1) a) $h(n) = \left\{ -\frac{1}{4}, -\frac{1}{2}, +\frac{1}{4}, +\frac{1}{2} \right\}$

$$H(\omega) = \sum_{n=0}^{3} h(n)\, e^{-j\omega n} = -\frac{1}{4} - \frac{1}{2} e^{-j\omega} + \frac{1}{4} e^{-j2\omega} + \frac{1}{2} e^{-j3\omega}$$

<u>no symmetry, <u>not</u> linear phase</u>

using Euler's identity:

$$H(\omega) = -\frac{1}{4} - \frac{1}{2}\left[\cos(\omega) - j\sin(\omega)\right] + \frac{1}{4}\left[\cos(2\omega) - j\sin(2\omega)\right] + \frac{1}{2}\left[\cos(3\omega) - j\sin(3\omega)\right]$$

so that we can collect terms as:

$$\mathrm{Re}\{H(\omega)\} = -\frac{1}{4} - \frac{1}{2}\cos(\omega) + \frac{1}{4}\cos(2\omega) + \frac{1}{2}\cos(3\omega)$$

$$\mathrm{Im}\{H(\omega)\} = \frac{1}{2}\sin(\omega) - \frac{1}{4}\sin(2\omega) - \frac{1}{2}\sin(3\omega)$$

thus

$$\phi(\omega) = \tan^{-1}\left\{ \frac{\frac{1}{2}\sin(\omega) - \frac{1}{4}\sin(2\omega) - \frac{1}{2}\sin(3\omega)}{-\frac{1}{4} - \frac{1}{2}\cos(\omega) + \frac{1}{4}\cos(2\omega) + \frac{1}{2}\cos(3\omega)} \right\}$$

1) b) $h(n) = \{1, -2, 1, 2, -1\}$

appears anti-symmetric, but center term is not zero (and $+1 \neq -1$) $\Rightarrow$ <u>not linear phase</u>

$$H(\omega) = 1 - 2e^{-j\omega} + 1e^{-2j\omega} + 2e^{-j3\omega} - 1e^{-j4\omega}$$

$$= 1 - 2[\cos(\omega) - j\sin(\omega)] + 1[\cos(2\omega) - j\sin(2\omega)]$$
$$+ 2[\cos(3\omega) - j\sin(3\omega)] - 1[\cos(4\omega) - j\sin(4\omega)]$$

so $\mathrm{Re}\{H(\omega)\} = 1 - 2\cos(\omega) + \cos(2\omega) + 2\cos(3\omega) - \cos(4\omega)$

$$\mathrm{Im}\{H(\omega)\} = 2\sin(\omega) - \sin(2\omega) - 2\sin(3\omega) + \sin(4\omega)$$

and

$$\phi(\omega) = \tan^{-1}\left\{\frac{2\sin(\omega) - \sin(2\omega) - 2\sin(3\omega) + \sin(4\omega)}{1 - 2\cos(\omega) + \cos(2\omega) + 2\cos(3\omega) - \cos(4\omega)}\right\}$$

1) c) $$h(n) = \underbrace{\sum_{k=1}^{16} \left(\frac{k}{16}\right) \delta(n-k)}_{=0 \text{ @ } k=0} - \underbrace{\sum_{k=0}^{15} \left(\frac{16-k}{16}\right) \delta(n-k)}_{=0 \text{ @ } k=16}$$

so $$h(n) = \sum_{k=0}^{16} \left(\frac{k-(16-k)}{16}\right) \delta(n-k) = \sum_{k=0}^{16} \left(\frac{2k}{16} - 1\right) \delta(n-k)$$

then $$h(n) = \left\{-1, \frac{-14}{16}, \frac{-12}{16}, \frac{-10}{16}, \frac{-8}{16}, \frac{-6}{16}, \frac{-4}{16}, \frac{-2}{16}, 0, \frac{+2}{16}, \frac{+4}{16}, \ldots, \frac{14}{16}, +1\right.$$

$\rightarrow$ anti-symmetric $\Rightarrow$ <u>linear phase</u>

$\rightarrow$ odd length $\Rightarrow N=17 \Rightarrow$ Type 3

therefore $$\phi(\omega) = -\omega\left(\frac{17-1}{2}\right) \pm \frac{\pi}{2} = \underline{-8\omega \pm \pi/2}$$

and $$\tau(\omega) = \frac{-\partial \phi(\omega)}{\partial \omega} = \underline{8 \text{ samples}}$$

2) $y(n) = 0.8\,y(n-1) + x(n) + 3x(n-1) + x(n-2) - x(n-4) + 2x(n-5)$

$\Rightarrow$ based on magnitude plot on next page, $\rightarrow$ <u>low-pass</u>

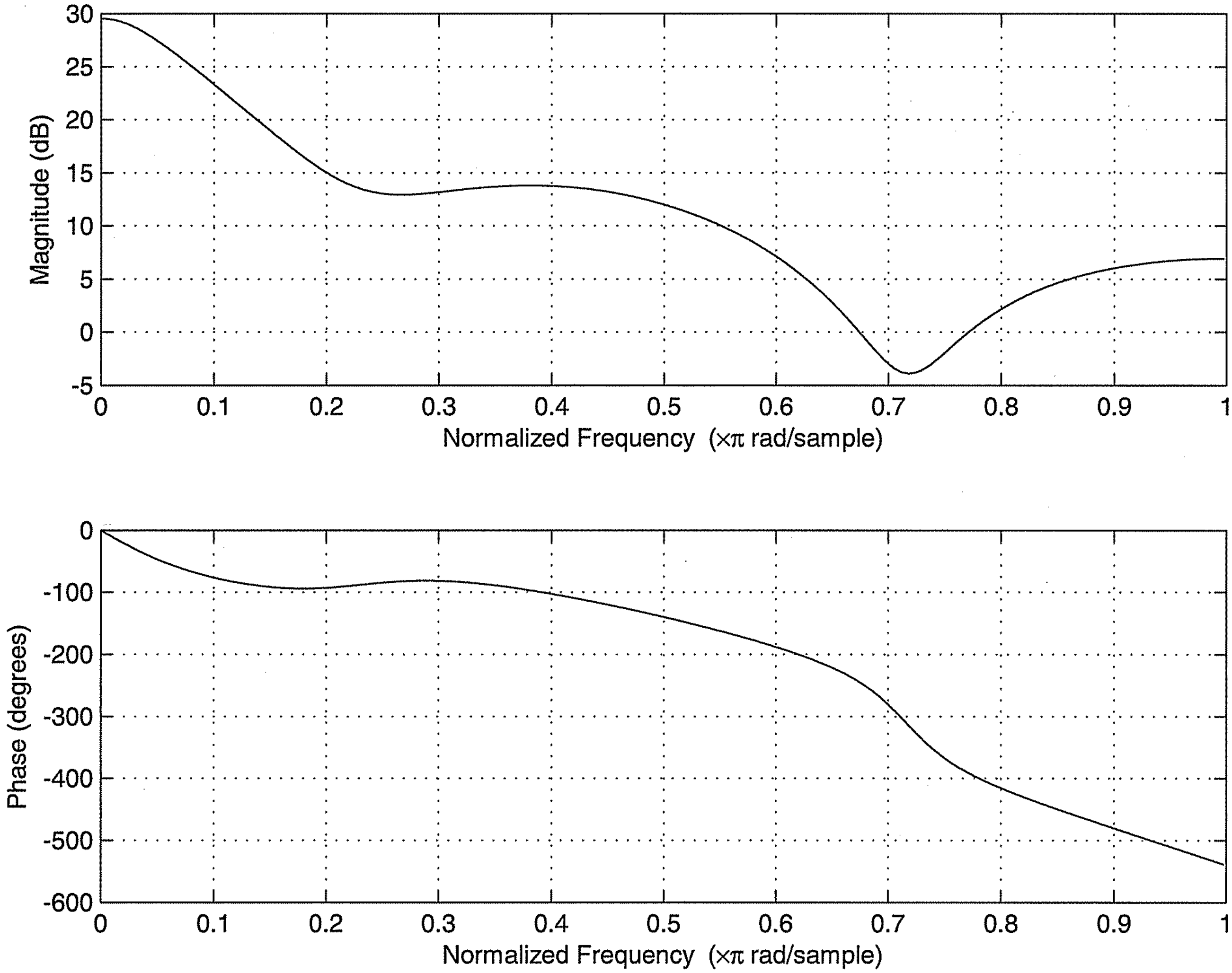
$\rightarrow$ transform to high-pass as

$$y(n) = -\,0.8\,y(n-1) + x(n) - 3x(n-1) + x(n-2) - x(n-4) - 2x(n-5$$

(-1) at odd indices

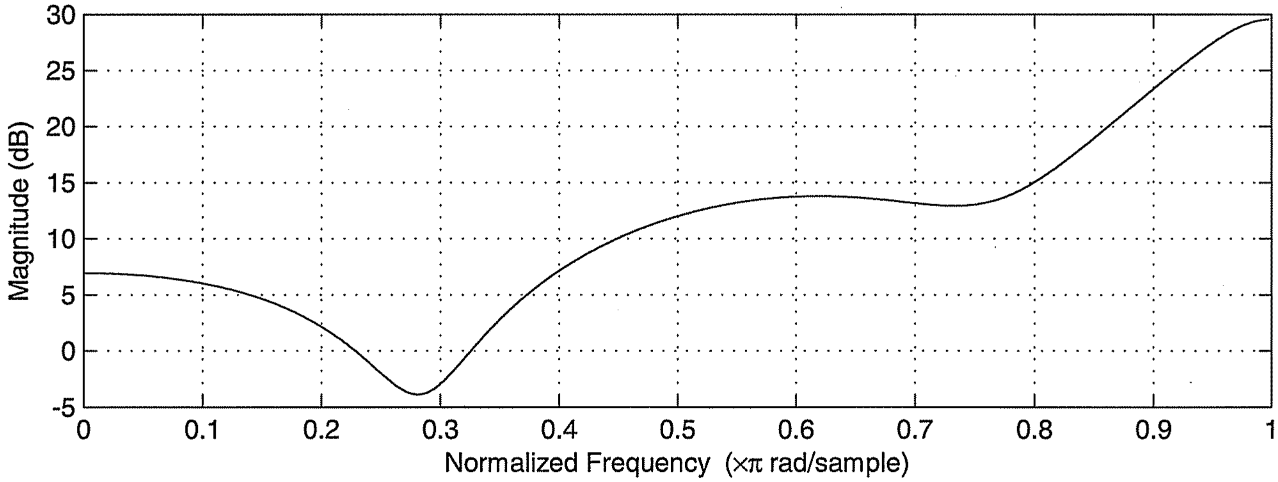
original filter

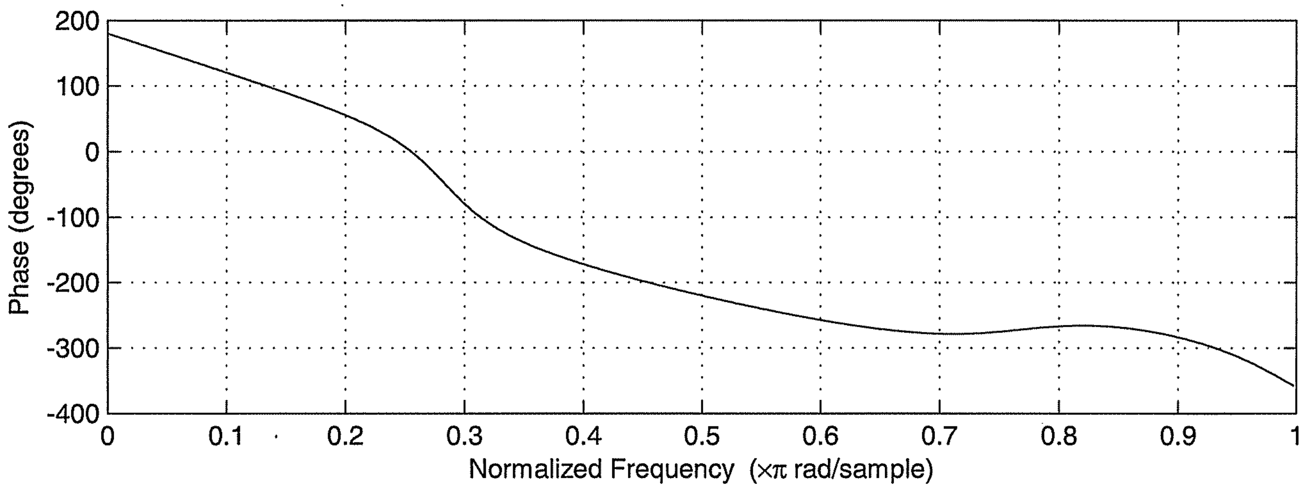
freqz([1 3 1 0 -1 2],[1 -0.8])



transformed filter

freqz([1 -3 1 0 -1 -2],[1 +0.8])





3) $$H(z) = \frac{z^2 + 1.44}{z^3 - 0.2z^2 - 0.48z} = \frac{(z+j1.2)(z-j1.2)}{z(z-0.8)(z+0.6)}$$

$\Rightarrow$ max @ $\omega = 0$ due to pole @ $z = +0.8$

$\Rightarrow$ min @ $\omega = \pm\frac{\pi}{2}$ due to zeros @ $z = \pm j1.2$

so $$|H(\omega=0)| = \left| \frac{(e^{j0}+j1.2)(e^{j0}-j1.2)}{e^{j0}(e^{j0}-0.8)(e^{j0}+0.6)} \right| = 7.625 = |H_{max}|$$

$$|H(\omega=\pi/2)| = \left| \frac{(e^{j\pi/2}+j1.2)(e^{j\pi/2}-j1.2)}{e^{j\pi/2}(e^{j\pi/2}-0.8)(e^{j\pi/2}+0.6)} \right| = 0.295 = |H_{min}|$$

so null depth $= 20\log_{10}\left(\frac{|H_{max}|}{|H_{min}|}\right) = \underline{28.26 \text{ dB}}$

4) $h(n) = \{-1, 2, 1, -2, 1\}$

$$h_e(n) = \frac{1}{2}\left[\{\underset{\uparrow}{-1}, 2, 1, -2, 1\} + \{1, -2, 1, 2, \underset{\uparrow}{-1}\}\right]$$

$$= \frac{1}{2}\{1, -2, 1, 2, -1-1, 2, 1, -2, 1\}$$

$$= \{\tfrac{1}{2}, -1, \tfrac{1}{2}, 1, \underset{\substack{\uparrow \\ n=0}}{-1}, 1, \tfrac{1}{2}, -1, \tfrac{1}{2}\} \Rightarrow \text{symmetric + odd length (Type 1)}$$

$$h_o(n) = \frac{1}{2}\left[\{\underset{\uparrow}{-1}, 2, 1, -2, 1\} - \{1, -2, 1, 2, \underset{\uparrow}{-1}\}\right]$$

$$= \frac{1}{2}\{-1, 2, -1, -2, -1+1, 2, 1, -2, 1\}$$

$$= \{-\tfrac{1}{2}, 1, -\tfrac{1}{2}, -1, \underset{\substack{\uparrow \\ n=0}}{0}, 1, \tfrac{1}{2}, -1, \tfrac{1}{2}\} \Rightarrow \text{anti-symmetric + odd length (Type 3)}$$

$$H_e(\omega) = [-1 + 2\cos(\omega) + \cos(2\omega) - 2\cos(3\omega) + \cos(4\omega)]\, e^{j0} \leftarrow \text{from Type 1 in notes}$$

$$H_o(\omega) = [-2\sin(\omega) - \sin(2\omega) + 2\sin(3\omega) - \sin(4\omega)]\, e^{j0 + j\pi/2} \leftarrow \text{from Type 3 in notes}$$

by linearity, $\underset{total}{H(\omega)} = H_e(\omega) + H_o(\omega)$

so while $H_e(\omega)$ and $H_o(\omega)$ can each be expressed in the form $H(\omega) = \underbrace{A(\omega)}_{\text{all real}} e^{-j\alpha\omega + j\beta}$,

their sum $H_e(\omega) + H_o(\omega)$ cannot.

5) $$H(z) = \frac{(z+\frac{1}{4})(z-\frac{1}{4})(z+2)(z-4)}{z^2(z+j\frac{1}{4})(z-j\frac{1}{4})}$$ since stable (& poles inside unit circle), must be causal

decompose mixed-phase system as

$$B_1(z) = (z+\tfrac{1}{4})(z-\tfrac{1}{4})$$
$$B_2(z) = (z+2)(z-4)$$
$$A(z) = z^2(z+j/4)(z-j/4)$$

thus $$\frac{B_2(z^{-1})}{B_2(z^{-1})} = \frac{(z^{-1}+2)(z^{-1}-4)}{(z^{-1}+2)(z^{-1}-4)} \times \frac{z^2}{z^2} = \frac{(1+2z)(1-4z)}{(1+2z)(1-4z)} \times \frac{-1/8}{-1/8} = \frac{(z+\frac{1}{2})(z-1/4)}{(z+1/2)(z-1/4)}$$

so $$H_{min}(z) = \frac{B_1(z)\,B_2(z^{-1})}{A(z)} = \frac{(z+1/4)(z-1/4)(z+1/2)(z-1/4)}{z^2(z+j/4)(z-j/4)}$$

$$H_{ap}(z) = \frac{B_2(z)}{B_2(z^{-1})} = \frac{(z+2)(z-4)}{(z+1/2)(z-1/4)}$$ where $|H_{ap}(\omega)| = 1$ by definition

thus $|G(\omega)\,H(\omega)| = \text{constant}$ is achieved when

$$G(z) = \frac{1}{H_{min}(z)} = \frac{z^2(z+j/4)(z-j/4)}{(z+1/4)(z-1/4)(z+1/2)(z-1/4)}$$

stable because all poles inside the unit circle

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