difference

In Laplace transform notation,

$$\frac{dy(t)}{dt} \quad \text{has system function} \quad H(s) = s$$

while the digital system that produces $[y(n) - y(n-1)]/T$ (from the approximation) has system function

$$H(z) = \frac{1 - z^{-1}}{T}$$

Thus, the frequency domain equivalent relationship for the Backward Difference approximation is

$$s = \frac{1 - z^{-1}}{T}$$

In the same manner, the second-order derivative is approximated by:

$$\left.\frac{d^2 y_a(t)}{dt^2}\right|_{t=nT} = \frac{d}{dt}\left[\frac{dy_a(t)}{dt}\right]\Bigg|_{t=nT} \cong \frac{[y(nT) - y(nT - T)]/T - [y(nT - T) - y(nT - 2T)]/T}{T}$$

$$= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2}$$

likewise resulting in the frequency domain relationship

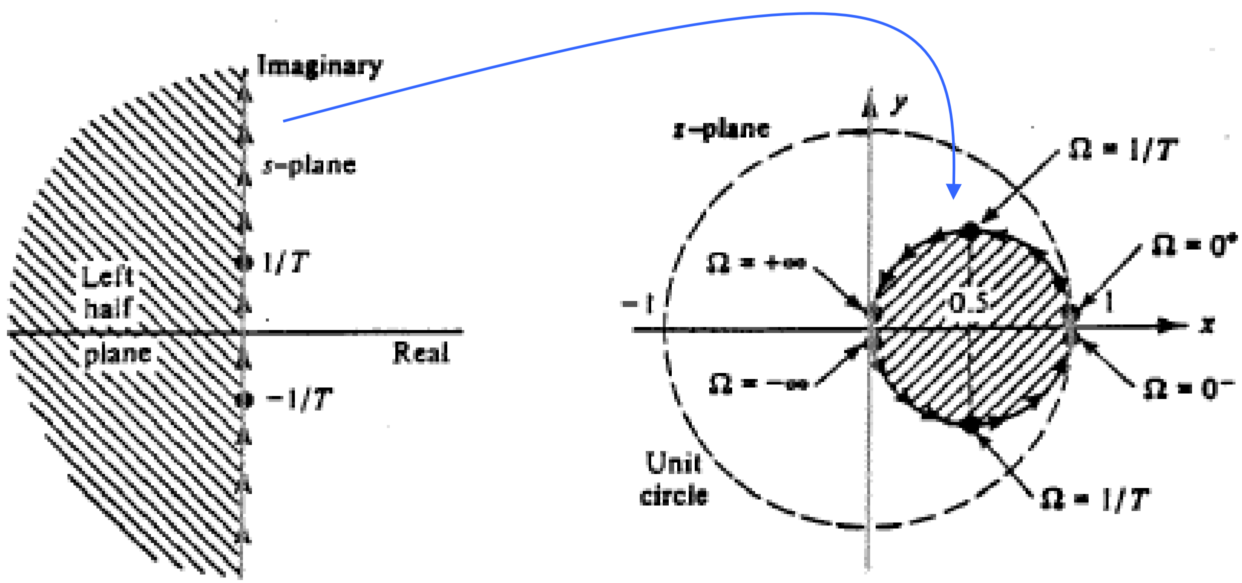
$$s^2 = \frac{1 - 2z^{-1} + z^{-2}}{T^2} = \left(\frac{1 - z^{-1}}{T}\right)^2$$

In general, it follows that the k^{th} derivative of $y_a(t)$ results in the frequency domain relationship

$$s^k = \left(\frac{1 - z^{-1}}{T}\right)^k$$

such that $H(z) = H_a(s)|_{s=(1-z^{-1})/T}$ which yields the mapping

stable analog filter becomes a stable digital filter



For a stable analog filter, the possible locations of the digital filter poles are confined to relatively small frequencies so the mapping is restricted to the design of lowpass and low frequency bandpass filters.

For example, it is not possible to directly map a highpass analog filter into a corresponding highpass digital filter.

To overcome this limitation, higher order approximations of the derivative having the form

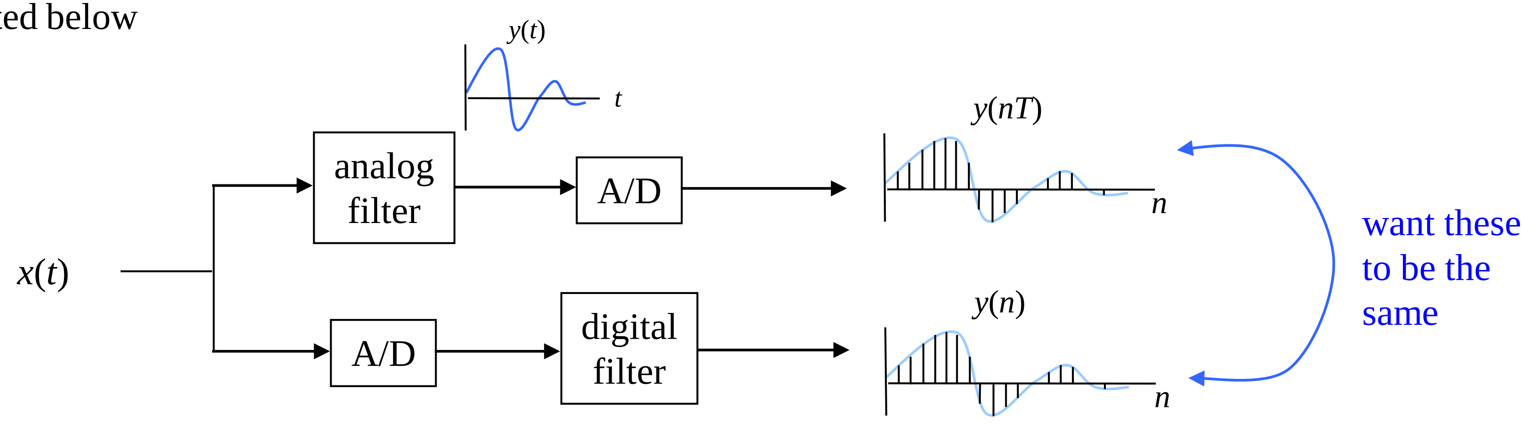
$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{1}{T}\sum_{k=1}^{L}\alpha_k \frac{y(nT+kT)-y(nT-kT)}{T}$$

have been proposed which both maintain stability and enable mapping to the interior of the entire unit circle.

However, the proper selection of the coefficients α_k pose a difficulty and, as we shall see, there exist much simpler ways to convert an analog IIR filter into a digital IIR filter.

8.3 IIR Filter Design by Impulse Invariance

A more useful approach for designing digital filters is called *time-invariant synthesis*. This method is illustrated below



Approach: - inject $x(t)$ into the analog filter

- inject $x(t)|_{t=nT} = x(nT) = x(n)$ into the digital filter

- adjust digital filter coefficients so that $y(t)|_{t=nT} = y(nT) = y(n)$

analog filter output → $y(t)|_{t=nT}$; digital filter output → $y(n)$

This approach optimizes the digital filter coefficients <u>with respect to a particular input</u>.

Let's consider the most often used time-invariant case which is when the input is an impulse.

Impulse Invariant Design

analog filter impulse response

We wish to adjust the digital filter coefficients so that $h(n) = h_a(t)|_{t=nT}$.

Note that we assume distinct poles. The formulation can also be generalized to multiple-order poles.

Given the analog filter, which can be described by

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

where p_k are the poles of the analog filter and c_k are coefficients from PFE. The inverse Laplace transform $\mathscr{L}^{-1}\{H_a(s)\}$ is found to be

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t} u(t)$$

Then sampling $h_a(t)$ periodically at $t = nT$ yields

$$h(n) = h_a(nT) = \sum_{k=1}^{N} c_k e^{p_k Tn} u(nT)$$

Taking the z-transform of $h(n)$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k e^{p_k T n} \right) z^{-n}$$

swap summation order by linearity and group "power of n" terms

$$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1} \right)^n$$

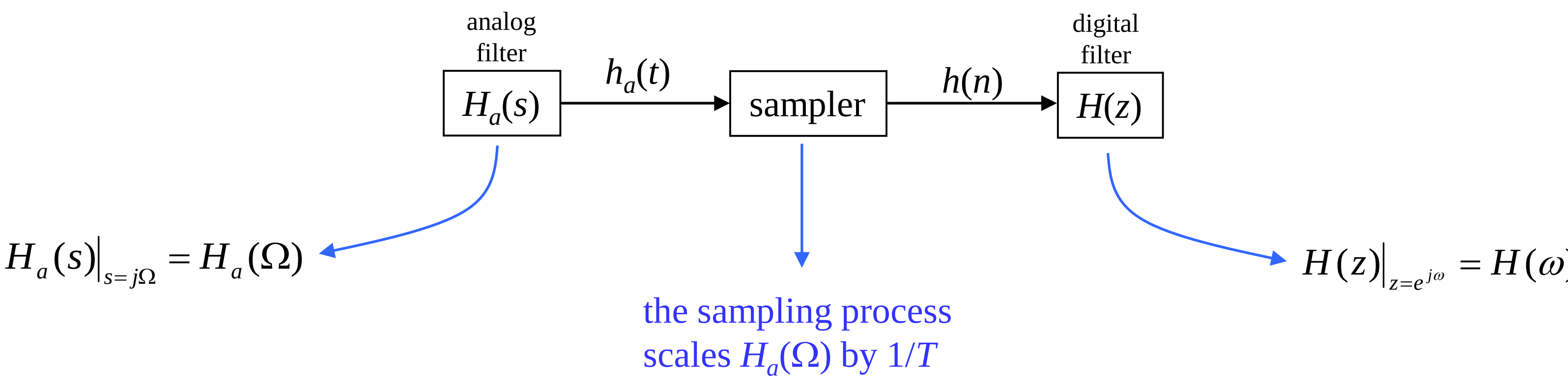
assuming stable analog filter (*i.e.* $p_k < 0$) to apply geometric sum formula

$$= \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

The discrete filter has poles at $z_k = e^{p_k T} \quad k = 1, 2, \ldots, N.$

Note that the same relationship does <u>not</u> hold between the analog and discrete zeros.

The impulse invariant design approach can be summarized as



For equivalence between the impulse responses of the digital and analog filters, we must multiply the digital filter transfer function by T , that is

$$H(z) = T\sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

Note that given c_k and p_k for the analog filter, and the sampling period T, we can simply substitute directly into $H(z)$ above.

Let's now look at the mapping properties (between the s and z planes) of the impulse invariance technique.

We previously established $\left\{ \begin{array}{l} z = e^{sT} \\ s = \sigma + j\Omega \end{array} \right\}$ which define a mapping from the s-plane to the z-plane.

We have shown this mapping to be

$$z = e^{sT} = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\omega} = r\,e^{j\omega}$$

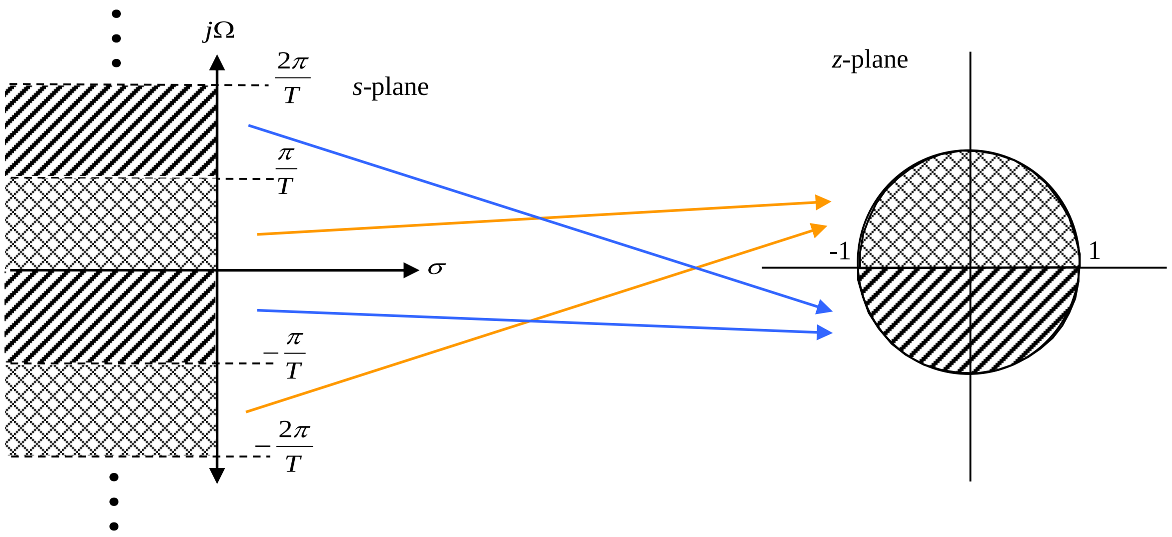
where

i) left half of s-plane ($\sigma < 0$) $\rightarrow |z| < 1$ inside the unit circle

ii) $j\Omega$ axis ($\sigma = 0$) $\rightarrow |z| = 1$ on the unit circle

iii) right half of s-plane ($\sigma > 0$) $\rightarrow |z| > 1$ outside the unit circle

iv) the $j\Omega$ axis is periodic in strips of $\frac{2\pi}{T}$



Each horizontal strip of the s-plane (for $\sigma < 0$) maps to a hemisphere inside the unit circle of the z-plane.

Thus mpulse invariance <u>does not</u> amount to a simple algebraic mapping of s-plane to z-plane.

In summary:

<u>advantages</u> of impulse invariant transformation

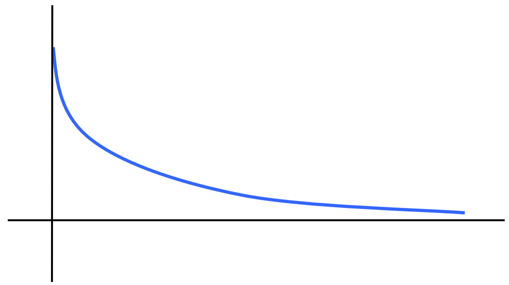
1) simple design procedure
2) stable analog filter → stable digital filter
3) impulse response characteristic is preserved

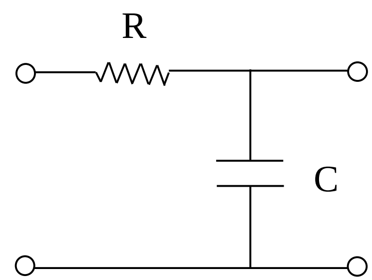
<u>disadvantages</u>

1) aliasing – results in $H(\omega)$ <u>not</u> being identical to $H_a(\Omega)$
2) only appropriate for lowpass and limited bandpass filter designs

Example: impulse invariant filter design

Given $h_a(t) = e^{-\alpha t}\, u(t)$

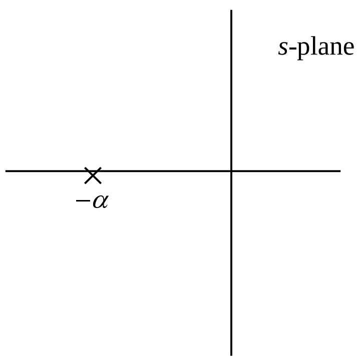


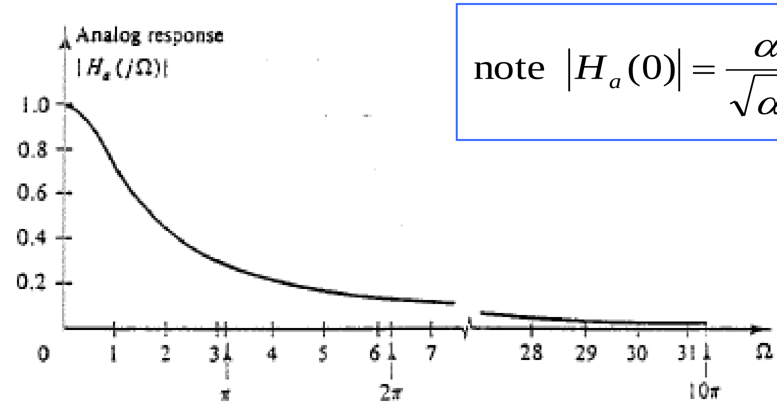


$$H(s) = \frac{1/RC}{s + 1/RC}$$

let $\alpha = 1/RC$

so $H(s) = \dfrac{\alpha}{s+\alpha}$ $\longrightarrow$ $|H(\Omega)| = \dfrac{\alpha}{\sqrt{\Omega^2 + \alpha^2}}$





note $|H_a(0)| = \dfrac{\alpha}{\sqrt{\alpha^2}} = 1$

For the digital impulse response to look the same,

$$h(n) = h_a(t)|_{t=nT} = e^{-\alpha nT}u(nT)$$

so the discrete-time transfer function is

$$H(z) = T\,\frac{\alpha}{1-e^{-\alpha T}z^{-1}}, \quad |z| > e^{-\alpha T}$$

and the frequency response is

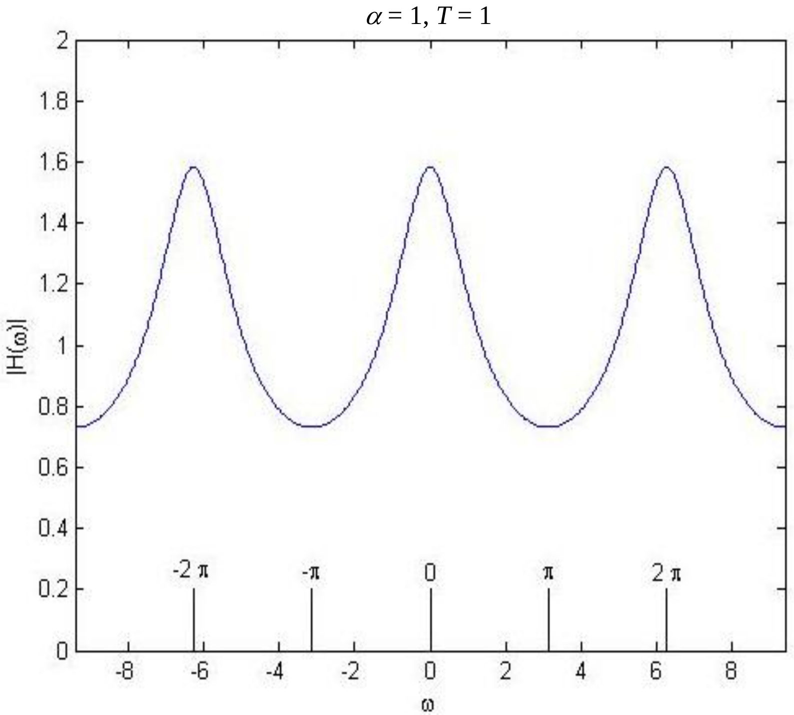
$$H(\omega) = T\,\frac{\alpha}{1-e^{-\alpha T}e^{-j\omega}}$$

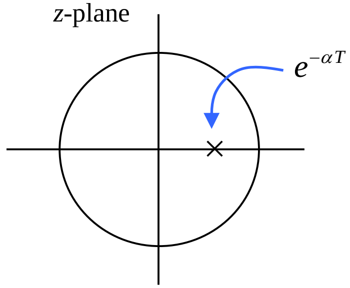
It can easily be shown that

$$|H(\omega)| = T\,\frac{\alpha}{\sqrt{1-2e^{-\alpha T}\cos\omega + e^{-2\alpha T}}}$$

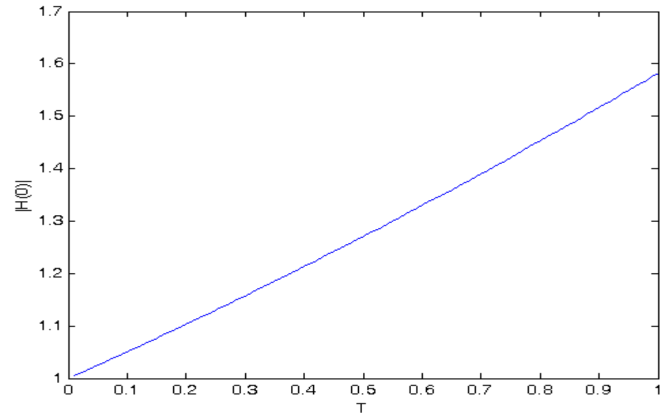
note

$$|H(0)| = T\,\frac{\alpha}{1-e^{-\alpha T}}$$

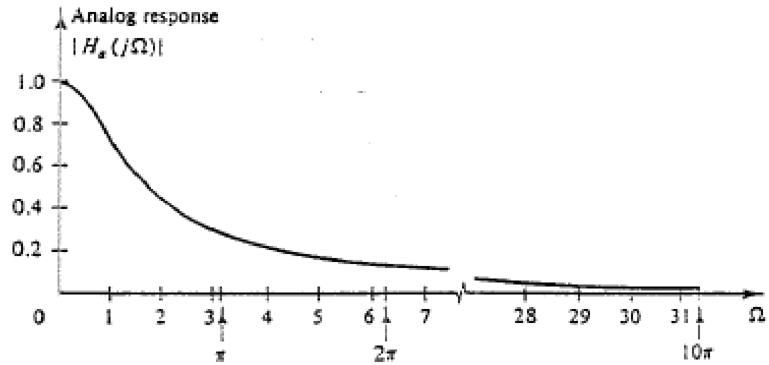


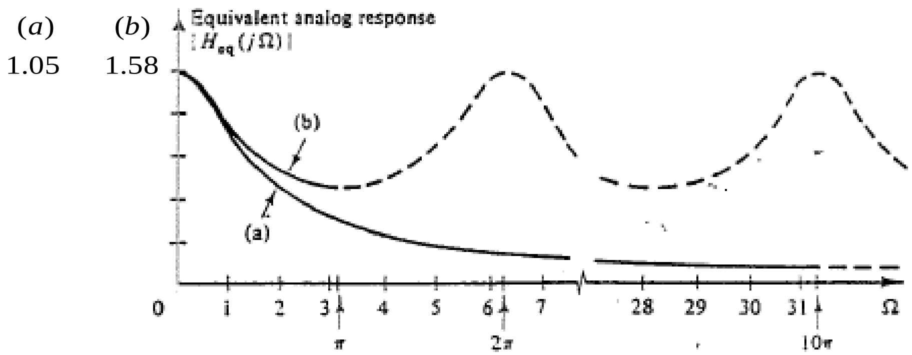


$$|H(0)| = T\,\frac{\alpha}{1-e^{-\alpha T}}$$



we want $|H(0)| \cong |H_a(0)| = 1$
so decrease T (increase sampling rate)
for a more accurate translation





Plots of $H_a(j\Omega)$ and $H_{eq}(j\Omega)$ for impulse invariant design. Curve (a): $\alpha = 1$, $T = 0.1$, curve (b): $\alpha = 1$, $T = 1$.

Finally, compute the recursion formula

$$H(z) = T\,\frac{\alpha}{1-e^{-\alpha T}z^{-1}} = \frac{Y(z)}{X(z)}$$

$$Y(z)[1-e^{-\alpha T}z^{-1}] = (\alpha T)\cdot X(z)$$

$$Y(z) = (\alpha T)\cdot X(z) + (e^{-\alpha T}z^{-1})\cdot Y(z)$$

$$y(n) = (\alpha T)\cdot x(n) + e^{-\alpha T}y(n-1)$$

8.4 IIR Filter Design with the Bilinear Transform

The bilinear transform provides the designer with an alternative technique for approximating analog filters with digital filters. This method is derived by approximating a first-order differential equation with a difference equation using the trapezoidal formula for numerical integration.

From a purely mathematical point of view, the bilinear transform can be viewed as one of many possible conformal mappings between two complex planes that have the desired properties, such as mapping the $j\Omega$ axis onto the unit circle and the left-side of the s-plane into the interior of the unit circle.

The bilinear transform mapping is:

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

in which the s-plane is mapped into the z-plane. Let's see how this mapping occurs.

Recall that $z = r\, e^{j\omega}$ and $s = \sigma + j\Omega$.

Then $s = \dfrac{2}{T}\dfrac{z-1}{z+1}$

$$= \frac{2}{T}\frac{r\,e^{j\omega}-1}{r\,e^{j\omega}+1}$$

multiply by $\dfrac{r\,e^{-j\omega}-1}{r\,e^{-j\omega}-1}$

$$= \frac{2}{T}\frac{r^2-1}{1+2r\cos\omega+r^2} + j\frac{2}{T}\frac{2r\sin\omega}{1+2r\cos\omega+r^2} = \sigma + j\Omega$$

Note that if $r < 1$, then $\sigma < 0$ and if $r > 1$, then $\sigma > 0$. So the left-hand side of the *s*-plane maps inside the unit circle in the *z*-plane and the right-hand side of the *s*-plane maps outside the unit circle in the *z*-plane.

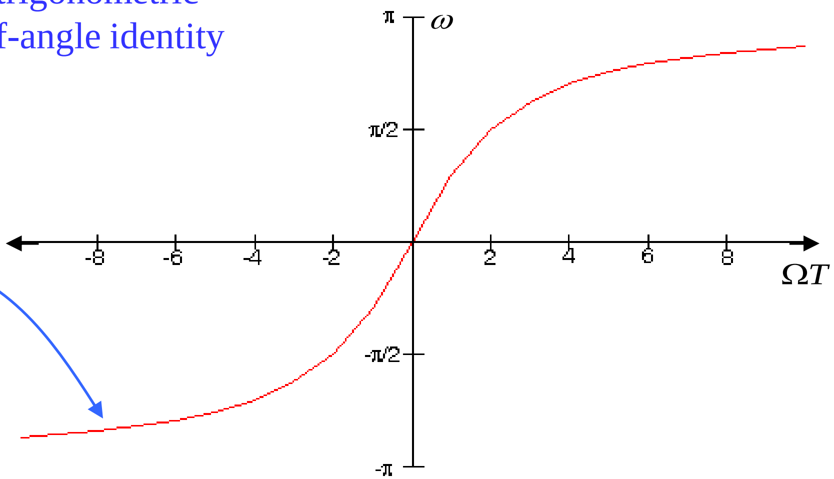
Also, when $r = 1$, then $\sigma = 0$ and

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega} = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

by trigonometric half-angle identity

or

$$\omega = 2\arctan\left(\frac{\Omega T}{2}\right)$$



The left-hand side of the s-plane maps to the interior of the unit circle in the z-plane.

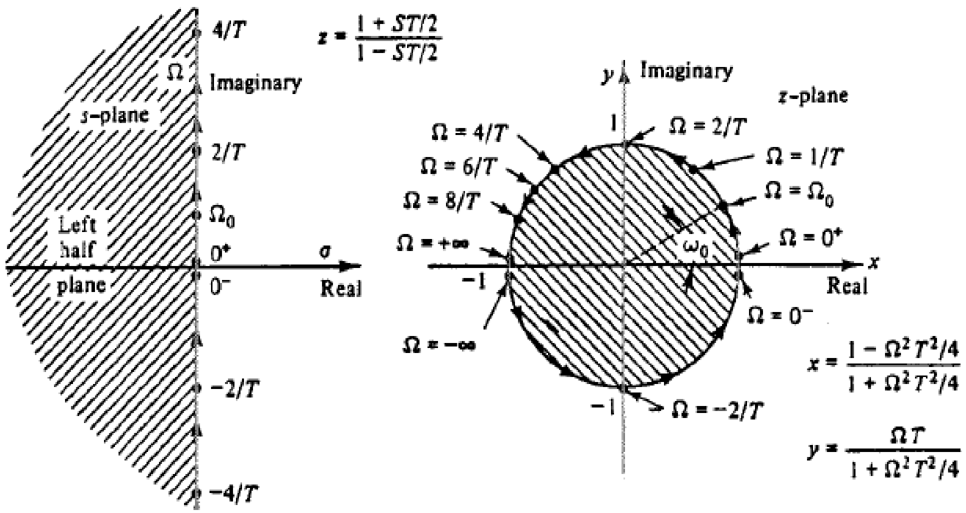
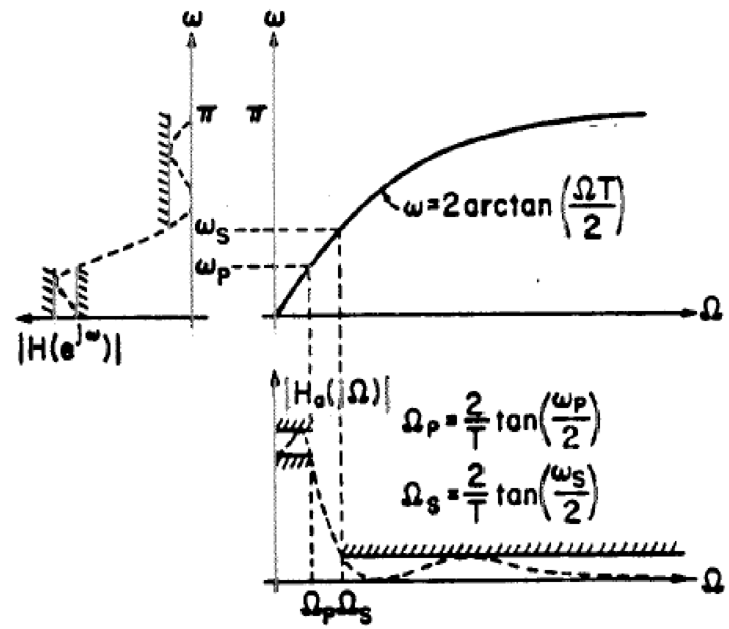


Figure 4.3 The image in the z-plane of the $j\Omega$ axis of the s-plane for the mapping $z = [1 + ST/2]/[1 - ST/2]$.



stable analog filter → stable digital filter

No aliasing problems since entire $j\Omega$ axis maps into the unit circle. However, there is some frequency distortion (especially at <u>higher frequencies</u>).

An Analog Filter Prototype – The Butterworth Filter

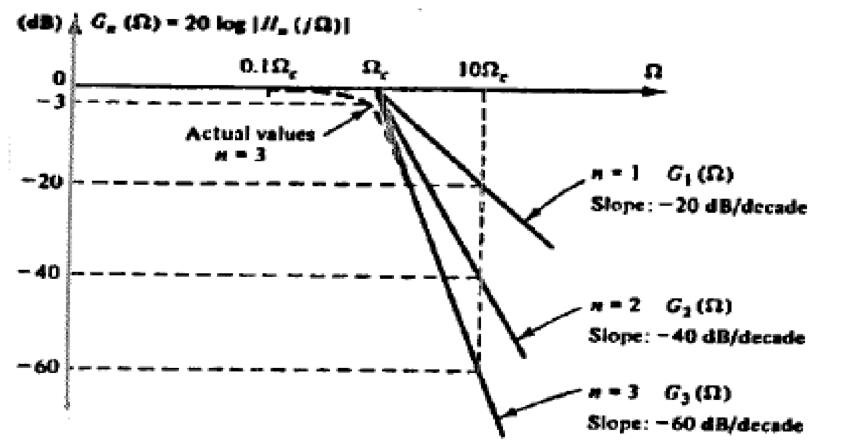
In this course we will concentrate on the Butterworth filter when illustrating digital filter design techniques.

Other commonly used filters:
Chebyshev filters (2 types)
Elliptic filters
Bessel filters

The normalized Butterworth model is

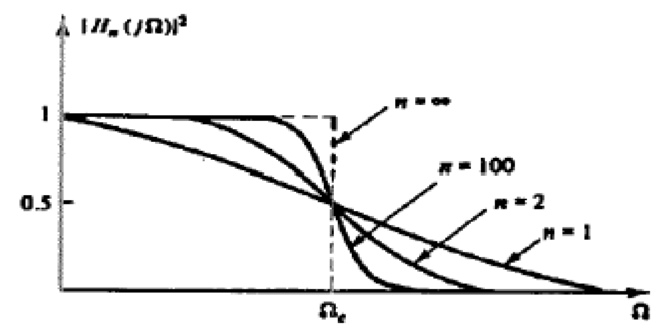
$$|H_a(\Omega)|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}} = \frac{1}{1+\kappa_p^2\left(\Omega/\Omega_p\right)^{2N}}$$

where Ω_c is the 3-dB cutoff frequency and Ω_p and κ_p are the passband edge frequency and passband edge value, respectively.



Filter gain plot for analog Butterworth filters of various orders *n*.

1. $|H_n(j\Omega)|^2 \big|_{\Omega=0} = 1$ for all *n*.
2. $|H_n(j\Omega)|^2 \big|_{\Omega=\Omega_c} = \frac{1}{2}$ for all finite *n*.
   This implies that
   $|H_n(j\Omega)| \big|_{\Omega=\Omega_c} = 0.707$
   and $20 \log |H_n(j\Omega)| \big|_{\Omega=\Omega_c} = -3.0103$.
3. $|H_n(j\Omega)|^2$ is a monotonically decreasing function of Ω.
4. As *n* gets larger, $|H_n(j\Omega)|^2$ approaches an ideal low-pass frequency response.
5. $|H_n(j\Omega)|^2$ is called maximally flat at the origin since all order derivatives exist and are zero.



The magnitude squared frequency response for a Butterworth filter.

Since $H_a(s)\,H_a(-s)$ evaluated at $s = j\Omega$ is equal to $|H_a(\Omega)|^2$, it follows that

$$H_a(s)H_a(-s) = \frac{1}{1+\left(\dfrac{s}{j\Omega_c}\right)^{2N}}$$

pole locations are where $1+\left(\dfrac{s}{j\Omega_c}\right)^{2N} = 0 \;\Rightarrow\; s_{pole} = \left(-1\right)^{1/2N}(j\Omega_c)$

Recall "n^{th} roots of unity" in which

$$\left(r\,e^{j\theta}\right)^{1/N} = \left[r\,e^{j(\theta+2\pi k)}\right]^{1/N} = r^{1/N}\,e^{j(\theta+2\pi k)/N} \qquad k = 0,\,1,\,\ldots,\,N-1$$

Thus, the $2N$ poles are located at

$$s_k = e^{j(\pi+2\pi k)/2N}\,e^{j\pi/2}\,\Omega_c \qquad k = 0,1,\,\ldots,\,2N-1$$

poles occur in the s-plane on a circle of radius Ω_c at equally spaced points

For example, the case where $N = 3$ yields

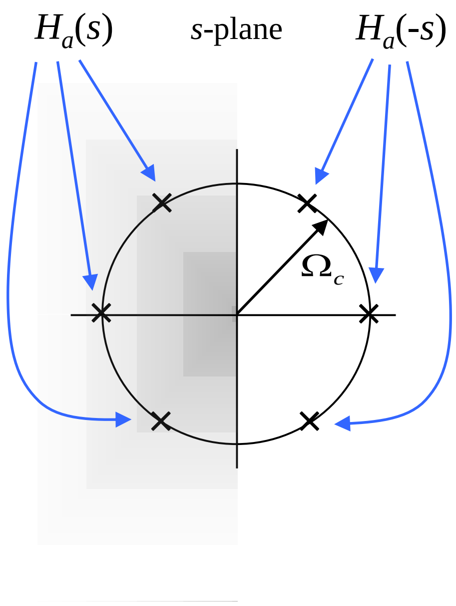
$$s_{pole} = \left(-1\right)^{1/6} (j\Omega_c)$$

Hence, $$s_k = e^{j(\pi+2\pi k)/6}\, e^{j\pi/2}\, \Omega_c \qquad k = 0, 1, \ldots, 5$$

$$= e^{j(4\pi+2\pi k)/6}\, \Omega_c \qquad k = 0, 1, \ldots, 5$$

$$= \left\{e^{j2\pi/3}, e^{j\pi}, e^{j4\pi/3}, e^{j5\pi/3}, e^{j2\pi}, e^{j7\pi/3}\right\}\Omega_c$$

$$= \left\{e^{j2\pi/3}, -1, e^{-j2\pi/3}, e^{-j\pi/3}, 1, e^{j\pi/3}\right\}\Omega_c$$



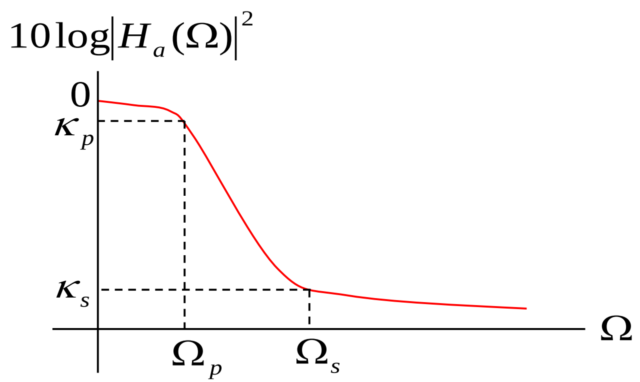
Poles spaced π/N radians apart and are symmetric about the $j\Omega$ axis.

As a general approach to designing a Butterworth filter consider the following design specs:

(passband spec) $0 \geq 10\log|H_a(\Omega)|^2 \geq \kappa_p \qquad \Omega \leq \Omega_p$

(stopband spec) $10\log|H_a(\Omega)|^2 \leq \kappa_s \qquad \Omega \geq \Omega_s$

These requirements can be illustrated as



where $|H_a(\Omega)|^2 = \dfrac{1}{1+(\Omega/\Omega_c)^{2N}}$ is the normalized Butterworth filter model.

Note that only 2 parameters (Ω_c and N) are needed to specify the filter.

Solving the passband and stopband specs with equality yields

$$-10\log\left[1+\left(\Omega_p/\Omega_c\right)^{2N}\right] = \kappa_p \qquad \text{and} \qquad -10\log\left[1+\left(\Omega_s/\Omega_c\right)^{2N}\right] = \kappa_s$$

solving for $\left(\Omega_p/\Omega_c\right)^{2N} = 10^{-\kappa_p/10} - 1$ and $\left(\Omega_s/\Omega_c\right)^{2N} = 10^{-\kappa_s/10} - 1$

divide one with another to cancel Ω_c and obtain

$$\left(\Omega_p/\Omega_s\right)^{2N} = \frac{10^{-\kappa_p/10} - 1}{10^{-\kappa_s/10} - 1}$$

and then solve for N as

$$N = \left\lceil \frac{\log\left[\left(10^{-\kappa_p/10} - 1\right)/\left(10^{-\kappa_s/10} - 1\right)\right]}{2\log\left(\Omega_p/\Omega_s\right)} \right\rceil$$

number of Butterworth filter pole pairs

Once N is selected, decide whether the Ω_p or Ω_s requirement is to be met exactly (the other requirement will then be achieved and exceeded due to the $\lceil \bullet \rceil$ operator used in computing N)

To meet the Ω_p requirement (and exceed the Ω_s requirement), set

$$\Omega_c = \frac{\Omega_p}{\left(10^{-\kappa_p/10} - 1\right)^{1/(2N)}}$$

To meet the Ω_s requirement (and exceed the Ω_p requirement), set

$$\Omega_c = \frac{\Omega_s}{\left(10^{-\kappa_s/10} - 1\right)^{1/(2N)}}$$

We now have the necessary parameters (Ω_c and N) that specify the filter. We shall use the filter order N to select the normalized Butterworth filter and then transform the normalized filter using the specific desired cutoff frequency Ω_c .

For some of the common orders of the Butterworth filter, the normalized filter response $H_n(s)$ is specified below. The normalized filter response is a lowpass template which we may transform into a particular filter realization (*i.e.* lowpass, highpass, etc..). Note normalized $\Omega_c = 1$.

$N=1$	$1/(s+1)$
$N=2$	$1/\left(s^2+\sqrt{2}\,s+1\right)$
$N=3$	$1/\left((s+1)(s^2+s+1)\right)$
$N=4$	$1/\left((s^2+0.7654s+1)(s^2+1.8478s+1)\right)$
$N=5$	$1/\left((s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)\right)$
$N=6$	$1/\left((s^2+0.5176s+1)(s^2+\sqrt{2}\,s+1)(s^2+1.9318s+1)\right)$
$N=7$	$1/\left((s+1)(s^2+0.4450s+1)(s^2+1.2470s+1)(s^2+1.8022s+1)\right)$
$N=8$	$1/\left((s^2+0.3986s+1)(s^2+1.1110s+1)(s^2+1.6630s+1)(s^2+1.9622s+1)\right)$

Transformations in the analog domain

To obtain the desired analog filter from the normalized filter response, we apply the appropriate analog filter transformation below:

normalized lowpass to lowpass $$s \rightarrow \frac{s}{\Omega_c}$$

normalized lowpass to highpass $$s \rightarrow \frac{\Omega_c}{s}$$

where Ω_c is the desired 3-dB cutoff. Also,

normalized lowpass to bandpass $$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

normalized lowpass to bandstop $$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

where Ω_l and Ω_u are the lower and upper band edge frequencies, respectively.

Designing a digital filter to meet analog frequency specs

1) Translate the frequency specs from analog to digital using $\omega = \Omega T$.

2) Pre-warp to obtain the normalized analog frequency specs using $\tilde{\Omega} = (2/T)\tan(\omega/2)$.
Note: we can set $T = 1$ here without loss of generality.

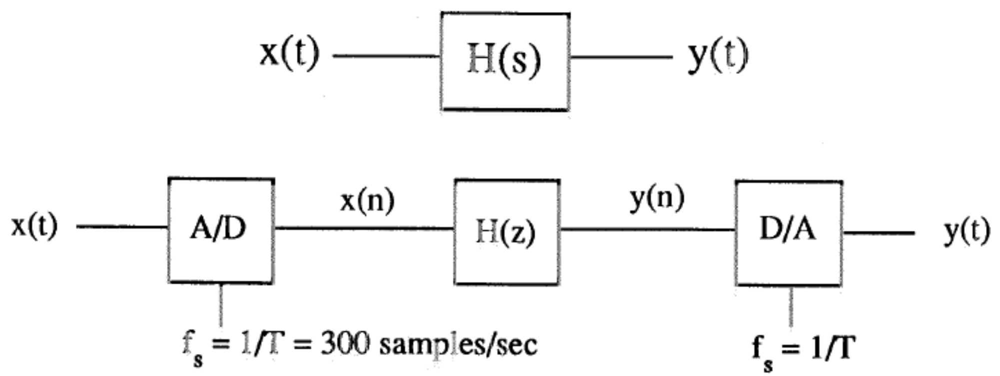
3) Determine N and $\tilde{\Omega}_c$ for Butterworth filter that meet design criteria using pre-warped specs.

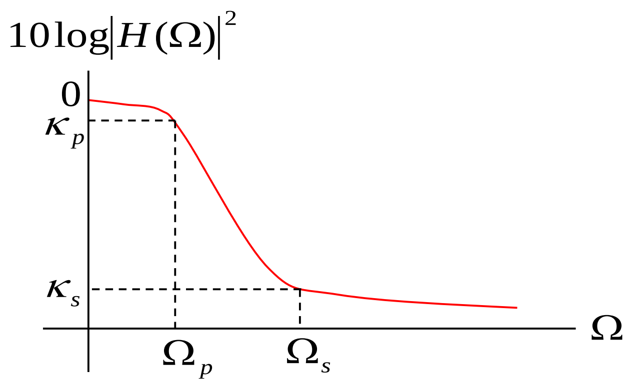
4) Select appropriate filter response from Butterworth filter table and apply analog transformation.

5) Apply bilinear transform to convert to a digital filter.
Note: again use $T = 1$ without loss of generality.

Note: we could also perform filter transformations in the digital domain (we did not cover this) which enables more general capability for Approximation of Derivatives and Impulse Invariant filter design techniques.

Example: Lets design a digital filter H(z) which meets the same specs as the analog system, as shown below.





$$\Omega_p = 10\pi \quad \rightarrow \quad \kappa_p = -2 \text{ dB}$$

$$\Omega_s = 60\pi \quad \rightarrow \quad \kappa_s = -20 \text{ dB}$$

1) Translate to digital specs

$$\omega_p = \frac{10\pi}{300} = \frac{\pi}{30}$$

$$\omega_s = \frac{60\pi}{300} = \frac{\pi}{5}$$

2) Pre-warp to obtain the analog filter specs

$$\tilde{\Omega}_p = 2\tan\left(\frac{1}{2}\cdot\frac{\pi}{30}\right) = 0.1048 \qquad \tilde{\Omega}_s = 2\tan\left(\frac{1}{2}\cdot\frac{\pi}{5}\right) = 0.6498$$

3) Design Butterworth filter

$$N = \left\lceil \frac{\log\left[\left(10^{-(-2)/10}-1\right)/\left(10^{-(-20)/10}-1\right)\right]}{2\log(0.1048/0.6498)} \right\rceil = \lceil 1.4062 \rceil = 2$$

Let's satisfy the κ_p requirement and exceed the κ_s requirement

$$\tilde{\Omega}_c = \frac{\tilde{\Omega}_p}{\left(10^{-\kappa_p/10}-1\right)^{1/2N}} = \frac{0.1048}{\left(10^{-(-2)/10}-1\right)^{1/(2\cdot 2)}} = 0.1198$$

4) From table of Butterworth normalized filters (for N = 2) apply analog lowpass transformation

$$H_a(s) = \left.\frac{1}{\left(s^2+\sqrt{2}\,s+1\right)}\right|_{s=\frac{s}{0.1198}} = \frac{1}{\left(69.68\,s^2+11.80\,s+1\right)}$$

5) Finally, apply bilinear transform

$$H(z) = \left.\frac{1}{\left(69.68\,s^2 + 11.80\,s + 1\right)}\right|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{1 + 2\,z^{-1} + z^{-2}}{304 - 556\,z^{-1} + 256\,z^{-2}}$$

Hence, the resulting algorithm is

$$y(n) = \frac{1}{304}\,x(n) + \frac{2}{304}\,x(n-1) + \frac{1}{304}\,x(n-2) + \frac{556}{304}\,y(n-1) - \frac{256}{304}\,y(n-2)$$

note that the current output is dominated by the previous outputs and new input values have very small effect which is what we should expect for a lowpass system

8.5 Summary of IIR Filter Design Techniques

IIR Filter Design: Traditionally based on the transformation of known analog filter characteristics into equivalent digital filters.

- Impulse invariant approach
  - impulse response of original analog filter is preserved
  - amplitude-frequency response is not preserved (due to aliasing)
  - inherent aliasing makes this method unsuitable for highpass or bandpass systems
  - could be followed by a digital transformation to design highpass, etc..

- Bilinear Transform approach
  - yields very efficient filters (few coefficients for required performance)
  - can model Butterworth, Chebyshev, Elliptic, etc., filters
  - preserves amplitude-frequency response characteristic
  - does not preserve time-domain properties